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Asymptotic Behavior of Solution to Nonlinear Eigenvalue Problem

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Abstract: We study the following nonlinear eigenvalue problem: $-u''(t) = \lambda f(u(t))$, $u(t) > 0$, $t \in I := (-1,1)$, $u(\pm 1) = 0$, where $f(u) = \log(1+u)$ and $\lambda > 0$ is a parameter. Then λ is a continuous function of $\alpha > 0$, where α is the maximum norm $\alpha = \|u_\lambda\|_\infty$ of the solution u_λ associated with λ. We establish the precise asymptotic formula for L^1-norm of the solution $\|u_\alpha\|_1$ as $\alpha \to \infty$ up to the second term and propose a numerical approach to obtain the asymptotic expansion formula for $\|u_\alpha\|_1$.

Keywords: asymptotic expansion; L^1-norm of the solution; nonlinear eigenvalue problems

1. Introduction

We consider the following nonlinear eigenvalue problems

$$-u''(t) = \lambda f(u(t)), \quad t \in I := (-1,1), \tag{1}$$

$$u(t) > 0, \quad t \in I, \tag{2}$$

$$u(-1) = u(1) = 0, \tag{3}$$

where $\lambda > 0$ is a parameter. In this paper, we consider the case $f(u) = \log(1+u)$, which is motivated by the logarithmic Schroedinger equation (see [1]) and the Klein-Gordon equation with logarithmic potential, which has been introduced in the quantum field theory (see [2]). We know from [3] that, if $f(u)$ is continuous in $u \geq 0$ and positive for $u > 0$, then for a given $\alpha > 0$, there exists a unique classical solution pair (λ, u_α) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$ for any given $\alpha > 0$. Since $(u_\alpha, \lambda(\alpha))$ is constructed explicitly by time-map method (cf. [3], Theorem 2.1), λ is a continuous for $\alpha > 0$, we write as $\lambda = \lambda(\alpha)$ for $\alpha > 0$.

We introduce one of the most famous results for bifurcation curve, which was shown for the one-dimensional Gelfand problem, namely, the Equations (1)–(3) with $f(u) = e^u$. Then it was shown in [4] that it has the exact solution

$$u_\alpha(t) = \alpha + \log\left(\operatorname{sech}^2\left(\frac{\sqrt{2\lambda(\alpha)}}{2} t e^{\alpha/2}\right)\right), \tag{4}$$

where $\operatorname{sech} x = 1/\cosh x$. The related results have been obtained in [5]. Unfortunately, however, such explicit solution as (4) cannot be expected in general. From this point of view, one of the standard approach for the better understanding of the asymptotic shape of $u_\alpha(t)$ is to establish precise asymptotic expansion formula for $u_\alpha(t)$ as $\alpha \to \infty$. Indeed, in some cases, the asymptotic expansion formulas for $u_\alpha(t)$ up to the second term have been obtained. Regrettably, however, precise asymptotic expansion formula for u_α is also difficult to obtain from technical point of view of pure mathematics.

In this paper, in order to understand the asymptotic behavior of u_α, we establish the asymptotic expansion formula for $\|u_\alpha\|_1$ as $\alpha \to \infty$. The importance of this view point is that, $\|u_\alpha\|_p$ ($p \ge 1$) characterizes (or is related to) the many significant properties, such as the density of the objects in quantum physics, logistic equation in biology, and so on. Moreover, by using this formula, it is possible to obtain the approximate value of $\|u_\alpha\|_1$ numerically as accurate as they want. If the readers observe Theorem 2 and Section 3 below, they understand immediately that it is impossible to purchase more correct approximate value by hand calculation any more .

Now we state our main results.

Theorem 1. *Let $f(u) = \log(1+u)$. Consider (1)–(3). For an arbitrary fixed small constant $0 < \epsilon \ll 1$, let $w_{\alpha,\epsilon}(t) := u_\alpha(t) - \alpha(-t^2+1)$ defined on the compact interval $I_\epsilon := [-1+\epsilon, 1-\epsilon] \subset I$. Then as $\alpha \to \infty$,*

$$\frac{w_{\alpha,\epsilon}(t)}{\alpha} \to 0 \tag{5}$$

uniformly on I_ϵ

Theorem 2. *Let $f(u) = \log(1+u)$. Consider (1)–(3). Then as $\alpha \to \infty$,*

$$\|u_\alpha\|_1 = 2\alpha \left\{ \frac{2}{3} + \left(-\frac{4}{3}\log 2 + \frac{8}{9} \right) \frac{1}{\log(1+\alpha)} + O\left(\frac{1}{(\log(1+\alpha))^2} \right) \right\}. \tag{6}$$

The leading term $4\alpha/3$ in the right-hand side of (6) comes from (5) immediately. The most important point of (6) is to give the procedure to obtain the asymptotic expansion formula for $\|u_\alpha\|_1$ as correct as we want by using computer-assisted method, although we only obtain up to the second term of $\|u_\alpha\|_1$, since the calculation is purchased by hand.

The proof depends on the time-map argument and the precise asymptotic formula for $\lambda(\alpha)$ as $\alpha \to \infty$.

2. Proof of Theorem 1

We put $v_\alpha(t) := u_\alpha(t)/\alpha$. It is known that if $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1)–(3), then

$$u_\alpha(t) = u_\alpha(-t), \quad 0 \le t \le 1, \tag{7}$$

$$u_\alpha(0) = \max_{-1 \le t \le 1} u_\alpha(t) = \alpha, \tag{8}$$

$$u'_\alpha(t) > 0, \quad -1 < t < 0. \tag{9}$$

We recall the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to \infty$.

Theorem 3. **([6]).** *Let $f(u) = \log(1+u)$ and consider (1)–(3). Then as $\alpha \to \infty$,*

$$\begin{aligned} \sqrt{\lambda(\alpha)} \quad = \quad & \sqrt{\frac{2\alpha}{\log(1+\alpha)}} \left\{ 1 - \frac{1}{2}(4\log 2 - 3)\frac{1}{\log(1+\alpha)} \right. \\ & \left. + \frac{3}{8}(5 - 8\log 2 + C_1)\frac{1}{(\log(1+\alpha))^2} \right\} + R_3, \end{aligned} \tag{10}$$

where C_1 is a constant explicitly represented by elementary definite integrals, R_3 is the remainder term satisfying

$$b_3(\log(1+\alpha))^{-3} \le |R_3| \le b_3^{-1}((\log(1+\alpha))^{-3}, \tag{11}$$

where $0 < b_3 < 1$ is a constant independent of $\alpha \gg 1$.

For any arbitrary fixed small $\epsilon > 0$, we set $I_\epsilon := [-1+\epsilon, 1-\epsilon]$. Then it is obvious that there is a constant $\delta_\epsilon > 0$ independent of $\alpha \gg 1$ such that $0 < \delta_\epsilon \le v_\alpha(t) \le 1$ for $t \in I_\epsilon$. Indeed, if there is a subsequence of $\{v_\alpha\}$, which is denoted by $\{v_\alpha\}$ again, such that $v_\alpha(-1+\epsilon) \to 0$ as $\alpha \to \infty$, then $v_\alpha(t)$ is not convex on $[-1, 0]$ for $\alpha \gg 1$. By (1) and Theorem 3, for any $t \in I_\epsilon$, we have

$$\begin{aligned} -\alpha v_\alpha''(t) &= \frac{2\alpha}{\log(1+\alpha)}(1+o(1))\log(1+\alpha v_\alpha(t)) \\ &= \frac{2\alpha}{\log(1+\alpha)}(1+o(1))(\log\alpha + \log v_\alpha(t)). \end{aligned} \tag{12}$$

By this, we see that as $\alpha \to \infty$

$$-v_\alpha''(t) \to 2 \tag{13}$$

uniformly on I_ϵ. By this, we easily see that $v_\alpha'(t) \to 2t$ and $v_\alpha(t) \to 1 - t^2$ for $t \in I_\epsilon$, which implies (5). Thus the proof is complete. □

3. Proof of Theorem 2

In this section, let $\alpha \gg 1$. In what follows, we denote by C the various positive constants independent of α. We modify the time-map method used in [6]. By (1), we have

$$\{u_\alpha''(t) + \lambda(\log(1+u_\alpha(t))\}\, u_\alpha'(t) = 0.$$

By this, (8) and putting $t = 0$, we obtain

$$\begin{aligned} &\frac{1}{2}u_\alpha'(t)^2 + \lambda\{u_\alpha(t)\log(1+u_\alpha(t)) - u_\alpha(t) + \log(1+u_\alpha(t))\} = \text{const.} \\ &= \lambda\{\alpha\log(1+\alpha) - \alpha + \log(1+\alpha)\}. \end{aligned}$$

This along with (9) implies that for $-1 \le t \le 0$,

$$u_\alpha'(t) = \sqrt{2\lambda}\sqrt{\alpha\log(1+\alpha) - u_\alpha(t)\log(1+u_\alpha(t)) + \xi(u_\alpha(t))}, \tag{14}$$

where

$$\xi(u) := \log(1+\alpha) - \log(1+u) - (\alpha - u). \tag{15}$$

By this and putting $u_\alpha(t) = \alpha s^2$, we obtain

$$\begin{aligned} \|u_\alpha\|_1 &= 2\int_{-1}^0 u_\alpha(t)dt = \sqrt{\frac{2}{\lambda}}\int_{-1}^0 \frac{u_\alpha(t)u_\alpha'(t)}{\sqrt{\alpha\log(1+\alpha) - u_\alpha(t)\log(1+u_\alpha(t)) + \xi(u_\alpha(t))}}dt \\ &= \sqrt{\frac{2}{\lambda}}\int_0^1 \frac{2\alpha^2 s^3}{\sqrt{\alpha(1-s^2)\log(1+\alpha) + \alpha s^2 A_\alpha(s) + \xi(\alpha s^2)}}ds \\ &:= \sqrt{\frac{2}{\lambda}}\frac{2\alpha^{3/2}}{\sqrt{\log(1+\alpha)}}\int_0^1 \frac{s^3}{\sqrt{1-s^2}}\frac{1}{\sqrt{1+g_\alpha(s)}}ds, \end{aligned} \tag{16}$$

where

$$A_\alpha(s) := \log(1+\alpha) - \log(1+\alpha s^2), \tag{17}$$

$$g_\alpha(s) := \frac{1}{\log(1+\alpha)}\frac{s^2}{(1-s^2)}A_\alpha(s) + \frac{\xi(\alpha s^2)}{\alpha(1-s^2)\log(1+\alpha)}. \tag{18}$$

For $0 \le s \le 1$, we have

$$\begin{aligned}
\left|\frac{1}{\log(1+\alpha)}\frac{s^2}{(1-s^2)}A_\alpha(s)\right| &\le \frac{s^2}{1-s^2}\frac{1}{\log(1+\alpha)}\int_{\alpha s^2}^{\alpha}\frac{1}{1+x}dx \\
&\le \frac{1}{\log(1+\alpha)}\frac{\alpha s^2}{1+\alpha s^2} \\
&\le \frac{1}{\log(1+\alpha)} \ll 1,
\end{aligned} \tag{19}$$

$$\left|\frac{\xi(\alpha s^2)}{\alpha(1-s^2)\log(1+\alpha)}\right| \le \frac{2}{\log(1+\alpha)} \ll 1. \tag{20}$$

By (16)–(20) and Taylor expansion, we obtain

$$\begin{aligned}
\|u_\alpha\|_1 &= \sqrt{\frac{2}{\lambda}}\frac{2\alpha^{3/2}}{\sqrt{\log(1+\alpha)}}\int_0^1\frac{s^3}{\sqrt{1-s^2}}\left\{1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{n!2^n}g_\alpha(s)^n\right\}ds \qquad (21)\\
&= \sqrt{\frac{2}{\lambda}}\frac{2\alpha^{3/2}}{\sqrt{\log(1+\alpha)}}\int_0^1\frac{s^3}{\sqrt{1-s^2}}\left\{1-\frac{1}{2}g_\alpha(s)+O\left(\frac{1}{(\log(1+\alpha))^3}\right)\right\}ds,
\end{aligned}$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1, (-1)!! = 1$. We have

$$\int_0^1\frac{s^3}{\sqrt{1-s^2}}ds = \frac{2}{3}. \tag{22}$$

Lemma 1. *As $\alpha \to \infty$*

$$\begin{aligned}
J_1 &:= \int_0^1\frac{s^3}{(1-s^2)^{3/2}}g_\alpha(s)ds \qquad (23)\\
&= \frac{1}{\log(1+\alpha)}\left(\frac{16}{3}\log 2-\frac{34}{9}\right)+O\left(\frac{1}{\alpha\log(1+\alpha)}\right).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
J_1 &= \frac{1}{\log(1+\alpha)}\int_0^1\frac{s^5}{(1-s^2)^{3/2}}A_\alpha(s)ds+\frac{1}{\alpha\log(1+\alpha)}\int_0^1\frac{s^3}{(1-s^2)^{3/2}}\xi_\alpha(\alpha s^2)ds \\
&:= \frac{1}{\log(1+\alpha)}J_{11}+\frac{1}{\alpha\log(1+\alpha)}J_{12}.
\end{aligned} \tag{24}$$

It was shown in [6] that as $\alpha \to \infty$,

$$0 \le \sin^2\theta A_\alpha(\sin\theta) = -2\sin^2\theta\log(\sin\theta)+O\left(\frac{1}{\alpha}\right). \tag{25}$$

Moreover, it is clear that for $0 \le \theta \le \pi/2$

$$\sin^2\theta-\frac{1}{\alpha} \le \sin^2\theta\left(\frac{\alpha\sin^2\theta}{1+\alpha\sin^2\theta}\right) \le \sin^2\theta. \tag{26}$$

We put $s = \sin\theta$. By (25), (26) and integration by parts, we obtain

$$\begin{aligned} J_{11} &= \int_0^{\pi/2} \frac{\sin^5\theta}{\cos^2\theta} A_\alpha(\sin\theta) d\theta \\ &= \Big[\tan\theta(\sin^5\theta A_\alpha(\sin\theta))\Big]_0^{\pi/2} \\ &\quad - 5\int_0^{\pi/2} \sin^5\theta A_\alpha(\sin\theta) d\theta + 2\int_0^{\pi/2} \frac{\alpha\sin^2\theta}{1+\alpha\sin^2\theta}\sin^5\theta d\theta \\ &= J_{111} + 10\int_0^{\pi/2} \sin^5\theta\log(\sin\theta) d\theta + 2\int_0^{\pi/2}\sin^5\theta d\theta + O(\alpha^{-1}) \\ &:= J_{111} + 10J_{112} + 2J_{113} + O(\alpha^{-1}). \end{aligned} \tag{27}$$

By l'Hôpital's rule, we obtain

$$\lim_{\theta\to\pi/2} \tan\theta(\sin^5\theta A_\alpha(\sin\theta)) = \lim_{\theta\to\pi/2} \frac{A_\alpha(\sin\theta)}{\cos\theta} = \lim_{\theta\to\pi/2} \frac{2\alpha\cos\theta}{1+\alpha\sin^2\theta} = 0. \tag{28}$$

This implies that $J_{111} = 0$. By (25), we have

$$\begin{aligned} J_{112} &= \int_0^{\pi/2} \sin^5\theta\log(\sin\theta) d\theta \\ &= \int_0^{\pi/2} (1-\cos^2\theta)^2 \log\sqrt{1-\cos^2\theta}\sin\theta d\theta \\ &= \frac{1}{2}\int_0^1 (1-x^2)^2\log(1-x^2) dx = \frac{8}{15}\log 2 - \frac{94}{225}. \end{aligned} \tag{29}$$

By (26), we have

$$J_{113} = \int_0^{\pi/2} \sin^5\theta d\theta + O(\alpha^{-1}) = \frac{8}{15} + O(\alpha^{-1}). \tag{30}$$

By this, we have

$$J_{11} = \frac{16}{3}\log 2 - \frac{28}{9} + O(\alpha^{-1}). \tag{31}$$

We put $s = \sin\theta$. Then by integration by parts, (25) and (26), we have

$$\begin{aligned} J_{12} &= \int_0^{\pi/2} \frac{1}{\cos^2\theta}\sin^3\theta(A_\alpha(\sin\theta) - \alpha\cos^2\theta) d\theta \\ &= \big[\tan\theta(\sin^3\theta A_\alpha(\sin\theta))\big]_0^{\pi/2} - 3\int_0^{\pi/2}\sin^3\theta A_\alpha(\sin\theta) d\theta + 3\alpha\int_0^{\pi/2}\sin^3\theta d\theta \\ &\quad +2\int_0^{\pi/2}\sin^3\theta\frac{\alpha\sin^2\theta}{1+\alpha\sin^2\theta} d\theta - 5\alpha\int_0^{\pi/2}\sin^5\theta d\theta \\ &= -\frac{2}{3}\alpha + 6\int_0^{\pi/2}\sin^3\theta\log(\sin\theta) d\theta + 2\int_0^{\pi/2}\sin^3\theta d\theta + O(\alpha^{-1}) \\ &= -\frac{2}{3}\alpha + 6\left(\frac{2}{3}\log 2 - \frac{5}{9}\right) + \frac{4}{3} + O(\alpha^{-1}) \\ &= -\frac{2}{3}\alpha + 4\log 2 - 2 + O(\alpha^{-1}). \end{aligned} \tag{32}$$

By (24), (31) and (32), we obtain

$$J_1 = \frac{1}{\log(1+\alpha)}\left(\frac{16}{3}\log 2 - \frac{34}{9}\right) + O\left(\frac{1}{\alpha\log(1+\alpha)}\right). \tag{33}$$

Thus the proof is complete. □

Proof of Theorem 2. By Theorem 3, Lemma 1, (21) and direct calculation, we obtain Theorem 2. Thus the proof is complete. □

Remark 1. *Unfortunately, the author is not familiar with computing an analytical formula through computer packages. All the calculations in this paper were obtained by hand. As the authors mentioned in Section 1, if the suitable computer packages will be found, then it will be possible to compute the term*

$$J_n := \int_0^1 \frac{s^3}{(1-s^2)^{3/2}} g_\alpha(s)^n ds \qquad (n = 2, 3, \cdots). \tag{34}$$

By this, (21) and computer, it will be possible to calculate the approximate value of $\|u_\alpha\|_1$ as correct as we want.

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