Article

A Note on the Characterization of Two-Dimensional Quasi-Einstein Manifolds

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Abstract: In this article, we aim to introduce new classes of two-dimensional quasi-Einstein pseudo-Riemannian manifolds with constant curvature. We also give a classification of 2D quasi-Einstein manifolds of warped product type working in local coordinates. All the results are obtained by elementary methods.

Keywords: Ricci soliton; quasi-Einstein metric; curvature; potential function

1. Introduction

Nonlinear differential equations arise in wide areas of research in pure and applied sciences. In this note, we study some non-linear differential equations whose solutions are quasi-Einstein metrics that appear in mathematical physics [1,2]. In our approach, we will use the techniques of mathematical analysis, more suitable in this case, as in article [3]. The quasi-Einstein metric is a generalization of the Einstein metric, it contains gradient Ricci solitons and it is also closely related to the construction of the warped product Einstein metrics. The study of quasi-Einstein metrics was initiated by Chaki and Maity in [4]. In [5], Chaki and Ghoshal studied some global properties of quasi-Einstein manifolds, while in [6], De and Ghosh gave some examples, proven their existence, and underline some properties. As applications, gravitational instantons are defined to be solutions of the quasi-Einstein equations [7,8].

To set forth our research in this article, we organized our discussion in two parts.

In the first part of this research, we consider a pseudo-Riemannian manifold $(M, \bar{g})$, which has constant sectional curvature K , [9]. Given a smooth real valued mapping F , defined on M and μ a real constant, suppose that the tensor having the components $h_{ij} = F_{,ij} - \mu F_{,i} F_{,j}$ is non-degenerate and has constant signature. With these assumptions, we establish a link between the Christoffel symbols Γ_{ij}^k of h and the Christoffel symbols $\bar{\Gamma}_{ij}^k$ of the initial metric $\bar{g}$.

When $\bar{g}$ has constant sectional curvature K , it is known that $\bar{R}_{jl} = (n - 1) K \bar{g}_{jl}$. The relation which defines the quasi-Einstein pseudo-Riemannian metrics:

$$\bar{R}_{ij} + F_{,ij} - \mu F_{,i} F_{,j} = \rho \bar{g}_{ij}$$

can be written as:

$$F_{,ij} - \mu F_{,i} F_{,j} = [\rho - (n - 1) K] \bar{g}_{ij}.$$

This means that the Christoffel symbols Γ_{ij}^k of h and the Christoffel symbols $\bar{\Gamma}_{ij}^k$ of $\bar{g}$ have to be identical. The equalities:

$$\Gamma_{ij}^k = \bar{\Gamma}_{ij}^k, \quad \forall i, j, k = \overline{1, n}$$

imply necessary conditions such that the initial metric $\bar{g}$, with constant curvature K , becomes a quasi-Einstein pseudo-Riemannian metric.

As application, we shall show that $\mathbb{R}^2$, endowed with the pseudo-Riemann metric of Kruskal-type, having null sectional curvature, is a quasi-Einstein manifold.

The second part of this research aims to continue the study of A. Pitea, recently published in [10,11]. Following this research, we introduce the explicit form of the sectional curvature under the necessary conditions that a generalized Poincaré metric $\bar{g}(x, y) = \text{diag}(g(y), f(y))$ becomes quasi-Einstein. We show that if the sectional curvature K is constant, then the metric $\bar{g}$ becomes a quasi-Einstein one.

As application, we consider the warped product Riemannian metrics $\bar{g}(x, y) = \text{diag}(g(y), 1)$, with constant sectional curvature, and we obtain a complete classification of those metrics which are quasi-Einstein too.

2. A New Class of Quasi-Einstein Metrics

Finding quasi-Einstein metrics classes has attracted the attention of much research, since there were studied in different forms in the recent past [2,7,10–13].

Consider a smooth n -dimensional pseudo-Riemannian manifold $(M^n, \bar{g})$ and fix $F : M \rightarrow \mathbb{R}$ a smooth real valued function on M . One denotes with $\nabla^2 F$ its Hessian with respect to $\bar{g}$.

A natural extension of the Ricci tensor $\bar{Ric}$ of $\bar{g}$ is the μ -Bakry–Emery Ricci tensor:

$$\bar{Ric}_F^\mu := \bar{Ric} + \nabla^2 F - \mu dF \otimes dF$$

and hence, if F is constant then $\bar{Ric}_F^\mu = \bar{Ric}$. For more details on the μ -Bakry–Emery Ricci tensor see [14,15].

The pseudo-Riemannian manifold $(M, \bar{g})$ is called *quasi-Einstein* if the μ -Bakry–Emery Ricci tensor is a constant multiple of the metric tensor:

$$\bar{Ric}_F^\mu = \rho \bar{g}, \rho \in \mathbb{R}. \quad (1)$$

In local coordinates, the relation (1) becomes a system of differential equations:

$$\bar{R}_{ij} + F_{,ij} - \mu F_{,i} F_{,j} = \rho \bar{g}_{ij} \quad (2)$$

for some real constants μ and ρ , where [13]:

$$F_{,i} = \frac{\partial F}{\partial x^i}; \quad F_{,ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} - \bar{\Gamma}_{ij}^r F_{,r}; \quad F_{,ijk} = \frac{\partial F_{,ij}}{\partial x^k} - \bar{\Gamma}_{ki}^l F_{,lj} - \bar{\Gamma}_{kj}^l F_{,li}.$$

For $\mu = 0$ the relation (1) defines *gradient Ricci solitons*, [16,17].

Suppose that the tensor of components $h_{ij} = F_{,ij} - \mu F_{,i} F_{,j}$ is non-degenerate and has constant signature. Then h is a new pseudo-Riemannian metric, which has the Levi–Civita connection ∇_h and the Christoffel symbols Γ_{ij}^k , see [18,19]. We will derive the symbols Γ_{ij}^k in terms of the $\bar{\Gamma}_{ij}^k$ of $\bar{g}_{ij}$:

Theorem 1. Let h^{pk} be the contravariant components of the pseudo-Riemannian metric h and $\bar{R}_{ijk}^m$ the curvature tensor field of $\bar{g}$. Then the Christoffel symbols Γ_{ij}^p are given by:

$$\Gamma_{ij}^p = \bar{\Gamma}_{ij}^p + \frac{1}{2} h^{pk} \left[F_{,ijk} - 2\mu F_{,ij} F_{,k} + \left(\bar{R}_{ikj}^l + \bar{R}_{jki}^l \right) F_{,l} \right].$$

Proof. From:

$$\begin{aligned}\frac{\partial h_{ij}}{\partial x^k} &= \frac{\partial}{\partial x^k} (F_{,ij} - \mu F_{,i} F_{,j}) \\ &= \frac{\partial}{\partial x^k} \left(\frac{\partial^2 F}{\partial x^i \partial x^j} - \bar{\Gamma}_{ij}^r F_{,r} \right) - \mu \left(\frac{\partial^2 F}{\partial x^k \partial x^i} F_{,j} + F_{,i} \frac{\partial^2 F}{\partial x^k \partial x^j} \right) \\ &= \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} - \frac{\partial \bar{\Gamma}_{ij}^r}{\partial x^k} F_{,r} - \bar{\Gamma}_{ij}^r F_{,kr} - \bar{\Gamma}_{ij}^r \bar{\Gamma}_{kr}^l F_{,l} - \mu \left(\frac{\partial^2 F}{\partial x^k \partial x^i} F_{,j} + F_{,i} \frac{\partial^2 F}{\partial x^k \partial x^j} \right)\end{aligned}$$

and the similar formulas for $\frac{\partial h_{jk}}{\partial x^i}$ and $\frac{\partial h_{ik}}{\partial x^j}$, we have:

$$\begin{aligned}2h_{pk}\Gamma_{ij}^p &= \frac{\partial h_{jk}}{\partial x^i} + \frac{\partial h_{ik}}{\partial x^j} - \frac{\partial h_{ij}}{\partial x^k} = \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} + \left(\frac{\partial \bar{\Gamma}_{ij}^l}{\partial x^k} - \frac{\partial \bar{\Gamma}_{jk}^l}{\partial x^i} - \frac{\partial \bar{\Gamma}_{ik}^l}{\partial x^j} \right) F_{,l} + \\ &+ \left(\bar{\Gamma}_{ij}^r \bar{\Gamma}_{kr}^l - \bar{\Gamma}_{ik}^r \bar{\Gamma}_{jr}^l - \bar{\Gamma}_{jk}^r \bar{\Gamma}_{ir}^l \right) F_{,l} + \left(\bar{\Gamma}_{ij}^r F_{,kr} - \bar{\Gamma}_{jk}^r F_{,ir} - \bar{\Gamma}_{ik}^r F_{,jr} \right) - \\ &- 2\mu \frac{\partial^2 F}{\partial x^i \partial x^j} F_{,k}.\end{aligned}$$

Substituting $\frac{\partial^2 F}{\partial x^i \partial x^j}$ by $F_{,ij} + \bar{\Gamma}_{ij}^r F_{,r}$ and using the formula for $F_{,ijk}$ we obtain:

$$\begin{aligned}2h_{pk}\Gamma_{ij}^p &= F_{,ijk} + \left(\bar{\Gamma}_{ki}^l F_{,lj} + \bar{\Gamma}_{kj}^l F_{,li} + \bar{\Gamma}_{ij}^l F_{,kl} \right) + \frac{\partial \bar{\Gamma}_{ij}^l}{\partial x^k} F_{,l} + \bar{\Gamma}_{ij}^r \bar{\Gamma}_{kr}^l F_{,l} + \\ &+ \left(\frac{\partial \bar{\Gamma}_{ij}^l}{\partial x^k} - \frac{\partial \bar{\Gamma}_{jk}^l}{\partial x^i} - \frac{\partial \bar{\Gamma}_{ik}^l}{\partial x^j} + \bar{\Gamma}_{ij}^r \bar{\Gamma}_{kr}^l - \bar{\Gamma}_{ik}^r \bar{\Gamma}_{jr}^l - \bar{\Gamma}_{jk}^r \bar{\Gamma}_{ir}^l \right) F_{,l} + \\ &+ \left(\bar{\Gamma}_{ij}^r F_{,kr} - \bar{\Gamma}_{jk}^r F_{,ir} - \bar{\Gamma}_{ik}^r F_{,jr} \right) - \left(2\mu F_{,ij} + 2\mu \bar{\Gamma}_{ij}^r F_{,r} \right) F_{,k}.\end{aligned}$$

We reduce the terms $\bar{\Gamma}_{ki}^l F_{,lj}$ with $\bar{\Gamma}_{ik}^r F_{,jr}$ and $\bar{\Gamma}_{kj}^l F_{,li}$ with $-\bar{\Gamma}_{jk}^r F_{,ir}$ and we find:

$$\begin{aligned}2h_{pk}\Gamma_{ij}^p &= F_{,ijk} - 2\mu F_{,ij} F_{,k} + 2\bar{\Gamma}_{ij}^l (F_{,kl} - \mu F_{,k} F_{,l}) + \left(\bar{R}_{ikj}^l + \bar{R}_{jki}^l \right) F_{,l} \\ &= 2\bar{\Gamma}_{ij}^l h_{kl} + F_{,ijk} - 2\mu F_{,ij} F_{,k} + \left(\bar{R}_{ikj}^l + \bar{R}_{jki}^l \right) F_{,l}.\end{aligned}$$

Finally, we obtain the formula:

$$\Gamma_{ij}^p = \bar{\Gamma}_{ij}^p + \frac{1}{2} h^{pk} \left[F_{,ijk} - 2\mu F_{,ij} F_{,k} + \left(\bar{R}_{ikj}^l + \bar{R}_{jki}^l \right) F_{,l} \right],$$

and the proof is complete. $\square$

In the following, we consider the case when $(M, \bar{g})$ has constant sectional curvature K and then:

$$\bar{R}_{jki}^l = K \left(\bar{g}_k^l \bar{g}_{ji} - \bar{g}_i^l \bar{g}_{jk} \right), \quad \bar{R}_{ij} = \bar{R}_{ilj}^l = (n-1) \bar{g}_{ij}.$$

The relation (2) takes the form:

$$F_{,ij} - \mu F_{,i} F_{,j} = [\rho - (n-1)K] \bar{g}_{ij}.$$

From Theorem 1 we get that $\Gamma_{ij}^p = \bar{\Gamma}_{ij}^p$, for all $i, j, p = \overline{1, n}$ if and only if:

$$F_{,ijk} - 2\mu F_{,ij}F_{,k} + K \left(2\bar{g}_{ij}F_{,k} - \bar{g}_{ik}F_{,j} - \bar{g}_{jk}F_{,i} \right) = 0, \quad \forall i, j, p = \overline{1, n}. \quad (3)$$

Example 1. We consider an open subset M of $\mathbb{R}^2$ endowed with the pseudo-Riemannian metric of Kruskal-type $\bar{g}(x, y) = \begin{pmatrix} 0 & \alpha(x)\beta(y) \\ \alpha(x)\beta(y) & 0 \end{pmatrix}$, where $\alpha, \beta: \mathbb{R} \rightarrow \mathbb{R}^*$ are smooth functions. In addition we assume that: $\alpha(x) > 0, \beta(y) > 0$ and $\mu > 0$. From [20] we have $K = 0$ and the Christoffel symbols:

$$\Gamma_{11}^1 = \frac{\alpha'(x)}{\alpha(x)}, \Gamma_{22}^2 = \frac{\beta'(y)}{\beta(y)}, \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{11}^2 = \Gamma_{22}^1 = 0.$$

The system of differential Equation (3) becomes:

$$\frac{\partial}{\partial x} \left[\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] - 2 \frac{\alpha'(x)}{\alpha(x)} \left[\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] = 2\mu \left[\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] \frac{\partial F}{\partial x} \quad (4)$$

$$\frac{\partial}{\partial y} \left[\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] = 2\mu \left[\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] \frac{\partial F}{\partial y} \quad (5)$$

$$\frac{\partial^3 F}{\partial x^2 \partial y} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial^2 F}{\partial x \partial y} = 2\mu \frac{\partial^2 F}{\partial x \partial y} \frac{\partial F}{\partial x} \quad (6)$$

$$\frac{\partial^3 F}{\partial x \partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial^2 F}{\partial x \partial y} = 2\mu \frac{\partial^2 F}{\partial x \partial y} \frac{\partial F}{\partial y} \quad (7)$$

$$\frac{\partial}{\partial x} \left[\frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} \right] = 2\mu \left[\frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} \right] \frac{\partial F}{\partial x} \quad (8)$$

$$\frac{\partial}{\partial y} \left[\frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} \right] - 2 \frac{\beta'(y)}{\beta(y)} \left[\frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} \right] = 2\mu \left[\frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} \right] \frac{\partial F}{\partial y}. \quad (9)$$

By integrating with respect to y the Equation (5), we obtain

$$\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} = e^{2\mu F} e^{a(x)},$$

where a is a function which depends on x only. Substituting in Equation (4) and then integrating with respect to x , we get $\ln \left[e^{a(x)} e^{2\mu F} \right] - 2 \ln \alpha(x) = 2\mu F + b(y)$, where b is a function, which depends on y only. The last equation leads us to $e^{a(x)} = \alpha^2(x) e^{b(y)}$, hence $\frac{e^{a(x)}}{\alpha^2(x)} = e^{b(y)}$ must be a positive constant. Then $a(x) = c + 2 \ln \alpha(x)$, and $b(y) = c$, so

$$\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} = e^{2\mu F} e^c \alpha^2(x).$$

In a similar manner, from the last two equations we obtain

$$\frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} = e^{2\mu F} e^c \beta^2(y).$$

From the Equation (6) we have $\frac{\partial^2 F}{\partial x \partial y} = \alpha(x) e^{2\mu F} e^{p(y)}$ and from the Equation (7) we deduce $\frac{\partial^2 F}{\partial x \partial y} = \beta(y) e^{2\mu F} e^{s(x)}$. The last equalities lead us to $\frac{e^{s(x)}}{\alpha(x)} = \frac{e^{p(y)}}{\beta(y)}$, so these expressions must be a positive constant r .

Therefore $e^{s(x)} = r\alpha(x)$, $e^{p(y)} = r\beta(y)$ and $\frac{\partial^2 F}{\partial x \partial y} = r\alpha(x)\beta(y)e^{2\mu F}$. Turning back to the initial system (2) which defines quasi-Einstein metrics, we have

$$F_{,11} - \mu F_{,1} F_{,1} = \rho \bar{g}_{11}, \quad (10)$$

$$F_{,12} - \mu F_{,1} F_{,2} = \rho \bar{g}_{12}, \quad (11)$$

$$F_{,22} - \mu F_{,2} F_{,2} = \rho \bar{g}_{22}. \quad (12)$$

The Equation (10) leads us to

$$\frac{\partial F}{\partial x} = \pm \frac{1}{\sqrt{\mu}} e^{\frac{\epsilon}{2}} e^{\mu F} \alpha(x).$$

The Equation (12) is equivalent to

$$\frac{\partial F}{\partial y} = \pm \frac{1}{\sqrt{\mu}} e^{\frac{k}{2}} e^{\mu F} \beta(y).$$

The Equation (11)

$$r\alpha(x)\beta(y)e^{2\mu F} - \mu \frac{\partial F}{\partial x} \frac{\partial F}{\partial y} = \rho\alpha(x)\beta(y)$$

becomes $e^{2\mu F} \left(r \pm e^{\frac{\epsilon}{2}} e^{\frac{k}{2}} \right) = \rho$. If $r \pm e^{\frac{\epsilon}{2}} e^{\frac{k}{2}} \neq 0$, then F is a constant, a trivial case. So we have to impose $r - e^{\frac{\epsilon}{2}} e^{\frac{k}{2}} = 0$ and $\rho = 0$. Then we choose $\frac{\partial F}{\partial x} = -\frac{1}{\sqrt{\mu}} e^{\mu F} e^{\frac{\epsilon}{2}} \alpha(x)$ and $\frac{\partial F}{\partial y} = -\frac{1}{\sqrt{\mu}} e^{\mu F} e^{\frac{k}{2}} \beta(y)$. The Equation (10) becomes

$$\frac{\partial F}{\partial x} e^{-\mu F} = -\frac{1}{\sqrt{\mu}} e^{\frac{\epsilon}{2}} \alpha(x),$$

hence

$$e^{-\mu F} = e^{\frac{\epsilon}{2}} \sqrt{\mu} \int \alpha(x) dx + t(y).$$

The relation (11) can be written

$$\frac{\partial F}{\partial y} e^{-\mu F} = -\frac{1}{\sqrt{\mu}} e^{\frac{k}{2}} \beta(y),$$

hence

$$e^{-\mu F} = e^{\frac{k}{2}} \sqrt{\mu} \int \beta(y) dy + u(x).$$

These last two equations lead us to

$$e^{\frac{\epsilon}{2}} \sqrt{\mu} \int \alpha(x) dx - u(x) = e^{\frac{k}{2}} \sqrt{\mu} \int \beta(y) dy - t(y).$$

Since the left-hand side is a function with one only variable x , and the right-hand side is a function with one only variable y , then

$$\begin{cases} e^{\frac{\epsilon}{2}} \sqrt{\mu} \int \alpha(x) dx - u(x) = -l \\ e^{\frac{k}{2}} \sqrt{\mu} \int \beta(y) dy - t(y) = -l, \end{cases}$$

l real constant. Thus

$$u(x) = e^{\frac{\epsilon}{2}} \sqrt{\mu} \int \alpha(x) dx + l, t(y) = e^{\frac{k}{2}} \sqrt{\mu} \int \beta(y) dy + l$$

and

$$e^{-\mu F} = \sqrt{\mu} \left(e^{\frac{\epsilon}{2}} \int \alpha(x) dx + e^{\frac{k}{2}} \int \beta(y) dy \right) + l.$$

Finally, the potential function is

$$F(x, y) = \frac{-1}{\mu} \ln \left| \sqrt{\mu} \left(e^{\frac{c}{2}} \int \alpha(x) dx + e^{\frac{k}{2}} \int \beta(y) dy \right) + l \right|,$$

which corresponds to $\mu \neq 0$ and $\rho = 0$.

3. Generalizing a Question of Besse

An important question was stated by Besse [21], which was how to determine examples of Einstein manifolds, which are warped products [16]. A natural generalization asks to finding examples of quasi-Einstein manifolds of warped product type. In the following, we continue the study started by Pitea [10], by finding new classes of explicit quasi-Einstein Riemannian manifolds, endowed with generalized Poincaré metrics, which also have constant sectional curvature. We consider the manifold M as an open subset of $\mathbb{R}^2$, endowed with metric of diagonal type $\bar{g}(x, y) = \text{diag}(g(y), f(y))$, where g and f are strictly positive smooth functions [16,22,23]. In order to find g , f and F which satisfy (2), Pitea [10] shows that in the case when the potential function F depends only on y , F has the form:

$$F(y) = 2\rho \int \frac{f(y)g(y)}{g'(y)} dy + \ln \left(\frac{|g'(y)|}{\sqrt{f(y)g(y)}} \right).$$

Also, Pitea [10] proves that if we introduce new functions $h = \frac{g'}{g}$, $p = \frac{f'}{f}$, then from (2) h and p must satisfy the equation:

$$2\rho \left(\frac{f}{h} \right)' + \left(\frac{h'}{h} \right)' - \frac{1}{2}p' - \mu \left(2\rho \frac{f}{h} + \frac{h'}{h} + \frac{1}{2}h - \frac{1}{2}\rho \right)^2 = \rho f + \frac{1}{4}h^2 + \rho p f \frac{1}{h} + \frac{1}{2}p \frac{h'}{h} - \frac{1}{4}p^2. \quad (13)$$

Now, we introduce the expression of the sectional curvature:

$$K = \frac{1}{f(y)} \left[- \left(\frac{g'(y)}{2g(y)} \right)' + \frac{g'(y)}{2g(y)} \left(\frac{f'(y)}{2f(y)} - \frac{g'(y)}{2g(y)} \right) \right]$$

in the Equation (13). It follows $K = \frac{h}{2f} \left(-\frac{h'}{h} + \frac{p-h}{2} \right)$, hence

$$\frac{h'}{h} = \frac{p-h}{2} - 2K \frac{f}{h}.$$

Substituting in (13) we obtain:

$$\begin{aligned} & \left(\frac{2\rho f}{h} + \frac{p-h}{2} - \frac{2Kf}{h} \right)' - \frac{1}{2}p' - \mu \left(\frac{2\rho f}{h} + \frac{p-h}{2} - \frac{2Kf}{h} + \frac{h-p}{2} \right)^2 \\ & = \rho f + \frac{1}{4}h^2 + \rho p f \frac{1}{h} + \frac{p}{2} \left(\frac{p-h}{2} - 2K \frac{f}{h} \right) - \frac{1}{4}p^2, \end{aligned}$$

which is equivalent to:

$$\left(\frac{2f(\rho - K)}{h} \right)' - \frac{1}{2}h' - \mu \frac{4f^2(\rho - K)^2}{h^2} = \rho f + \frac{1}{4}h^2 + \frac{\rho p f - K p f}{h} - \frac{p h}{4}.$$

Substituting again h' by $\frac{h(p-h)}{2} - 2Kf$ we find:

$$2(\rho - K) \left(\frac{f}{h} \right)' - 4\mu(\rho - K)^2 \frac{f^2}{h^2} = (\rho - K)f + \frac{pf(\rho - K)}{h}$$

or:

$$(\rho - K) \left[2 \frac{f'}{h} - \frac{2f}{h} \frac{h'}{h} - 4\mu(\rho - K) \left(\frac{f}{h} \right)^2 - f - \frac{pf}{h} \right] = 0.$$

But $\frac{h'}{h} = \frac{p-h}{2} - 2K\frac{f}{h}$ and $pf = f'$. Finally, we get:

$$(\rho - K)(K - \mu\rho + \mu K) = 0. \quad (14)$$

The sectional curvature K must be either $K = \rho$ or $K = \frac{\mu\rho}{1+\mu}$, $\mu \neq -1$. Therefore the use of K from (14) seems to be the most suitable way to find quasi-Einstein manifolds with constant sectional curvature, endowed with generalized Poincaré metrics.

Hence we expressed the relation (13) according to sectional curvature K , obtaining the relation (14) and the following theorem.

Theorem 2. An open subset M of $\mathbb{R}^2$ with the metric $\bar{g}(x, y) = \text{diag}(g(y), f(y))$, having the constant sectional curvature $K = \rho$ or $K = \frac{\mu\rho}{1+\mu}$, $\mu \neq -1$ is a quasi-Einstein Riemannian manifold corresponding to the real constants ρ and μ .

CASE STUDY. In the following, we shall determine all metrics of the form $\bar{g}(x, y) = \text{diag}(g(y), 1)$ with constant sectional curvature. Taking $f(y) = 1$ in the above formula we obtain the sectional curvature:

$$K = -\frac{1}{2} \left(\frac{g'(y)}{g(y)} \right)' - \frac{1}{4} \left(\frac{g'(y)}{g(y)} \right)^2,$$

and the expression of potential function:

$$F(y) = 2\rho \int \frac{g(y)}{g'(y)} dy + \ln \left(\left| \frac{g'(y)}{\sqrt{g(y)}} \right| \right).$$

The relation $h = \frac{g'}{g}$ yields:

$$2h' = -h^2 - 4K. \quad (15)$$

The solutions of Equation (15), obtained in the general case, enable us to classify the two dimensional quasi-Einstein Riemannian metrics $\bar{g}(x, y) = \text{diag}(g(y), 1)$ with constant sectional curvature K .

CLASS 1. If $-h^2 - 4K = 0$ then $h' = 0$. Hence $h(y) = c$ a real constant, the metric component $g(y) = e^{cy}$ and the potential function is $F(y) = \frac{4\rho + c^2}{2c}y + \ln|c|$. If $K = -\rho$ then $\rho = -\frac{c^2}{4}$ and $F(y) = \ln|c|$ is trivial. If $K = \frac{\mu\rho}{1+\mu} = -\frac{c^2}{4}$ then $F(y) = -\frac{c^2}{4}y + \ln|c|$.

Therefore, we just proved the following proposition.

Proposition 1. The Riemannian manifold $(\mathbb{R}^2, \bar{g}(x, y) = \text{diag}(e^{cy}, 1))$ with sectional curvature $K = -\frac{c^2}{4}$ is a quasi-Einstein manifold corresponding to the real constants ρ and μ , with $\frac{\mu\rho}{1+\mu} = -\frac{c^2}{4} < 0$, $\mu \notin \{-1, 0\}$ and the potential function:

$$F(y) = \frac{-c}{2\mu}y + \ln|c|.$$

CLASS 2. If $-h^2 - 4K \neq 0$ then we may write $\frac{2h'}{-h^2 - 4K} = 1$ or $\frac{h'}{h^2 + 4K} = -\frac{1}{2}$.

CLASS 2.1. If $K > 0$, then we integrate $\frac{h'(y)}{h^2(y) + (2\sqrt{K})^2} = -\frac{1}{2}$ and we have $h(y) = -2\sqrt{K}\tan(y\sqrt{K})$, $g(y) = \cos^2 y\sqrt{K}$ and the potential function

$$F(y) = \left(\frac{-\rho}{K} + 1\right) \ln|\sin y\sqrt{K}| + \ln(2\sqrt{K}).$$

If $K = \rho$ then $F(y) = \ln(2\sqrt{K})$ is trivial. If $K = \frac{\mu\rho}{1+\mu}$, then $F(y) = -\frac{1}{\mu} \ln|\sin y\sqrt{K}| + \ln(2\sqrt{K})$.

Therefore, we just proved the following proposition.

Proposition 2. The Riemannian manifold $(\mathbb{R} \times [-\frac{\pi}{2K}, \frac{\pi}{2K}], \bar{g}(x, y) = \text{diag}(\cos^2 y\sqrt{K}, 1))$ with the sectional curvature $K > 0$ is a quasi-Einstein manifold corresponding to the real constants ρ and μ with $\frac{\mu\rho}{1+\mu} = K > 0$, $\mu \notin \{-1, 0\}$ and the potential function:

$$F(y) = \frac{-1}{\mu} \ln|\sin y\sqrt{K}| + \ln(2\sqrt{K}).$$

CLASS 2.2. If $K = 0$ then $\frac{h'}{h^2} = -\frac{1}{2}$ hence $h(y) = -\frac{2}{y}$, $g(y) = cy^2$, $c > 0$ and the potential function:

$$F(y) = \frac{\rho}{2}y^2 + \ln 2\sqrt{c}.$$

If $K = \rho = 0$ then $F(y) = \ln 2\sqrt{c}$ is trivial. If $K = \frac{\mu\rho}{1+\mu} = 0$ then we choose $\mu = 0$ and in this case we obtain a gradient Ricci soliton.

Hence we just demonstrated the following proposition.

Proposition 3. The Riemannian manifold $(\mathbb{R}^2 \setminus \{0\}, \bar{g}_c(x, y) = \text{diag}(cy^2, 1))$, $c > 0$ with the sectional curvature $K = 0$, is quasi-Einstein corresponding to the real constants ρ and $\mu = 0$ and with the potential function:

$$F(y) = \frac{\rho}{2}y^2 + \ln 2\sqrt{c}.$$

In fact, $(\mathbb{R}^2 \setminus \{0\}, \bar{g}_c)$ is a gradient Ricci soliton; more precisely, $\bar{g}_1$ is the Euclidean metric in polar coordinates.

CLASS 2.3. If $K < 0$ then we have $\frac{h'}{h^2 - (2\sqrt{-K})^2} = -\frac{1}{2}$, hence $\left| \frac{h(y) - 2\sqrt{-K}}{h(y) + 2\sqrt{-K}} \right| = e^{-2\sqrt{-K}y}$ or $\frac{h(y) - 2\sqrt{-K}}{h(y) + 2\sqrt{-K}} = \pm e^{-2\sqrt{-K}y}$.

SUBCLASS 2.3.1. If $\frac{h(y) - 2\sqrt{-K}}{h(y) + 2\sqrt{-K}} = e^{-2\sqrt{-K}y}$ then:

$$h(y) = 2\sqrt{-K} \frac{e^{2\sqrt{-K}y}}{e^{2\sqrt{-K}y}} = 2\sqrt{-K} \frac{\text{ch}(\sqrt{-K}y)}{\text{sh}(\sqrt{-K}y)},$$

$$g(y) = c \text{sh}^2(\sqrt{-K}y), \quad c > 0$$

and the potential function:

$$F(y) = \left(-\frac{\rho}{K} + 1\right) \ln |\text{ch}\sqrt{-K}y| + \ln(2\sqrt{-cK}).$$

If $K = \rho = 0$ then $F(y) = \ln(2\sqrt{-cK})$ is trivial. If $K = \frac{\mu\rho}{1+\mu} < 0$ then $F(y) = -\frac{1}{\mu} \ln |\text{ch}\sqrt{-K}y| + \ln(2\sqrt{-cK})$.

We just showed the following proposition.

Proposition 4. The Riemannian manifold $(\mathbb{R}^2, \bar{g}(x, y) = \text{diag}(c \text{sh}^2(\sqrt{-K}y), 1))$, $c > 0$ with sectional curvature $K < 0$ is a quasi-Einstein corresponding to the real constants ρ and μ with $\frac{\mu\rho}{1+\mu} = K$, $\mu \notin \{-1, 0\}$ and the potential function:

$$F(y) = -\frac{1}{\mu} \ln |\text{ch}\sqrt{-K}y| + \ln(2\sqrt{-cK}).$$

SUBCLASS 2.3.2. If $\frac{h(y) - 2\sqrt{-K}}{h(y) + 2\sqrt{-K}} = -e^{-2\sqrt{-K}y}$, then:

$$h(y) = 2\sqrt{-K} \frac{\text{sh}(\sqrt{-K}y)}{\text{ch}(\sqrt{-K}y)}, \quad g(y) = c \text{ch}^2(\sqrt{-K}y), \quad c > 0$$

and the potential function:

$$F(y) = \left(-\frac{\rho}{K} + 1\right) \ln |\text{sh}\sqrt{-K}y| + \ln(2\sqrt{-cK}).$$

If $K = \rho < 0$, then $F(y) = \ln(2\sqrt{-cK})$ is trivial. If $K = \frac{\mu\rho}{1+\mu} < 0$ then $F(y) = -\frac{1}{\mu} \ln |\text{sh}\sqrt{-K}y| + \ln(2\sqrt{-cK})$.

Thus, we obtain the following result.

Proposition 5. The Riemannian manifold $(\mathbb{R}[0, \infty), \bar{g}(x, y) = \text{diag}(c \text{ch}^2(\sqrt{-K}y), 1))$, $c > 0$ with the sectional curvature $K < 0$ is quasi-Einstein corresponding to the real constants ρ and μ , with $\frac{\mu\rho}{1+\mu} = K$, $\mu \notin \{-1, 0\}$ and the potential function:

$$F(y) = -\frac{1}{\mu} \ln |\text{sh}\sqrt{-K}y| + \ln(2\sqrt{-cK}).$$

Remark 1. The pseudo-Riemannian metric $\bar{g}(x, y) = \text{diag}(c \text{ch}^2(\sqrt{-K}y), 1)$ corresponding to $c = -1$ is exactly the two-dimensional version of anti de Sitter metric of General Relativity, a very important model in that theory.

4. Conclusions

In our work, we provided new classes of two-dimensional quasi-Einstein manifolds, endowed with generalized Poincaré metrics and having constant sectional curvature K .

These metrics generalize some remarkable metrics (for instance, Kruskal-type metric and anti de Sitter metric) and have applications in Theoretical and Experimental Physics [24–30], where geometric methods are used for physical modeling.

This use is due to the fact that the Ricci tensor for these manifolds looks like that of a perfect fluid model of general relativity.

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