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Article

# On the Affine Image of a Rational Surface of Revolution

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**Abstract:** We study the properties of the image of a rational surface of revolution under a nonsingular affine mapping. We prove that this image has a notable property, namely that all the *affine normal lines*, a concept that appears in the context of *affine differential geometry*, created by Blaschke in the first decades of the 20th century, intersect a fixed line. Given a rational surface with this property, which can be algorithmically checked, we provide an algorithmic method to find a surface of revolution, if it exists, whose image under an affine mapping is the given surface; the algorithm also finds the affine transformation mapping one surface onto the other. Finally, we also prove that the only rational *affine surfaces of rotation*, a generalization of surfaces of revolution that arises in the context of affine differential geometry, and which includes surfaces of revolution as a subtype, affinely transforming into a surface of revolution are the surfaces of revolution, and that in that case the affine mapping must be a similarity.

**Keywords:** surface of revolution; affine differential geometry; affine equivalence

## 1. Introduction

Surfaces of revolution are classical objects in differential geometry, generated by rotating a curve around a fixed line, called the axis of revolution of the surface. These surfaces appear often in nature, in architecture, and in many common human artifacts, and are widely used in Geometric Design. Additionally, when the surface of revolution is rational, i.e., admitting a parametrization whose components are quotients of bivariate polynomials (a *rational parametrization* ), the strong structure of the surface allows to perform easily certain operations like implicitizing [1], reparametrizing the surface over the real numbers [2], or analyzing the surjectivity of the parametrization [3]. We recall that every rational surface is *algebraic*, i.e., it is the zeroset of a trivariate polynomial.

In this paper we study how rational surfaces of revolution are transformed when a nonsingular affine mapping is applied. The resulting surface is certainly rational too, but in general it is not a surface of revolution. However, some properties of this image can be discovered when elements of *affine differential geometry* are used. Classical differential geometry studies objects and notions that behave well when an orthogonal transformation is applied: for instance, normal lines transform accordingly, and the Gauss curvature is preserved. Affine differential geometry [4,5], started by Blaschke in the first decades of the 20th century, however, studies objects and notions that behave well when we consider matrix transformations of the special linear group $\mathbf{SL}_3(\mathbb{R})$, i.e., the group of matrices with determinant equal to 1. Thus, in the context of affine differential geometry, for instance, normals and Gauss curvature are replaced by *affine* normals and *affine* curvature, which have good properties when transformations of the special linear group are applied.

In the context of affine differential geometry, *affine surfaces of rotation* [6,7], which generalize classical surfaces of revolution, are introduced. These surfaces can be of three different subtypes, *elliptic*, *hyperbolic* and *parabolic*, the first of them being the classical surfaces of revolution. Theoretical properties of algebraic affine surfaces of rotation are treated in some recent papers: the elliptic case is studied in [8], the hyperbolic case is addressed in [9], and the parabolic in [10]. Furthermore, an algorithm for recognizing algebraic affine surfaces of revolution is provided in [11]. In this regard, a necessary condition, although not sufficient, for a surface to be an affine surface of rotation is that all the affine normal lines of the surface intersect a fixed line, called the *affine axis* of rotation. If the affine normal lines satisfy this property, we say that the surface is ANIL (Affine Normal lines Intersecting a Line). In particular, surfaces of revolution are ANIL surfaces.

Using notions of affine differential geometry and *Plücker coordinates* (see [12]) as fundamental tools, we prove that the image of every rational ANIL surface, and therefore of every rational surface of revolution, under a nonsingular affine mapping is also ANIL. Furthermore, we also provide an algorithmic method to find, given a rational ANIL surface, a rational surface of revolution affinely transforming onto the first surface, and to compute the mapping itself. This is useful because, as we mentioned before, certain operations like implicitizing, reparametrizing over the reals or studying surjectivity can be efficiently performed on surfaces of revolution; via the affine mapping relating the surface of revolution and the given ANIL surface, the results of these operations can be carried to the original ANIL surface.

Additionally, we also explore under what conditions the image of a rational surface of revolution under a nonsingular affine mapping is an affine rotation surface. We prove that this is only possible when the affine rotation surface is another surface of revolution and the mapping is a similarity, i.e., the composition of a rigid motion and a scaling. This shows that there are in fact many ANIL surfaces which however are not affine surfaces of rotation, since the image of any surface of revolution under an affine mapping that is not a similarity is an ANIL surface, but not an affine surface of rotation. The observation is of interest since up to our knowledge, the only known examples of ANIL surfaces to this date are affine surfaces of rotation and *affine spheres*, i.e., surfaces where all the affine normals intersect at one point, called the *center* of the sphere. Affine spheres do not need to be affine surfaces of rotation [11], and their nature is preserved by affine mappings.

The structure of the paper is the following. In Section 2, we recall several notions and results on affine differential geometry, and Plücker coordinates. In Section 3, we prove that the image of a rational surface of revolution is an ANIL surface. In Section 4, we develop an algorithmic method to compute a surface of revolution affinely equivalent to a given ANIL rational surface, and to find the affine mapping between the surfaces. In Section 5, we address the conditions for the affine image of a rational surface of revolution to be an affine surface of rotation. We close in Section 6, where we present our conclusions.

## 2. Preliminaries

In this section we consider several preliminary notions on affine differential geometry and line geometry. Along the section, we let $S \subset \mathbb{R}^3$ be a rational surface. For certain technical reasons, which will be clear later, we assume that $S$ is not a developable surface, i.e., isometric to the plane, so $S$ has Gaussian curvature not identically equal to zero.

### *2.1. Affine Rotation Surfaces*

In this subsection we recall several notions and results on affine differential geometry and a special class of surfaces, called *affine rotation surfaces*, which appear in the context of affine differential geometry and generalize surfaces of revolution. First, we recall from [13,14] some notions from affine differential geometry. The *affine co-normal vector* at each point of $S$ is defined as

$$\nu = |K|^{-\frac{1}{4}} \cdot \mathbf{N}, \tag{1}$$

where $\mathbf{N}$ is the unit Euclidean normal vector, and $K$ is the Gaussian curvature. The affine co-normal vector is not defined when $K$ is zero.

The *affine normal vector* to $S$ at a point $p \in S$ is

$$\xi(p) = [\nu(p), \nu_u(p), \nu_v(p)]^{-1} \left(\nu_u(p) \times \nu_v(p)\right), \tag{2}$$

where $\bullet_u, \bullet_v$ represent the partial derivatives of $\bullet$ with respect to the variables $u, v$, and $[\bullet, \bullet_u, \bullet_v]$ represents the determinant of $\bullet, \bullet_u, \bullet_v$. The *affine normal line* at $p \in S$ is the line through $p$, parallel to the affine normal vector. Denoting by $\mathbf{SL}_3(\mathbb{R})$ the special linear group, i.e., the group of matrices with determinant equal to 1, the affine normal lines are known to be *covariant* under affine transformations of $\mathbf{SL}_3(\mathbb{R})$ (see Prop. 3 in [13]): this means that if $h$ represents an affine transformation of the special linear group and $\mathcal{L}_p$ represents the affine normal line at $p$, then $h(\mathcal{L}_p)$ coincides with $\mathcal{L}_{h(p)}$. Sometimes we will refer to this property as the *covariance property* of affine normal lines.

Also in the context of affine differential geometry, *affine rotation groups* are introduced. An *affine rotation group* is a uniparametric matrix group that is a subgroup of $\mathbf{SL}_3(\mathbb{R})$, and which leaves invariant exactly one line in 3-space, called the *affine axis of rotation*. Lee [6] shows that there are only three different types of such subgroups; in an appropriate coordinate system, these types correspond to the following uniparametric matrix groups:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ \theta & 1 & 0 \\ \frac{\theta^2}{2} & \theta & 1 \end{pmatrix}. \tag{3}$$

In the three cases of Equation (3), the invariant line is the $z$-axis. We name the rotations defined in each case as *elliptic* (left-most matrix, which defines a classical rotation about the $z$-axis), *hyperbolic* (center matrix, which defines a hyperbolic rotation about the $z$-axis), and *parabolic* (right-most matrix). The surfaces which, after perhaps an orthogonal change of coordinates $\mathcal{T}$, are invariant under one of the matrix groups in Equation (3) are called *affine rotation surfaces*; furthermore, in this case the preimage under $\mathcal{T}$ of the $z$-axis is called the *affine axis of rotation* of the surface. We say that an affine rotation surface is of elliptic, hyperbolic or parabolic type depending on the form of the matrix group. If the surface is algebraic, then we say that the surface is an *algebraic affine rotation surface*. Notice that the affine rotation surfaces of elliptic type are the classical surfaces of revolution.

Every affine rotation surface about the $z$-axis can be parametrized locally around a regular point using differentiable functions $f(s), g(s)$ as

$$\mathbf{x}(\theta, s) = \boldsymbol{Q}_\theta \cdot [f(s), 0, g(s)]^T, \tag{4}$$

where $[f(s), 0, g(s)]^T$ parametrizes a *directrix curve* and $\boldsymbol{Q}_\theta$ corresponds to one of the uniparametric matrix groups in Equation (3). We will refer to this representation as the *standard* form of the surface. Using the standard form, the curves $\mathbf{x}(\theta_0, s)$ are called *meridians*, while the curves $\mathbf{x}(\theta, s_0)$ are called *parallel curves*. In particular, the directrix is a meridian. Moreover, according to [6], the parallel curves are (a) in the elliptic case, circles centered on the $z$-axis, contained in planes normal to the $z$-axis; (b) in the hyperbolic case, equilateral hyperbolae centered on the $z$-axis, contained in planes normal to the $z$-axis, with parallel asymptotes; (c) in the parabolic case, parabolas placed in planes normal to the $x$-axis, with parallel axes, whose major axis is parallel to the $z$-axis.

Affine normals can be used to characterize affine rotation surfaces [11]. Before providing this characterization, we need to introduce two more properties.

**Definition 1.** *Let $S$ be a surface which under an orthogonal change of coordinates $\mathcal{T}$, can be locally parametrized as $\mathbf{y}(\theta, s) = \boldsymbol{A}_\theta \cdot [f(s), 0, g(s)]^T$, where $\boldsymbol{A}_\theta$ is a $3 \times 3$ matrix depending on a parameter $\theta$, and let $\mathcal{A}$ be the preimage of the z-axis under the transformation $\mathcal{T}$. We say that $S$ has the* shadow line *property with respect to the line $\mathcal{A}$, if along every meridian $\mathbf{y}(\theta_0, s)$ the tangents to the parallel curves $\mathbf{y}(\theta, s_0)$ are parallel.*

For instance, one can see that surfaces of revolution always have the shadow line property with respect to its axis of revolution.

**Definition 2.** *We say that a non-developable surface $S$ is* ANIL *(Affine Normals Intersecting a same Line), or that $S$ has the* ANIL property*, if all the affine normal lines of $S$ intersect a same line $\mathcal{A}$, called the* axis *of $S$.*

Then we have the following theorem (see [11] for a proof), which characterizes affine rotation surfaces.

**Theorem 1.** *The surface $S$ is an affine rotation surface with affine axis $\mathcal{A}$ if and only if the following two conditions hold: (1) $S$ is* ANIL*, with axis $\mathcal{A}$; (2) $S$ has the shadow line property with respect to the line $\mathcal{A}$.*

From Theorem 1, it is clear that every affine rotation surface is ANIL. The converse, however, is not true: in [11] it is observed that there exist *affine spheres*, i.e., surfaces where all the affine normal lines intersect at one point (for instance, ellipsoids), called the center of the sphere, which are not affine rotation surfaces. Since all the affine normals of an affine sphere intersect at the center of the sphere, the affine normals obviously intersect every line through the center, so every affine sphere is an ANIL surface. In this paper, however, we will discover that there are many ANIL surfaces which are not affine rotation surfaces, or affine spheres: in fact, in Section 5 we will see that the images of surfaces of revolution under most nonsingular affine mappings are exactly like this.

### *2.2. Plücker Coordinates*

Theorem 1 can be used to device an algorithm for detecting whether a given algebraic surface is an affine rotation surface, and to find the affine axis, in the affirmative case [11]. In order to do so, the key question is to efficiently exploit Condition (1) in Theorem 1, i.e., the fact that all the affine normal lines intersect the affine axis. This can be done using *Plücker coordinates* [12,15], which we recall in this subsection.

Plücker coordinates provide an alternative way to represent straight lines. A line $L \subset \mathbb{R}^3$ is completely determined when we know a point $P \in L$ and a vector $\boldsymbol{w}$ parallel to $L$. Therefore, we often write $L = (P, \boldsymbol{w})$. Now let $\overline{\boldsymbol{w}} = \boldsymbol{P} \times \boldsymbol{w}$, where $\boldsymbol{P}$ here denotes the vector connecting the point $P$ with the origin of the coordinate system. Then the *Plücker coordinates* of $L$ are $(\boldsymbol{w}, \overline{\boldsymbol{w}}) \in \mathbb{R}^6$. Notice that by construction $\boldsymbol{w} \cdot \overline{\boldsymbol{w}} = 0$; this equation defines a quadric in $\mathbb{R}^6$ known as the *Klein quadric*.

Plücker coordinates of lines are unique up to multiplication by a constant nonzero factor. Moreover $\overline{\boldsymbol{w}}$ is independent of the choice of the point $P \in L$, since if $Q \in L$, then $(\boldsymbol{Q} - \boldsymbol{P}) \times \boldsymbol{w} = 0$. Furthermore, given the Plücker coordinates $(\boldsymbol{w}, \overline{\boldsymbol{w}})$ of $L$, we can recover a point $P$ on $L$ from the relationship

$$\boldsymbol{P} \times \boldsymbol{w} = \overline{\boldsymbol{w}}, \tag{5}$$

by writing $\boldsymbol{P} = (x, y, z)$ and solving the system of linear Equations (5) for $x, y, z$. An alternative to solving this system of linear equations is simply to compute the pedal point $\boldsymbol{w} \times \overline{\boldsymbol{w}} \langle \boldsymbol{w}, \boldsymbol{w} \rangle^{-1}$ on the line $(\boldsymbol{w}, \overline{\boldsymbol{w}})$.

Let $(\alpha, \beta)$ be the Plücker coordinates of a line in $\mathbb{R}^3$, and consider all the lines $(\boldsymbol{w}, \overline{\boldsymbol{w}})$, written in Plücker coordinates, such that

$$\alpha \cdot \overline{\boldsymbol{w}} + \beta \cdot \boldsymbol{w} = 0. \tag{6}$$

This equation (see [15,16]) expresses the condition that the lines $(\boldsymbol{w}, \overline{\boldsymbol{w}})$ intersect the line $(\alpha, \beta)$, so these lines span a hyperplane of $\mathbb{R}^6$. Thus, Equation (6) provides an efficient way of managing Condition (1) in Theorem 1, and therefore of detecting whether a given rational surface is ANIL: given a rational surface rationally parametrized by $\mathbf{x}(t,s)$, one can compute the affine normal line at several points $\mathbf{x}(t_i, s_i)$, where $(t_i, s_i) \in \mathbb{R}^2$. From Equation (6), each point gives a linear condition on the Plücker coordinates $(\alpha, \beta)$ of a potential line $\mathcal{A}$, intersected by all the affine normal lines of the surface. Solving the linear system of equations corresponding to all these linear conditions, the coordinates $(\alpha, \beta)$ can be efficiently computed. This, for instance, is used in [11] in order to detect whether an algebraic surface is an affine surface of rotation.

## 3. Affine Image of a Rational Surface of Revolution (I)

The goal of this section is to prove that the image under a nonsingular affine mapping $f$ of a surface of revolution about an axis $\mathcal{A}$ is an ANIL surface of axis $\widehat{\mathcal{A}} = f(\mathcal{A})$. Notice that from Theorem 1, this is a necessary condition for a surface to be an affine surface of rotation. Later, in Section 5, we will explore in what cases the image of a surface of revolution under a nonsingular affine mapping is an affine surface of rotation.

In order to do this, we let $S \subset \mathbb{R}^3$ be a rational ANIL surface, rationally parametrized by $\mathbf{x}(t,s)$, where $t, s$ are parameters, and we let $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, where $\boldsymbol{A} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ and $\boldsymbol{b} \in \mathbb{R}^3$. We denote $\widehat{S} = f(S)$. By definition, if $S$ is a surface of revolution about an axis $\mathcal{A}$ then $S$ is an affine rotation surface of elliptic type about the affine axis $\mathcal{A}$. Hence, by Theorem 1 all the affine normals of $S$ intersect the line $\mathcal{A}$, so $S$ is an ANIL surface of axis $\mathcal{A}$.

For now we will assume that $S$ is not developable; some considerations about developable surfaces will be made at the end of this section. Observe that a developable surface (see [17]) can always be, at least locally, parametrized as $\mathbf{y}(u,v) = \boldsymbol{a}(u) + v\boldsymbol{c}(u)$ where $[\boldsymbol{a}'(u), \boldsymbol{c}(u), \boldsymbol{c}'(u)] = 0$, so the vectors $\{\boldsymbol{a}'(u), \boldsymbol{c}(u), \boldsymbol{c}'(u)\}$ are coplanar. Since the images of these vectors under a nonsingular mapping $g(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ are also coplanar, a surface is developable if and only if its image under a nonsingular mapping $g(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ is also developable. Since translations are isometries, and therefore preserve the property of being developable, we deduce that a surface is developable if and only if the image of the surface under every nonsingular affine mapping $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ is also developable. In particular, and since we are assuming that $S$ is not developable, $\widehat{S}$ is not developable either. Thus, the affine normal lines of both $S$ and $\widehat{S}$ are well defined.

Furthermore, we will need the following technical lemma.

**Lemma 1.** *Let $\boldsymbol{A} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ be nonsingular. Then $\boldsymbol{A} = k\boldsymbol{B}$, where $k \in \mathbb{R}$ and $\det(\boldsymbol{B}) = 1$.*

**Proof.** Let $k = \sqrt[3]{\det(\boldsymbol{A})}$. Since $\det(\boldsymbol{A}) \neq 0$, $k \neq 0$ too. Let $\boldsymbol{B} = \frac{1}{k}\boldsymbol{A}$. Then $\det(\boldsymbol{B}) = \frac{1}{k^3} \cdot \det(\boldsymbol{A}) = \frac{1}{k^3} \cdot k^3 = 1$. □

In order to show that $\widehat{S} = f(S)$ is an ANIL surface, we first consider the image $\tilde{S}$ of $S$ under a homothety $\tilde{f}(\boldsymbol{x}) = k\boldsymbol{x}$ with $k \in \mathbb{R} - \{0\}$; we denote $\tilde{S} = \tilde{f}(S)$.

**Lemma 2.** *Let $S \subset \mathbb{R}^3$ be an ANIL surface of axis $\mathcal{A}$ rationally parametrized by $\mathbf{x}(t,s)$ which is not developable, and let $\tilde{S}$ be the image of $S$ under a homothety $\tilde{f}(\boldsymbol{x}) = k\boldsymbol{x}$, $k \in \mathbb{R} - \{0\}$. Then $\tilde{S} = \tilde{f}(S)$ is an ANIL surface of axis $\tilde{\mathcal{A}} = \tilde{f}(\mathcal{A})$.*

**Proof.** First we need to consider the relationship between the affine normal lines of $S$ and $\tilde{S}$. In order to do this, observe that $\mathbf{y}(t,s) = k\mathbf{x}(t,s)$ parametrizes $\tilde{S}$. Let us denote by $K_{\mathbf{x}}, \mathbf{N}_{\mathbf{x}}, K_{\mathbf{y}}, \mathbf{N}_{\mathbf{y}}$ the Gauss curvatures and unitary normal vectors of $S$ and $\tilde{S}$. And let us also denote by $\mu_{\mathbf{x}}, \xi_{\mathbf{x}}, \mu_{\mathbf{y}}, \xi_{\mathbf{y}}$ the affine co-normal vectors and the affine normal vectors of $S$ and $\tilde{S}$. One can check that

$$\mathbf{N_y} = \mathbf{N_x},\ K_{\mathbf{y}} = \frac{1}{k^2}K_{\mathbf{x}},\ \mu_{\mathbf{y}} = \sqrt{|K_{\mathbf{x}}|}\mu_{\mathbf{x}},\ \xi_{\mathbf{y}} = \frac{1}{\sqrt{|K_{\mathbf{x}}|}}\xi_{\mathbf{x}}. \tag{7}$$

Notice that these equalities describe the relationship between the unitary normal, co-normal and affine normal vector of $\tilde{S}$ at the point $\mathbf{y}(t,s)$, and the corresponding vector of $S$ at the point $\mathbf{x}(t,s)$; similarly for the Gaussian curvatures. Furthermore, since by hypothesis $S$ is not developable, the affine normal vectors of both $S$ and $\tilde{S}$ are well defined.

Now let $P \in \mathcal{A}$, and let $\overline{\boldsymbol{w}}$ be a vector parallel to $\mathcal{A}$. Then $(\alpha, \beta) = (\overline{\boldsymbol{w}}, \boldsymbol{P} \times \overline{\boldsymbol{w}}) = (\alpha, \beta)$, where $\boldsymbol{P}$ denotes the vector connecting the point $P$ and the origin of the coordinate system, are the Plücker coordinates of the line $\mathcal{A}$. Thus, the Plücker coordinates of the line $\tilde{\mathcal{A}} = \tilde{f}(\mathcal{A})$ are $(\overline{\boldsymbol{w}}, k\boldsymbol{P} \times \overline{\boldsymbol{w}}) = (\alpha, k\beta)$.

Since $S$ is an ANIL surface about the axis $\mathcal{A}$, from Equation (6) we have

$$\alpha(\mathbf{x} \times \xi_{\mathbf{x}}) + \beta\xi_{\mathbf{x}} = 0. \tag{8}$$

Taking into account that the Plücker coordinates of the line $\tilde{\mathcal{A}} = \tilde{f}(\mathcal{A})$ are $(\overline{\boldsymbol{w}}, k\boldsymbol{P} \times \overline{\boldsymbol{w}}) = (\alpha, k\beta)$, and using Equations (7) and (8), we get

$$\alpha(\mathbf{y} \times \xi_{\mathbf{y}}) + k\beta\xi_{\mathbf{y}} = \alpha\left(k\mathbf{x} \times \frac{1}{\sqrt{|K_{\mathbf{x}}|}}\xi_{\mathbf{x}}\right) + k\beta\frac{1}{\sqrt{|K_{\mathbf{x}}|}}\xi_{\mathbf{x}} = \frac{K_{\mathbf{x}}}{\sqrt{|K_{\mathbf{x}}|}}[\alpha(\mathbf{x} \times \xi_{\mathbf{x}}) + \beta\xi_{\mathbf{x}}] = 0 \tag{9}$$

Hence, again from Equation (6) we conclude that the affine normal lines of $\tilde{S}$ all intersect the line $\tilde{\mathcal{A}}$. □

Now we consider the image $S^\star$ of $S$ under a translation $f^\star(\boldsymbol{x}) = \mathbf{x} + \boldsymbol{b}$, with $\boldsymbol{b} \in \mathbb{R}^3$.

**Lemma 3.** *Let $S \subset \mathbb{R}^3$ be a* ANIL *surface of axis $\mathcal{A}$ rationally parametrized by $\mathbf{x}(t,s)$ which is not developable, and let $S^\star$ be the image of $S$ under a translation $f^\star(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{b}$, $\boldsymbol{b} \in \mathbb{R}^3$. Then $S^\star = f^\star(S)$ is an* ANIL *surface of axis $\mathcal{A}^\star = f^\star(\mathcal{A})$.*

**Proof.** Observing that $\mathbf{y}(t,s) = \mathbf{x}(t,s) + \boldsymbol{b}$ parametrizes $S^\star$, we get that

$$\mathbf{N_y} = \mathbf{N_x},\ K_{\mathbf{y}} = K_{\mathbf{x}},\ \mu_{\mathbf{y}} = \mu_{\mathbf{x}},\ \xi_{\mathbf{y}} = \xi_{\mathbf{x}}, \tag{10}$$

where these equalities describe the relationships between the unitary normal, co-normal and affine normal vector of $S^\star$ at the point $\mathbf{y}(t,s)$, and the corresponding vector of $S$ at the point $\mathbf{x}(t,s)$; similarly for the Gaussian curvatures. Furthermore, since by hypothesis $S$ is not developable, the affine normal vectors of both $S$ and $S^\star$ are well defined. Then we argue as in the proof of Lemma 1. □

Finally we can prove the main result of this section.

**Theorem 2.** *Let $S \subset \mathbb{R}^3$ be an* ANIL *surface of axis $\mathcal{A}$ rationally parametrized by $\mathbf{x}(t,s)$ which is not developable, and let $\widehat{S}$ be the image of $S$ under a nonsingular affine mapping $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, $\boldsymbol{A} \in \mathcal{M}_{3\times 3}(\mathbb{R})$, $\boldsymbol{b} \in \mathbb{R}^3$. Then $\widehat{S} = f(S)$ is an* ANIL *surface of axis $\widehat{\mathcal{A}} = f(\mathcal{A})$.*

**Proof.** By Lemma 1, $\boldsymbol{A} = k\boldsymbol{B}$, where $k \in \mathbb{R} - \{0\}$ and $\det(B) = 1$; thus, $f(\boldsymbol{x}) = k\boldsymbol{B}\boldsymbol{x} + \boldsymbol{b}$. Let $f^\dagger$ be the linear mapping defined by $f^\dagger(\boldsymbol{x}) = \boldsymbol{B}\boldsymbol{x}$, and let $S^\dagger$ be the image of $S$ under $f^\dagger$, i.e., $S^\dagger = f^\dagger(S)$. By the covariance property of affine normal lines, the affine normal lines of $S^\dagger$ are the images of the affine normal lines of $S$ under $f^\dagger$. Since by hypothesis all the affine normal lines of $S$ intersect $\mathcal{A}$, and since linear mappings preserve incidence, all the affine normal lines of $S^\dagger$ intersect the line $\mathcal{A}^\dagger = f^\dagger(\mathcal{A})$. Then the result follows from Lemma 2 and Lemma 3. □

**Corollary 1.** *Let $S \subset \mathbb{R}^3$ be a rational surface of revolution about an axis $\mathcal{A}$, and assume that $S$ is not developable. Let $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, $\boldsymbol{A} \in \mathcal{M}_{3\times3}(\mathbb{R})$ nonsingular, $\boldsymbol{b} \in \mathbb{R}^3$. Then the image of $S$ under the mapping $f$ is an* ANIL *surface of axis $f(\mathcal{A})$.*

Furthermore, since affine mappings preserve incidence, we also have the following corollary of Theorem 2 on affine spheres.

**Corollary 2.** *Let $S \subset \mathbb{R}^3$ be an affine sphere of center $\mathbf{c}$ rationally parametrized by $\mathbf{x}(t,s)$ which is not developable, and let $\widehat{S}$ be the image of $S$ under a nonsingular affine mapping $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, $\boldsymbol{A} \in \mathcal{M}_{3\times3}(\mathbb{R})$, $\boldsymbol{b} \in \mathbb{R}^3$. Then $\widehat{S} = f(S)$ is an affine sphere of center $\widehat{\mathbf{c}} = f(\mathbf{c})$.*

*The Case of Developable Surfaces*

Let $S \subset \mathbb{R}^3$ be a developable surface, in which case the Gaussian curvature is zero. Since the affine normal line is not defined when the Gassian curvature is zero, the notion of an ANIL surface is not applicable to these surfaces. However, some considerations can be done in the case when $S$ is a surface of revolution. Without loss of generality we assume that the axis of revolution of $S$ is the $z$-axis. A first obvious possibility is that $S$ is a cylinder of revolution, and therefore a quadric. If $S$ is not a cylinder of revolution, then $S$ admits (see Section 15.1 of [17]) an, at least local, parametrization of $S$ as

$$\mathbf{x}(\rho,\gamma) = (\rho\cos\gamma, \rho\sin\gamma, h(\rho)).$$

Additionally, imposing that the Gaussian curvature of $S$ is identically zero, one can see (e.g., Section 15.3 of [17]) that $h(\rho) = C_1\rho + C_2$, with $C_1, C_2$ constants, $C_1$ nonzero, so $S$ is a cone of revolution: indeed, eliminating $\rho, \gamma$ in

$$x = \rho\cos\gamma,\ y = \rho\sin\gamma,\ z = C_1\rho + C_2,$$

one gets $x^2 + y^2 = \left(\frac{z-C_2}{C_1}\right)^2$, which shows that $S$ is a cone of revolution. Since affine mappings preserve incidence and parallelism, one deduces that the image of a developable surface of revolution under a nonsingular affine mapping is either cylindrical, i.e., a ruled surface whose generatrices are all of them parallel, or conical, i.e., a ruled surface whose generatrices intersect at a point, named the vertex of the surface. Furthermore, since affine mappings preserve the degree of the surface, it must also be a quadric.

## 4. Computing a Surface of Revolution Affinely Equivalent to an ANIL Surface

Given an ANIL surface $S_1 \subset \mathbb{R}^3$, rationally parametrized by $\mathbf{x}(t,s)$, we aim to find an algorithm to solve the following problem: find, if it exists, a rational surface of revolution $S_2 \subset \mathbb{R}^3$ which is *affinely equivalent* to $S_1$, i.e., such that there is a nonsingular affine mapping $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, where $\boldsymbol{A} \in \mathcal{M}_{3\times3}(\mathbb{R})$ and $\boldsymbol{b} \in \mathbb{R}^3$, satisfying that $f(S_1) = S_2$. We say that $f$ is an *affine equivalence* between $S_1, S_2$. Notice that certainly $S_2$ is not unique, since by composing $f$ with any *similarity* $h$, the surface $(h \circ f)(S_1)$ is also a surface of revolution affinely equivalent to $S_1$; recall that similarities are the composition of a congruence (also called rigid motion, a mapping preserving distances) and a homothety (which preserves angles and scales the objects).

In order to solve the problem, it is useful to recall the following theorem, characterizing algebraic surfaces of revolution. In this theorem we consider classical normals, and not affine normals. We will need to apply this theorem on the surface $S_2$ we are seeking. Notice that by hypothesis $S_1$ is ANIL; since the notion of an ANIL surface is not applicable to developable surfaces, $S_1$ is not developable, and therefore $S_2$ is not developable either. In particular, $S_2$ is not cylindrical.

**Theorem 3.** *Let $S \subset \mathbb{R}^3$ be an algebraic surface which is not cylindrical. Then $S$ is a surface of revolution about an axis $\mathcal{A}$ if and only if all the normals to the surface intersect the axis $\mathcal{A}$.*

**Proof.** See Theorem 4.2.1 and Lemma 4.2.2 of [12]. □

Observe that in Theorem 3 the hypothesis of $S$ being algebraic is necessary: if the surface is not algebraic, the condition in the theorem implies that the surface is either a surface of revolution, or a *helical surface*, i.e., a surface invariant under a *helical motion* (see Section 3.1.2 of [12]). Helical motions are the mappings in $\mathbb{R}^3$ that can be written in a certain system of coordinates as $T(\boldsymbol{x}) = \boldsymbol{Q}_\theta \boldsymbol{x} + [0, 0, p\theta]^T$, where $\boldsymbol{Q}_\theta$ is the left-most matrix in Equation (3), and $p \neq 0$ ($p$ is called the *pitch*). However, helical surfaces are not algebraic. Notice also in Theorem 3 that the condition on the shadow line property is not necessary. As it also happened with Theorem 1 and affine rotation surfaces, using Plücker coordinates one can use Theorem 3 to build an efficient algorithm for detecting surfaces of revolution (see e.g., [16]). In our case, Theorem 3 will be key in order to solve the problem we are addressing.

We still need some additional observations. First, by applying if necessary a translation followed by a rotation about a line, we can assume that the axis of $S_1$ is the $z$-axis. Furthermore, we can also assume that the axis of revolution of the surface $S_2$ we are looking for is the $z$-axis as well: since the composition of nonsingular affine mappings is a nonsingular affine mapping, if there exists a surface of revolution affinely equivalent to $S_1$, then there also exists a surface of revolution about the $z$-axis with the same property (one just needs to apply a congruence to reach this surface). Finally, since the composition of $S_2$ with any translation by a vector parallel to the $z$-axis also provides a surface of revolution about the $z$-axis, we can assume that the affine equivalence transforming $S_1$ into $S_2$ fixes the origin, so that $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$. Our problem, then, is to find the matrix $\boldsymbol{A}$: after computing $\boldsymbol{A}$, the surface $S_2$ is immediately obtained.

Now if $S_1$ is parametrized by $\mathbf{x}(t,s)$ and $f(S_1) = S_2$, then $\mathbf{y}(t,s) = \boldsymbol{A}\mathbf{x}(t,s)$ is a parametrization of $S_2$. In order to use Theorem 3, we consider the (classical) normals to $S_2$. Since $\mathbf{y}_t = \boldsymbol{A}\boldsymbol{x}_t$, $\mathbf{y}_s = \boldsymbol{A}\boldsymbol{x}_s$, and taking into account the well-known formula $M\boldsymbol{a} \times M\boldsymbol{b} = \det(M)M^{-T}(\boldsymbol{a} \times \boldsymbol{b})$ for $M \in \mathcal{M}_{3\times 3}(\mathbb{R})$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, we get

$$\boldsymbol{y}_t \times \boldsymbol{y}_s = \det(\boldsymbol{A})\boldsymbol{A}^{-T}(\boldsymbol{x}_t \times \boldsymbol{x}_s). \tag{11}$$

The Plücker coordinates of a generic normal line of $S_1$ are $(\alpha, \beta) = (\boldsymbol{x}_t \times \boldsymbol{x}_s, \boldsymbol{x} \times \boldsymbol{x}_t \times \boldsymbol{x}_s)$. From Equation (11), the Plücker coordinates of a generic normal line of $S_2$ are

$$(\boldsymbol{y}_t \times \boldsymbol{y}_s, \boldsymbol{y} \times \boldsymbol{y}_t \times \boldsymbol{y}_s) = (\det(\boldsymbol{A})\boldsymbol{A}^{-T}(\boldsymbol{x}_t \times \boldsymbol{x}_s), \boldsymbol{A}\boldsymbol{x} \times \det(\boldsymbol{A})\boldsymbol{A}^{-T}(\boldsymbol{x}_t \times \boldsymbol{x}_s)). \tag{12}$$

Notice that since $\mathbf{y}(t,s)$ parametrizes a surface of revolution about the $z$-axis, $k\mathbf{y}(t,s)$ with $k \in \mathbb{R} - \{0\}$ parametrizes another surface of revolution about the $z$-axis too; we can prove it from Theorem 3, taking into account the relationship between the normals of the surfaces parametrized by $\mathbf{y}(t,s)$ and $k\mathbf{y}(t,s)$. This implies that we can assume $\det(\boldsymbol{A}) = 1$. Therefore, and calling $\alpha = \boldsymbol{x}_t \times \boldsymbol{x}_s$, we get that the Plücker coordinates of a generic normal line of $S_2$ are

$$(\boldsymbol{A}^{-T}\alpha, \boldsymbol{A}\boldsymbol{x} \times \boldsymbol{A}^{-T}\alpha) \tag{13}$$

Additionally, the Plücker coordinates of the $z$-axis, which is the axis of revolution of $S_2$, are $(\mathbf{0}, \mathbf{k})$, where $\mathbf{0} = (0,0,0)$ and $\mathbf{k} = (0,0,1)$. From Theorem 3, all normals to $S_2$ intersect the $z$-axis. Using Plücker coordinates, from Equation (6) this condition is translated into

$$\boldsymbol{A}^{-T}\alpha \cdot \mathbf{0} + (\boldsymbol{A}\boldsymbol{x} \times \boldsymbol{A}^{-T}\alpha) \cdot \mathbf{k} = 0. \tag{14}$$

Let $\mathbf{Co}(\boldsymbol{A})$ be the cofactor matrix of $\boldsymbol{A}$. Since we are assuming that $\det(\boldsymbol{A}) = 1$, $A^{-T} = \mathbf{Co}(\boldsymbol{A})$. Then, Equation (14) is equivalent to

$$(\boldsymbol{A}\boldsymbol{x} \times \mathbf{Co}(\boldsymbol{A})\alpha) \cdot \mathbf{k} = [\boldsymbol{A}\boldsymbol{x}, \mathbf{Co}(\boldsymbol{A})\alpha, \mathbf{k}] = 0. \tag{15}$$

By Theorem 2, $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ must preserve the $z$-axis, so $\boldsymbol{A}\mathbf{k} = \lambda\mathbf{k}$ for $\lambda \neq 0$. Since additionally $\det(\boldsymbol{A}) = 1$, we get that

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{16}$$

where $a_{33}(a_{11}a_{22} - a_{12}a_{21}) = 1$. Since $a_{33} \neq 0$ (because otherwise $\boldsymbol{A}$ is singular), we can always assume that $a_{33} = 1$. Thus, we get

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}, \quad \mathbf{Co}(\boldsymbol{A}) = \begin{pmatrix} a_{22} & -a_{21} & a_{21}a_{32} - a_{22}a_{31} \\ -a_{12} & a_{11} & -a_{11}a_{32} + a_{12}a_{31} \\ 0 & 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix} \tag{17}$$

Substituting the expressions for $\boldsymbol{A}$ and $\mathbf{Co}(\boldsymbol{A})$ into Equation (15), and adding the equation

$$a_{11}a_{22} - a_{12}a_{21} = 1, \tag{18}$$

we get cubic equations in $a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}$ which define an algebraic variety $\mathcal{V} \subset \mathbb{C}^6$. Any real point of $\mathcal{V}$ provides a matrix $\boldsymbol{A}$ with the desired property. So throughout the section we have proven the following result. In turn, this result provides the Algorithm 1, which solves the problem considered in this section.

**Theorem 4.** *Let $S_1 \subset \mathbb{R}^3$ be an* ANIL *surface whose axis is the $z$-axis. Then $S_1$ is affinely equivalent to a surface of revolution if and only if $\mathcal{V} \cap \mathbb{R}^6 \neq \emptyset$.*

**Remark 1.** *In fact, if $\mathcal{V} \cap \mathbb{R}^6 \neq \emptyset$ then $\mathcal{V}$ must contain at least a real curve, since rotating a surface of revolution $S_2$ with the desired properties around the $z$-axis also yields a surface of revolution.*

In practice, instead of deriving a system of cubic equations directly from Equation (15), it is cheaper from the computational point of view to substitute points $(t_i, s_i)$ into Equation (15) to generate equations. The system of cubic equations derived this way can be solved by using computer algebra methods, e.g., Gröbner bases. In our case, we used the computer algebra system Maple 17, and the `Groebner` package.

**Algorithm 1** Revol.

**Require:** A non-developable ANIL surface $S_1$, rationally parametrized by $\mathbf{x}(t,s)$.
**Ensure:** A rational surface of revolution $S_2$ affinely equivalent to $S_1$, or a certificate of its non-existence.
1: Substitute the entries of $\boldsymbol{A}$ and $\mathbf{Co}(\boldsymbol{A})$ from Equation (17) into Equation (15).
2: Solve the cubic polynomial system $\mathcal{S}$ in $a_{ij}$, $i \in \{1,2,3\}, j \in \{1,2\}$, consisting of the equations derived in Step 1, and Equation (18).
3: **if** the system does not have any real solution **then**
4: **return** “`there is no surface of revolution affinely equivalent to the surface`”
5: **else**
6: pick a real solution $a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}$ of $\mathcal{S}$.
7: **return** the surface $S_2$ parametrized by $\mathbf{y}(t,s) = \boldsymbol{A}\mathbf{x}(t,s)$, where $\boldsymbol{A}$ is the matrix in the left-hand side of Equation (17) whose entries correspond to the solution in Step 6.
8: **end if**

**Example 1.** *Let $S_1$ be the sextic surface, rationally parametrized by*

$$\begin{aligned}\mathbf{x}(t,s) \quad = \quad & \Bigg(-\frac{2(s^3t^2 - s^3t + s^2t^2 - s^3 + s^2 + t^2 - t - 1)}{t^2+1}, -\frac{(s^3+1)(t^2-2t-1)}{t^2+1}, \\ & -\frac{(s^3+1)(t^2-4t-1)}{t^2+1}\Bigg).\end{aligned}$$

*Using Plücker coordinates, one can see that $S_1$ is an* ANIL *surface, and that the axis is the x-axis. Additionally, one can check that the implicit equation of the surface has the form*

$$F(x,y,z) = (x - 3y + z)^6 + l.o.t.,$$

*where l.o.t. stands for* lower order terms. *Since the form of highest order of an affine surface of rotation has a very specific structure (see Theorem 6 in [8], Theorem 6 in [10], Theorem 6 in [9]), we deduce that $S_1$ is not an affine surface of rotation. In order to compute cubic equations defining the variety $\mathcal{V}$, we consider Equation* (15) *for the points corresponding to $(t_i, s_i)$ with $t_i, s_i$ ranging from $-3$ to 3. The first of these equations is*

$$\begin{aligned}&912600a_{11}^2a_{32} - 912600a_{11}a_{12}a_{31} + 638820a_{11}a_{12}a_{32} - 638820a_{12}^2a_{31} + 912600a_{21}^2a_{32} \\ &-912600a_{21}a_{22}a_{31} + 638820a - 21a_{22}a_{32} - 638820a - 22^2a - 31 + 2332200a_{11}^2 \\ &+2464020a_{11}a_{12} + 582036a_{12}^2 + 2332200a - 21^2 + 2464020a_{21}a_{22} + 582036a_{22}^2 = 0.\end{aligned}$$

*Adding also Equation* (18)*, we get 50 cubic equations. Maple solves the polynomial system consisting of these equations in 0.265 s, and yields the following families of real solutions (there are also some complex solutions, which we do not list):*

$$a_{11} = \lambda, a_{12} = -\frac{1}{2}a_{21} - \frac{3}{2}\lambda, a_{22} = -\frac{3}{2}a_{21} + \frac{1}{2}\lambda, a_{31} = 1, a_{32} = -3,$$

*where $\lambda$ satisfies that $\lambda^2 + a_{21}^2 - 2 = 0$, and*

$$a_{11} = 0, a_{12} = -\frac{1}{2}\mu, a_{21} = \mu, a_{22} = -\frac{3}{2}\mu, a_{31} = 1, a_{32} = -3,$$

*where $\mu$ satisfies that $\mu^2 - 2 = 0$. Picking $a_{21} = 1$, $\lambda = 1$ in the first family, we get*

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & 0 \\ 1 & -3 & 1 \end{pmatrix}.$$

*The affine mapping $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ maps $S_1$ onto the surface $S_2$ parametrized by*

$$\mathbf{y}(t,s) = \left( \frac{(s^3+1)(t^2-1)}{t^2+1}, \frac{2(s^3+1)t}{t^2+1}, -2s^2 \right),$$

*which one can recognize as the surface of revolution generated by rotating the cubic curve parametrized by $(s^3 + 1, 0, -2s^2)$ about the $z$-axis.*

## 5. Affine Image of a Surface of Revolution (II)

In this section, we want to explore under what circumstances the image of a surface of revolution under a nonsingular affine mapping is an affine surface of rotation. In order to do this, we will use the preceding notations, and we will benefit from certain observations done in Section 4.

Let $S_1, S_2$ be two rational surfaces, none of them developable, $S_1$ an ANIL surface, $S_2$ a surface of revolution, related by a nonsingular affine mapping. Following the observations in Section 4, without loss of generality we can assume that the the affine axis of $S_1$ is the $z$-axis, the axis of revolution of $S_2$ is the $z$-axis as well, and that the nonsingular affine mapping transforming $S_1$ into $S_2$ has the form $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$. Even more, we can assume that the matrix $\boldsymbol{A}$ has the form in Equation (17), and that the entries of the matrix $\boldsymbol{A}$ also satisfy Equation (18). We will separately consider the cases when $S_1$ is an elliptic, hyperbolic or parabolic affine surface of rotation. We begin with the parabolic and the hyperbolic cases, and we conclude with the elliptic case. In what follows, the reader is invited to review the notion of *parallel curve* of an affine rotation surface, recalled in Section 2.1.

### 5.1. The Parabolic Case

If $S_1$ is a parabolic affine rotation surface about the $z$-axis, we can assume (see Section 2.1) that the parallel curves are placed in planes normal to the $x$-axis, i.e., planes $x = x_0$, $x_0 \in \mathbb{R}$, that we denote by $\Pi_{x_0}$. Furthermore, in that case the intersection $\Pi_{x_0} \cap S_1$ is a union of parabolas lying on planes parallel to the $yz$-plane, and whose major axes are parallel to the $z$-axis. We are interested in finding the images of the planes $\Pi_{x_0}$ under the mapping $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$. Thus, we have

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} a_{11}x_0 + a_{12}\lambda \\ a_{21}x_0 + a_{22}\lambda \\ a_{31}x_0 + a_{32}\lambda + \mu \end{pmatrix}. \tag{19}$$

Elliminating the parameters $\lambda, \mu$, and since $a_{11}a_{22} - a_{12}a_{21} = 1$, we get the plane $a_{22}x - a_{12}y - x_0 = 0$, that is parallel to the $z$-axis, and which we denote by $\widehat{\Pi}_{x_0}$. Since $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ is an affine mapping $\widehat{\Pi}_{x_0} \cap S_2$ must be a union of parabolas as well. Since $S_2$ is a surface of revolution about the $z$-axis, we deduce that $S_2$ is generated by rotating parabolas around the $z$-axis, so $S_2$ must be the union of several paraboloids of revolution. Because $S_2$ is rational and therefore irreducible, we get that $S_2$ must be a paraboloid of revolution, so $S_1 = f(S_2)$ must also be a paraboloid. But this is a contradiction, because from Corollary 5 in [10] the only quadrics that are affine surfaces of rotation of parabolic type are either cones (which are developable surfaces), or hyperboloids. Therefore, we have proved the following result.

**Theorem 5.** *The affine image of a rational surface of revolution that is not developable cannot be an affine surface of rotation of parabolic type.*

*5.2. The Hyperbolic Case*

Let $S_1$ be a hyperbolic affine surface of rotation about the $z$-axis. Then the parallel curves are placed in planes $z = z_0$, that we denote by $\Pi_{z_0}$. Proceeding as in Section 5.1, we can check that $f$ maps $\Pi_{z_0}$ onto the plane $\widehat{\Pi}_{z_0}$, defined by

$$A_{13}x + A_{23}y + (z - z_0) = 0, \tag{20}$$

where $A_{ij}$ represents the cofactor of the element $(i, j)$ of the matrix $\boldsymbol{A}$. Since the coefficient of $z$ in Equation (20) is nonzero, $\widehat{\Pi}_{z_0}$ is not parallel to the $z$-axis. Furthermore, since $f$ is affine and $S_2 \cap \Pi_{z_0}$ is a union of equilateral hyperbolae, $f(S_1 \cap \Pi_{z_0}) = S_2 \cap \widehat{\Pi}_{z_0}$ must also be a union of hyperbolae. Additionally, since $S_2$ is a surface of revolution we can see $S_2$ as generated by rotating $S_2 \cap \widehat{\Pi}_{z_0}$ around the $z$-axis. We want to see that this cannot be.

In order to do that, assume first that $A_{13} = A_{23} = 0$. Then, Equation (20) corresponds to a horizontal plane, i.e., normal to the $z$-axis. Since $S_2$ is a surface of revolution about the $z$-axis, the horizontal sections of $S_2$ are unions of circles centered at the points on the $z$-axis. Since $S_1 \cap \widehat{\Pi}_{z_0}$ is a union of hyperbolas, this cannot happen. So let us focus on the case where $A_{13}, A_{23}$ are not both zero, in which case the plane in Equation (20) is not horizontal. We need the following previous result.

**Lemma 4.** *Let $S$ be a rational surface of revolution about the $z$-axis, and let $\mathcal{D}$ be a rational planar curve contained in a planar section $S \cap \Pi$ of the surface $S$, where $\Pi$ is not normal to the $z$-axis. Then $S$ is the surface obtained by rotating $\mathcal{D}$ about the $z$-axis.*

**Proof.** By rotating $\mathcal{D}$ around the $z$-axis we get a rational surface $S' \subset S$. Since $S$ and $S'$ are rational and therefore irreducible, $S = S'$. □

Now assume that $S_2 = f(S_1)$, where $f$ is an affinity, is a surface of revolution about the $z$-axis, and consider two planes $\Pi_{z_0}$ and $\Pi_{z_1}$, defined by $z = z_0$ and $z = z_1$, where $z_0 \neq z_1$. Let $\widehat{\Pi}_{z_0}, \widehat{\Pi}_{z_1}$ be the images of $\Pi_{z_0}, \Pi_{z_1}$ under $f$. Notice that $S_1 \cap \Pi_{z_0}, S_1 \cap \Pi_{z_1}$ are unions of circles, so $\mathcal{C}_0 = f(S_1 \cap \Pi_{z_0})$, $\mathcal{C}_1 = f(S_1 \cap \Pi_{z_1})$ are unions of hyperbolas. Furthermore, since $z_0 \neq z_1$, $\mathcal{C}_0 \neq \mathcal{C}_1$.

From Lemma 4, and since $S_2$ is rational and therefore irreducible, the surface $S_2$ should be obtained both by rotating a rational component of $\mathcal{C}_0$ around $z$, and by rotating a rational component of $\mathcal{C}_1$ around $z$. We want to see that this is not possible, i.e., that by rotating such components we generate different surfaces, not the same surface. For simplicity, we will assume that $\mathcal{C}_0$ and $\mathcal{C}_1$ are hyperbolae, and not unions of hyperbolae; were this not the case, it suffices to consider one rational component in each case.

The situation is shown in Figure 1: in more detail, the notation in Figure 1 represents the following:

- $\mathcal{C}_0 = f(S_1 \cap \Pi_{z_0}), \mathcal{C}_1 = f(S_1 \cap \Pi_{z_1})$.
- $\mathcal{P}$ is a horizontal plane, i.e., normal to the $z$-axis, through one of the vertices of $\mathcal{C}_0$.

Furthermore, Figure 2 represents the plane $\mathcal{P}$ seen from above. The notation in Figure 2 represents the following:

- The point $\mathbf{P}$ is a vertex of $\mathcal{C}_0$. Furthermore, $\mathbf{P}$ is the only intersection of $\mathcal{P}$ with $\mathcal{C}_0$.
- The lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are the intersections of the planes $\widehat{\Pi}_{z_0}, \widehat{\Pi}_{z_1}$ with the horizontal plane $\mathcal{P}$. These lines are also shown in blue in Figure 1. Notice that since $\widehat{\Pi}_{z_0}, \widehat{\Pi}_{z_1}$ are parallel, $\mathcal{L}_1, \mathcal{L}_2$ are parallel too.
- The points $\mathbf{Q_1}, \mathbf{Q_2}$ are the intersections of the curve $\mathcal{C}_1$ with the plane $\widehat{\Pi}_{z_1}$; it could happen that $\mathbf{Q_1} = \mathbf{Q_2}$, or even that the intersection of $\mathcal{C}_1$ with the plane $\widehat{\Pi}_{z_1}$ was empty, but in those cases we would obtain contradictions as well.

- The point **C** is the intersection of the $z$-axis with the horizontal plane $\mathcal{P}$.
- The circle in red, $\tilde{C}$, is the circle through the point **P**, centered at **C**; this circle is also shown in red in Figure 1.

The following result certifies that the picture shown in Figure 2 is correct:

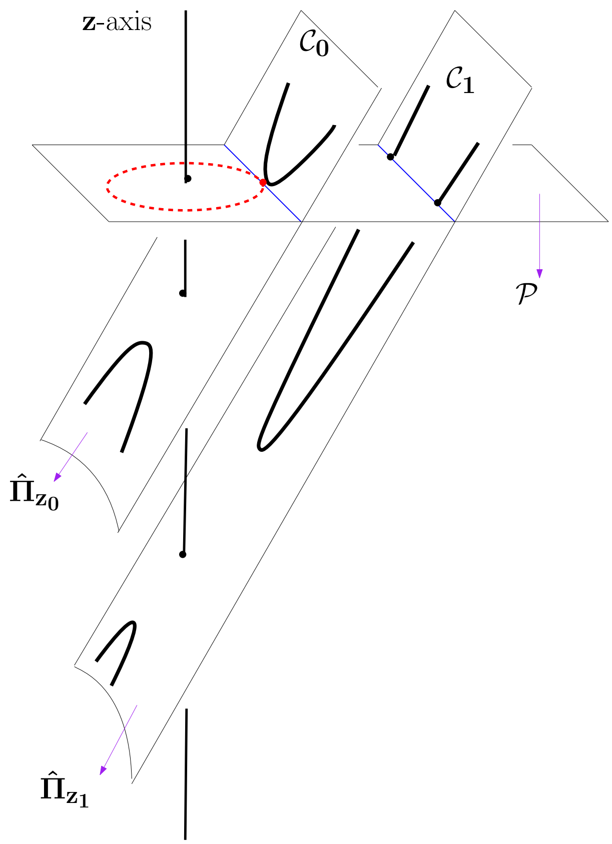


**Figure 1.** The case of hyperbolic affine rotation surfaces (I).

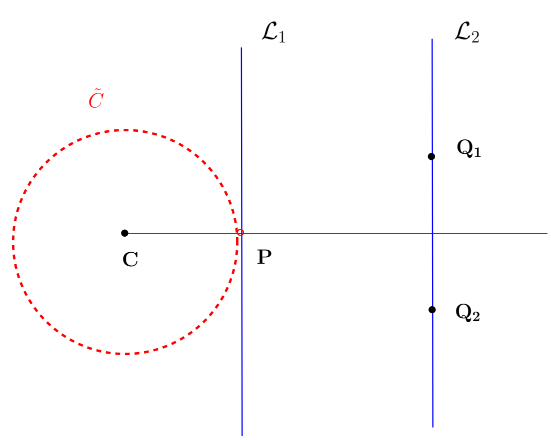


**Figure 2.** The case of hyperbolic affine rotation surfaces (II).

**Lemma 5.** *The line connecting* $\mathbf{C}$ *and* $\mathbf{P}$ *is perpendicular to the line* $\mathcal{L}_1$.

**Proof.** Assume that the line connecting $\mathbf{C}$, which is the intersection of the $z$-axis and the plane $\mathcal{P}$, and $\mathbf{P}$, which is the vertex of the hyperbola $\mathcal{C}_0$ contained in the plane $\mathcal{P}$, is not perpendicular to $\mathcal{L}_1$. Then the circle $\tilde{C}$ centered at $\mathbf{C}$ through $\mathbf{P}$ is not tangent to $\mathcal{L}_1$, and therefore there is another intersection point $\mathbf{P}' \neq \mathbf{P}$ of the circle $\tilde{C}$ with the line $\mathcal{L}_1$. However, since $S_2$ is generated by rotating the curve $\mathcal{C}_0$ around the $z$-axis, then $\mathbf{P}' \in S_2$. Even more, since $\mathcal{C}_0 = \widehat{\Pi}_{z_0} \cap S_2$ and $\mathbf{P}' \in \mathcal{L}_1 \subset \widehat{\Pi}_{z_0}$, we get that $\mathbf{P}' \in \mathcal{C}_0$. Furthermore, since $\mathcal{L}_1 \subset \mathcal{P}$, $\mathbf{P}' \in \mathcal{P}$, so $\mathbf{P}' \in \mathcal{C}_0 \cap \mathcal{P}$. However, since $\mathcal{P}$ is the horizontal plane through the vertex $\mathbf{P}$, the only point of $\mathcal{C}_0 \cap \mathcal{P}$ is $\mathbf{P}$, and therefore $\mathbf{P}' = \mathbf{P}$. □

We also need the following lemma.

**Lemma 6.** *Let* $d(\mathbf{C}, \mathbf{Q_i})$, *with* $i = 1, 2$, *denote the distance between* $\mathbf{C}, \mathbf{Q_i}$, *let* $d(\mathbf{C}, \mathcal{L}_j)$, *with* $j = 1, 2$, *denote the distance between the point* $\mathbf{C}$ *and the line* $\mathcal{L}_j$, *and let* $\overline{\mathbf{CP}}$ *denote the segment connecting* $\mathbf{C}$ *and* $\mathbf{P}$. *Then* $d(\mathbf{C}, \mathbf{Q_i}) > \overline{\mathbf{CP}}$.

**Proof.** Since $\mathcal{L}_1, \mathcal{L}_2$ are parallel, $d(\mathbf{C}, \mathcal{L}_2) > d(\mathbf{C}, \mathcal{L}_1)$. Furthermore, $d(\mathbf{C}, \mathbf{Q_i}) \geq d(\mathbf{C}, \mathcal{L}_2)$. Thus, $d(\mathbf{C}, \mathbf{Q_i}) > d(\mathbf{C}, \mathcal{L}_1)$. But from Lemma 5, $d(\mathbf{C}, \mathcal{L}_1) = \overline{\mathbf{CP}}$. □

**Corollary 3.** *The circle* $\tilde{C}$ *centered at* $\mathbf{C}$ *of radius* $\overline{\mathbf{CP}}$ *is not contained in the set generated by rotating* $\mathcal{C}_1$ *around the* $z$*-axis.*

Now we can prove the following result. Here we use the preceding notation, and the help of Figures 1 and 2.

**Theorem 6.** *The affine image of a rational surface of revolution that is not developable cannot be an affine surface of rotation of hyperbolic type.*

**Proof.** Without loss of generality, we reduce to the situation analyzed before. We have already seen that $A_{13}, A_{23}$ cannot be both zero, so we can assume that Equation (20) defines a plane which is neither horizontal, nor parallel to the $z$-axis, in which case we can use our last observations. In particular, if $S_2 = f(S_1)$, where $f$ is an affinity, is a surface of revolution about the $z$-axis, $S_2$ is generated by both curves $\mathcal{C}_0$ and $\mathcal{C}_1$, defined before. However, the surface obtained by rotating $\mathcal{C}_0$ about the $z$-axis contains the circle $\tilde{C}$. But from Corollary 3, $\tilde{C}$ is not contained in the surface generated by rotating $\mathcal{C}_1$ about the $z$-axis. Thus, rotating $\mathcal{C}_0$ and $\mathcal{C}_1$ around the $z$-axis provides different surfaces (Notice that by just moving the value $z_0$, the union of the corresponding circles $\tilde{C}$ gives rise to another surface not contained in $S_2$, so one can refine the argument to show that the surfaces generated by rotating $\mathcal{C}_0$ and $\mathcal{C}_1$ about the $z$-axis differ not in one curve, but in a whole 2-dimensional subset.), which contradicts our hypothesis. □

**Remark 2.** *Notice that the essence of the argument in the proof of Theorem 6 is not altered if the points* $\mathbf{Q_i}$ *coincide, or if* $\mathcal{C}_1 \cap \mathcal{P}$ *is empty.*

### *5.3. The Elliptic Case*

Assume now that $S_1$ is an affine surface of rotation of elliptic type, i.e., a surface of revolution. Thus, the sections $\Pi_{z_0}$ of $S_1$ with planes $z = z_0$ are unions of circles, which are transformed by $f(\boldsymbol{x}) = \boldsymbol{Ax}$ into unions of ellipses contained in planes $\widehat{\Pi}_{z_0}$ like Equation (20). If $A_{13}$ and $A_{23}$ are not both zero, then Equation (20) defines a plane not normal to the $z$-axis. In this case, we can argue as in Section 5.2 to see that this cannot happen: again, we prove that by considering the affine images of different sections of $S_1$ normal to the axis, we get planar curves, contained in $S_2$, which generate different surfaces when rotating about the $z$-axis. So we focus on the case $A_{13} = A_{23} = 0$. Here, we observe that $\widehat{\Pi}_{z_0}$ is also the plane $z = z_0$, so $f$ preserves the $z$-coordinate. Thus, the entries $a_{31}, a_{32}$ of the matrix $\boldsymbol{A}$ are both zero, so $\boldsymbol{A}$ can be written as a block matrix

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{21}$$

where $\mathbf{Q}$ defines a linear transformation $g(\boldsymbol{x}) = \mathbf{Q}\boldsymbol{x}$ of the plane, preserving the origin, where $\det(\mathbf{Q}) = 1$. Furthermore, since $S_2$ is by hypothesis a surface of revolution about the $z$-axis, and $f$ preserves the $z$-coordinate, we deduce that $f$ maps circles to circles, and therefore that $g$ maps circles centered at the origin onto circles centered at the origin. Then we have the following lemma.

**Lemma 7.** *With the preceding notation and hypotheses, $g(\boldsymbol{x}) = \mathbf{Q}\boldsymbol{x}$ defines a congruence of the plane.*

**Proof.** Let $\boldsymbol{x} = [x, y]^T$. Then the equation of a circle $C_r$ centered at the origin is

$$\boldsymbol{x}^T \cdot \boldsymbol{x} = r^2, \tag{22}$$

with $r > 0$. Since $g(\boldsymbol{x}) = \mathbf{Q}\boldsymbol{x}$ maps circles to circles and preserves the origin, $C_r$ is mapped onto the circle $C_R$ of equation

$$\boldsymbol{x}^T \mathbf{Q}^T \cdot \mathbf{Q}\boldsymbol{x} = R^2, \tag{23}$$

where $R > 0$ and $\boldsymbol{x}$ satisfies Equation (22). Multiplying Equation (22) by an appropriate $\lambda$, we get $\boldsymbol{x}^T \cdot \lambda \mathbf{I} \cdot \boldsymbol{x} = R^2$, where $\mathbf{I}$ denotes the $2 \times 2$ identity matrix. Subtracting this expression from Equation (23), we get that $\mathbf{Q}^T\mathbf{Q} = \lambda\mathbf{I}$. Finally, since $\det(\mathbf{Q}) = 1$, we deduce that $\lambda = 1$, so $\mathbf{Q}$ is orthogonal. Therefore $g(\boldsymbol{x})$ is an orthogonal transformation, so $g(\boldsymbol{x})$ defines a congruence. □

Lemma 7 provides the following corollary.

**Corollary 4.** *The image of a rational surface of revolution that is not developable under a nonsingular affine mapping, is another surface of revolution if and only if the affine mapping corresponds to a similarity.*

From the algorithmic point of view, notice that given two surfaces of revolution about the same axis, one can check whether the surfaces are similar by intersecting both surfaces with a same plane, say, the $yz$-plane, and then checking whether the resulting planar curves are similar. There are efficient algorithms for doing this: if the sections are rational, one can use the algorithm in [18]; if the sections are not rational, one can use the algorithm in [19].

Finally, we summarize all the results of the section in the following theorem.

**Theorem 7.** *The image of a surface of revolution under a nonsingular affine mapping is an affine surface of rotation if and only if the affine mapping defines a similarity, in which case the image is also a surface of revolution.*

**Corollary 5.** *The image of a non-developable rational surface of revolution under a nonsingular affine mapping that is not a similarity, is an* ANIL *surface that is not an affine surface of rotation.*

Notice that Corollary 5 comes to show that there are many ANIL surfaces that are not affine surfaces of rotation: in fact, the image of any surface of revolution under a non-orthogonal affine mapping is that way. Taking Corollary 2 also into account, we conclude that there are many ANIL surfaces that are not either affine surfaces of rotation, or affine spheres.

## 6. Conclusions

Throughout the paper we have proved that the image of a non-developable rational surface of revolution under a nonsingular affine mapping is an ANIL surface which is not an affine rotation surface except for certain, well-described, cases. Furthermore, given an ANIL surface, we have provided an algorithm to determine whether it is the affine image of a surface of revolution, and to recover it, if it exists.

One can wonder whether there exist ANIL surfaces which are not the image of a surface of revolution or an affine sphere. We do not have an answer to this question. Were the answer negative, it would be nice to identify notable surfaces with this property. These are problems that we leave here as open questions.

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