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How Does Pedagogical Flexibility in Curriculum Use Promote Mathematical Flexibility? An Exploratory Case Study

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Abstract: Flexibility has been increasingly valued in mathematics education to better prepare students for lives in the rapidly changing society of the future. Although there has been conjecture that teachers' flexibility plays a substantial role in facilitating students' mathematical flexibility, there has been little examination of how teachers can use a flexible curriculum to develop mathematical flexibility (MF) in authentic classroom environments. This paper elaborates the notion of flexible curriculum use, referred to as pedagogical flexibility (PF) in curriculum use, as the competence to expand pedagogical space and make alternative pedagogical decisions when planning and enacting a curriculum that differs from the routine practices provided in the intended and written curriculum. We develop a framework for PF in curriculum use to identify and characterize teachers' curriculum use to promote MF. In an explorative case study with one middle school teacher, we analyzed what and how specific aspects of PF in curriculum use promote potential and actual MF in the learning of central tendency measures. Findings indicate that the teacher could expand his pedagogical space by carefully differentiating the pedagogical considerations of the curriculum and could find alternative approaches by making associative and reflective connections among them. This provides insight into how PF in curriculum use can promote students' potential and actual MF.

Keywords: pedagogical flexibility; teacher curriculum use; mathematical flexibility; lesson planning; lesson implementation

1. Introduction

There have been growing claims in recent years that the vision and goal of mathematics education in the 21st century should be set to keep up with the rapid changes occurring in society as it moves into the future [1,2]. The rapid changes in the society of the future are often described as having volatility and uncertainty as their main characteristics [3]. To prepare students to face this uncertain future, educational objectives should aim at developing adaptive competencies rather than just mastering algorithms. In that vein, creativity is listed as one of the essential 21st-century skills [4–6] and acknowledged as vital to individual and social success [7]. Many researchers have viewed flexibility, an indispensable element of creativity, as critical to students' understanding of concepts, creative concept formation, problem solving, and a deeper understanding of procedural knowledge [8,9]. Flexibility is also necessary for teachers to adjust learning objectives, content, and methods to suit changing contexts and expectations accordingly [10–12]. It is extremely important for teachers to have the capacity to understand and pursue facilitation of new educational objectives by adapting rather than offloading curriculum materials in their daily teaching in the current wave of rapid change [11]. Thus, flexibility is considered not only a key competency for future generations, but also one of the core teacher competencies in curriculum use.

There have been two main streams of integrating flexibility into mathematics education: mathematical flexibility (MF) and pedagogical flexibility (PF). MF is related to changing perspectives in mathematical concepts, processes, representations, models, and so on [9,13,14]. MF is the ability to avoid having a fixed mathematical perspective, which hinders perception of novel ideas, while PF is related to teachers' decisions to make significant changes to teaching approaches when designing tasks and planning lessons [15,16]. MF has a long history of studies beginning in the early 20th century through examining mathematicians' and gifted learners' creative work in mathematics [17–21] and focusing on problem solving [22]. Less is known, however, about PF. Pioneering studies on PF indicate that different methods or solutions can be achieved in problem solving [16], and diverse students can have better opportunities to learn [10] if mathematics teachers activate PF in their classroom teaching when necessary.

However, PF has not yet been operationalized enough to reveal its explicit roles in effective and innovate teaching and its significance in teacher expertise in curriculum use. PF could play more extensive roles in other learning contexts, such as conceptual understanding and development of flexible thinking. In addition, PF could be sophisticated in the key stages of curriculum use such as reading and interpreting the curriculum, adaptive lesson planning, and adaptive lesson implementation. In this study, we explore how PF can extensively contribute to innovating classroom teaching and learning. Specifically, we investigate what and how flexibility in making pedagogical decisions is used in the curriculum in the lesson planning and implementation stages when representative values are taught to Grade 9 students. We further explore the relationships between teachers' PF and students' potential and actual MF development. Drawing on our own conceptual framing of PF, this study aimed in particular at investigating what PF a mathematics teacher shows when adaptively using curriculum and how this has an impact on students' learning by their MF experiences.

2. Conceptual Background

2.1. Conceptualization of Teacher Curriculum Use in the Research Context

Much of the body of research on teacher curriculum use has employed several distinctions in curriculum to clarify what and how teachers interact with a curriculum. One such is a differentiation of curriculum into official curriculum and operative curriculum, which was introduced by Remillard and Heck [23]. They defined official curriculum as the set of curricular aims and objectives, content of consequential assessments, and designated curriculum. In comparison, operative curriculum is defined to include teacher-intended curriculum, enacted curriculum, and student outcomes [23].

Applying Remillard and Heck's differentiation of curriculum [23] to the current research context, we consider the Korean national curriculum and the textbooks and teacher guides developed and authorized by the Ministry of Education as an official curriculum. The Korean National Mathematics Curriculum revised in 2015 (hereafter referred to as the 2015 Math Curriculum) [24] presents succinct explanations of the nature of mathematics, core mathematical competences to promote, content objectives, content organization information, achievement standards, and an overall perspective on effective instruction and evaluation [25]. For the current study, teacher curriculum use is conceived as beginning with profound reading and adapting of the 2015 Math Curriculum document and the written curriculum, in other words, the textbooks and teacher guides, and then proceeding to designing the teacher's intended curriculum and enacting it, with the end result being student outcomes (see Figure 1). We focus in particular on MF as a key student outcome.

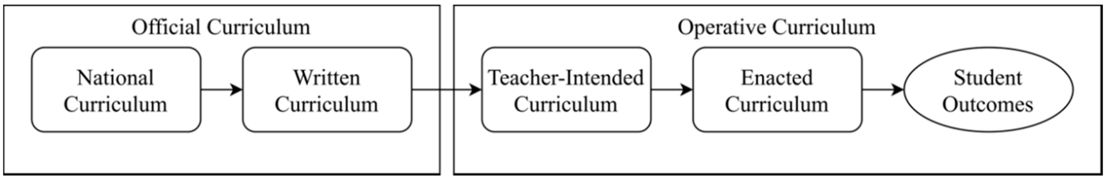

Figure 1. Korean mathematics teachers' curriculum use (Adapted from [23] (p. 709)).

2.2. *Significance of Mathematical Flexibility (MF) Experiences in Learning Mathematics*

The term "flexibility" is not used consistently across mathematics education literature. Although some definitions concern flexibility in terms of product (e.g., [22,26]), our focus is more on the process of MF, including decision-making, noticing, reasoning, and so forth. One description of flexibility from Krutetskii is "the ability to switch rapidly from...one train of thought to another" [19] (pp. 222–223). Similarly, Haylock [14] states that flexibility is the ability to overcome rigidity of thinking. A more refined characterization of flexible thinking is Krutetskii's framework of reversibility [27], which is the ability to establish two-way relationships that comes from the ability to make the transition from a direct association to its corresponding reverse association. Warner et al. [28] depicted a case of the transition of association in a mathematical domain. A problem given to a group of sixth grade students asked for the formula for the number of handshakes $h(n)$ given at an n-person party, given that each person shakes hands once with each other person. However, a student reversed this association when she questioned the possibility of finding the number of people n when the number of handshakes $h(n)$ was given. Reversibility is more complex when dealing with different types of mathematics registers, in other words, "the forms of meaning and styles of communication used by the mathematics disciplinary community" [29] (p. 29). To converse a representation to another and vice versa without changing the denoted objects flexibly, a student should associate two representations with direct and reverse orientation [30].

MF in mathematical learning has been explicitly highlighted as a cornerstone by several mathematics educators. For example, Gray and Tall [9] described flexible thinking as the ability to move between interpreting notation as a process of doing something procedurally and an object to think with and about conceptually. Mason [13] further argued that flexibility can be taught in mathematics classrooms by intertwining training in behavior with the education of awareness in order to enable attention to be complexly structured: This comes about through harnessing of emotions via the creation of intentional selves. He specifically indicated that flexibility of thinking comes from having connections, associations, heuristics, and topics come to mind while working on a problem and being sufficiently aware of these to be able to switch horses in midstream and shift attention to a new line of reasoning. Although some educators (e.g., [31]) conjecture that teachers' flexibility plays a substantial role in facilitating students' mathematical flexibility, the relationship between the two has been less frequently described in the mathematics education literature. One possible reason may be that the PF framework has not yet been sufficiently operationalized to explain how it has an impact on students' MF. Below, we review prior work and design our conceptual framework for PF.

2.3. *Related Concepts of Pedagogical Flexibility (PF) in Curriculum Use*

A group of researchers have shed light on the necessity of flexibility in teacher curriculum use, characterizing it as responsive and adaptive teacher-curriculum interaction [11,12,32,33]. Brown [11], after classifying teachers' ways of curriculum use into three types, offloading, adapting, and improvising, suggested framing teacher curriculum use as design work in that a teacher adaptively creates his/her own curriculum in order to achieve their didactical objectives by making alterations in the given curriculum materials. Sherin and Drake [12] recognized three types of curriculum use, reading, evaluating, and adapting, and investigated different teachers' use of curriculum and what impact it potentially has on student learning. Roth McDuffie et al. [32] focused on responsiveness in curriculum use using their own framework, which they referred to as curriculum noticing. They indicated curriculum noticing consists of three intertwined practices of attending, interpreting, and responding to curriculum in lesson planning and implementation while the teacher is making pedagogical decisions based on their orientation and interpretation of the teaching situation. Remillard [33] presented a participatory perspective of curriculum use where adaptive design underpins a teacher's work to effectively use curriculum. The above description of flexibility in curriculum use is related to teachers' ability and disposition to do adaptive lesson planning and classroom teaching wherein teachers anticipate and recognize students' conceptual development, inquiry, communication, and connection and then apply appropriate strategies flexibly.

There are a few studies taking concepts similar to PF into consideration for an explicit research focus. Ruthven et al. [15] coined the term interpretative flexibility to describe teachers' adaptation of educational resources based on teaching objectives and classroom environment. They reported how teachers' transformation of resources in their design work varies according to their mathematical and pedagogical considerations. Leikin and Dinur [16] presented the term teacher flexibility to highlight significant roles of teachers' responsive instruction in the classroom discussion while solving a mathematical problem. They investigated what changes a teacher made to her planned lesson trajectories as responses to unanticipated student reactions. Pepin et al. [10] proposed the concept of didactical flexibility to describe teachers' adaptations to curriculum in classroom teaching. Schoenfeld [34] similarly highlighted teachers' in-the-moment decision making, which is responsive in nature depending on their orientations, resources, and goals. Foster [35] presented a case of contingent teaching in which a teacher promoted learning by taking students' non-mathematical interruptions into consideration and effectively transforming them into mathematical ideas. This work has primarily focused on understanding instructional flexibility, which relates official curriculum and individual teachers' enactment in their classroom teaching. Few studies elaborate what a teacher's flexible curricular reasoning looks like and how it relates to instructional flexibility in association with developing students' MF. Our conception of PF in curriculum use encompasses a teacher's adaptive use of curriculum from official to operative (see Figure 1).

2.4. Associative and Reflective Relational Thinking as the Characteristics of PF in Curriculum Use

Teachers' curriculum use, in particular how teachers flexibly adapt official curriculum to build an innovative operative curriculum, relies on their profound mathematical knowledge, orientations to innovative teaching and beliefs in school mathematics reform [34], their own successful teaching experiences [12], effective curriculum noticing [32], and robust curriculum design capacity [10]. Like other similar concepts mentioned in the previous section, PF in the curriculum itself is not observable or measurable, but some kind of changes in curriculum use, in other words, alternative pedagogical considerations, can reveal and characterize teachers' PF. Ruthven et al. [15] posits that transitions in curriculum use vary according to what teachers take into consideration at certain moments. There are variety of considerations teachers can and should look at while planning and implementing lessons. Considerations teachers note in curriculum use can be mathematical (content-related) and non-mathematical (content-free); for instance, affective or social aspects are significantly considered to promote learning to make changes to the curriculum. Furthermore, each mathematical or non-mathematical consideration is related to itself or the other reflectively when making pedagogical decisions influenced and supported by curricular material and teacher resources. We use the metaphor pedagogical space (PS) to refer to the set of mathematical and non-mathematical considerations and the specific relations among them that a teacher has access to when making pedagogical decisions. We hypothesize a high-PF teacher seeks to extend their PS before and after curriculum use rather than being satisfied with the fixed or pre-authored PS described in the curricular materials. In other words, teachers who only follow the content and practices presented in the curricular material are viewed as having limited or low PF. Extension of PS by high-PF teachers include adding new pedagogical considerations and making reflective associations within and between them.

One important cognitive feature of making new pedagogical decisions is that they are reflective, relating to the sense that a teacher moves their focus from one aspect of pedagogical considerations to another and vice versa, which we refer to as a two-way relationship. Based on this conception of PF in curriculum use, we can think of three types of two-way relations that describe the way PF operates: (a) mathematical and mathematical (M-M), (b) mathematical and non-mathematical or non-mathematical and mathematical (M-N, N-M), and (c) non-mathematical and non-mathematical (N-N). Using this framework, we can investigate what PF a teacher uses when interacting with the curriculum. This framework is visualized in Figure 2.

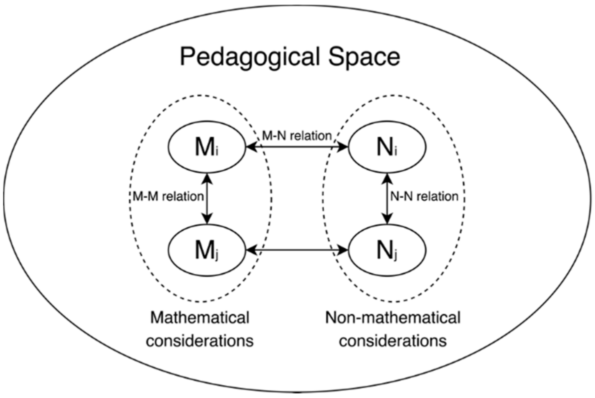

Figure 2. A conceptual framework for pedagogical flexibility (PF) in curriculum use.

M-M relations are constructed when teachers correlate and make two-way relations between concepts, ideas, representations, structure, definitions, and logic within mathematics with a pedagogical purpose to create alternative approaches to the curriculum or given curricular material. As a teacher reads curriculum materials [36] and associates their mathematical aspects, they come up with extended mathematical considerations with reflective associations between them and make pedagogical decisions that differ from the curricular material. For example, in Watson and Mason's [37] work, a teacher started teaching linear equations by writing down a simple equation ($x = 4$), which is not a representative example of linear equation. In order to teach how to solve it, he requested that students complicate the equation by performing the same operation on both sides. This is different from typical curricular approaches. We evaluate the teacher has having extended PS with two-way relations between pedagogical considerations that use worked-out examples and encouraging inquiry in solution methods of linear equation.

M-N or N-M relations are created when teachers move their attention back and forth between non-mathematical and mathematics-related aspects. For example, Foster [35] reported his effort as a teacher to make an impressive transition from the students' interest in their teacher's hair to a numerical inquiry about hair. Foster could have only sought topics that intrigued students from mathematical content given in the curriculum materials. However, he shifted the association to mathematical topics that related to the students' interests by quickly observing them. Responsive teaching by nature requires teachers to extend their PS in the moment and make transitions between non-mathematical and mathematical aspects, as Foster did [35].

Finally, N-N relations are created when teachers' non-mathematical considerations are re-organized. Research has shown that mathematics learning can be sustained by considering diverse non-mathematical aspects such as emotion [38,39], self-efficacy [40,41], equity [42], future societal environments [1], digital technology [43], and ecology [44]. These aspects are and can be correlated with each other in teachers' adaptive use of the curriculum. We expect to see this correlation when teachers make changes to classroom organization, provide emotive feedback, and use digital technology suggested in curricular materials based on their knowledge and recognition of their students' various background information and reactions.

The purpose of the study is not to show the effect of PF using the two-way relations between mathematics- and non-mathematics-related pedagogical considerations but to find detailed information on how they relate to lesson planning, implementation, and students' potential and actual MF when teaching and learning representative values. This gives information on teachers' possible ways of pedagogical reasoning when extensively using curricular materials. Because the adaptive use of curriculum to facilitate students' mathematical creativity has been shown to be very important, the findings of this study should provide a significant contribution to the research field on creative teaching of mathematics to promote students' MF.

3. Method

3.1. Participants and Settings

The official curriculum in Korea is characterized as centralized in that the overall vision and pedagogical decisions are made at the Ministry of Education level. It is concise in that there are overall directions in the national curriculum document and a compact set of concept definitions, worked examples, and exercises for key learning content in the textbooks, with moderate background information in the teacher guides that is widely open to teachers' interpretation. This study is a part of larger research on how middle school mathematics teachers understand the new vision of the national curriculum, how to adapt the textbooks to realize one aspect of this vision, mathematical creativity education, in their regular teaching by participating in a lesson study community of four teachers (Kim, T1–T3). The community was constructed in 2015 and has had regular meetings every other week to design and reflect on lessons together and watch videos of the members' classroom teaching. The teacher, Kim, with 14 years of teaching experience and holding a master's degree in mathematics education, participated in this study as the leader of the community. Researchers supported the community to read and adapt the official curriculum, search and read literature on nurturing mathematical creativity and statistical literacy, and reflect on lessons using video clips of lessons.

Kim's school was located in a low socioeconomic area, and the students' academic achievement level was very low. Kim aimed to improve his students' low performance and low interest in mathematics by nurturing their mathematical creativity. Twenty-six ninth grade students (14 boys and 12 girls, S1–S26) who were 14 to 15 years old participated in the study. Students worked in small groups of five (G1–G5). Kim's initial survey report showed that many of his students (20 out of 26) wrote that the break times between classes and lunch times are the happiest times in their school days, and a considerable number of students (18 out of 26) felt that they could not do mathematics well enough. Kim considered students' low interest in mathematics more serious than their low academic achievement. The three teachers in the lesson design community were all working in different schools, but Kim's feelings resonated with them in that they also wished to engage more low-interest students in doing mathematics. Hence, the most important goal for the community was to promote students' sustained learning of mathematics by nurturing mathematical creativity.

3.2. Data Collection and Analysis

Using a qualitative research paradigm [45], data collection lasted for 13 months from February 2017 to February 2018. Data sources for the chosen 11 lessons for teaching representative values (central tendency measures) were the lesson study community meetings, Kim's lesson plans and enactments, and interviews with Kim. Twelve community meetings (seven offline and five online) for the chosen lessons, which lasted 60 to 150 min, were audio recorded and transcribed. Kim's lesson plans (see Table 1 for the key prompts for each lesson) with his detailed anticipation of students' reactions and how to make pedagogical decisions to be responsive were documented. Eleven lesson enactments (each lesson lasted 45 min), including whole-class interactions and small-group activities with the key prompts of each lesson (Table 1), were video and audio recorded and transcribed. The first author interviewed Kim about his lesson plans and enactments for 30 to 60 min before and after the lessons, and these interviews were audio recorded and transcribed.

Table 1. Key prompts for each lesson in Kim's lesson plans.

	Key Prompts
Lesson 1	What kinds of data are there in the world? How can we use data to catch up with the changing world?
Lesson 2	What are the two kinds of data you are familiar with that are different in form and how they are different in form? What about two other kinds of data that have different forms? What types of data are they?
Lesson 3	Based on your examples, what types of data can have mean values and why? What would be your definition of representative values? What does mean value mean in representing data? What does mean value not mean in representing data?
Lesson 4	Based on your examples, what types of data can have a median and why? What does median mean and not mean in representing data? Based on your examples, what types of data can have a mode? What does mode mean and not mean in representing data?
Lesson 5	Design your own survey for 10 of your peers and plan your report using various representative values and graphical representations. Can you imagine when you would get the same mean value although you have added one more value to the original data set?
Lesson 6	Why are people not satisfied with a government report that the mean salary has increased? How can we critically read a newspaper article that has representative value information?
Lesson 7	Conduct your own survey amongst 30 friends and plan how to report your survey results using various representative values and graphical representations. Can you imagine when you would get the same representative values although you have added one more value to the original data set?
Lesson 8	Let's share what we determined in our survey of 30 friends. What can we tell about our friends using each representative value of the survey results and how can we evaluate the process and the findings of each survey in association with the meanings of the representative values?
Lesson 9	Reflecting on what and how you collected and interpreted the data, what can you summarize about mean, median, and mode as representative values? What were the most interesting things in learning about representative values?
Lesson 10	Reflecting on what you understood about the other teams' presentations, what can you summarize about mean, median, and mode as representative values? Which team's presentation do you think was the most effective in reporting the survey results?
Lesson 11	What are reasons for us to look at the world through data? What would be the differences between "the world with data" and "the world without data"? How can we use data to catch up with the changing world?

Data analysis was completed in three phases. In the first phase, we identified evidence of Kim's PF in his use of the official curriculum to promote students' MF during lesson planning. In other words, we determined what pedagogical points Kim considered most extensively in promoting students' MF during the first two lesson-design community discussions (2 and 23 February 2017) and in the first three interviews on his lesson plans (2 and 23 February 2017 and 9 March 2017). We parsed the transcripts into theme-related units and analyzed what points were considered in each unit. Using open coding, we described Kim's extension of PS to the set of extensive mathematical/non-mathematical considerations in promoting students' MF. We measured the frequency of each pedagogical point to reveal to what degree Kim paid attention to each. To ascertain coding consistency, the identified points and codes were shared among researchers and discussed until agreements were reached. The second phase of the analysis focused on Kim's PF in his use of the official curriculum to promote students' MF during lesson implementation. We again found the pedagogical points Kim considered most extensively in the lesson design community discussions and in the interviews on his lesson enactments. We repeated the entire process of the first analysis phase for the data on lesson enactments. In the third phase of the data analysis, we examined the relationships between Kim's PF in curriculum use during lesson planning and lesson implementation to promote students' MF by comparing the categorized extensive pedagogical considerations and the frequency measures. The final codes and categories with our draft interpretations were checked by the participants to verify our accurate understanding of their responses [45].

4. Findings

We first give the results of the analysis of Kim's extended PS with mathematical/non-mathematical considerations identified in his comments made in the community discussions and interviews. We found nine key extensions of mathematical aspects (M1–M9) and eight non-mathematical aspects (N1–N8) of official and operative curriculum that Kim was concerned about. Second, we present the findings of the investigation of potential MF chances that were entailed by Kim's PF during lesson planning. Third, we report how students' actual MF was promoted by Kim's extended PS, in other words, his high PF.

4.1. Kim's Extended PS for Teaching Representative Values

At the first community meeting on 2 February 2017, Kim and the three other teachers read and evaluated the 2015 Math Curriculum. They understood that the 2015 Math Curriculum put a lot of emphasis on students' growth not only in knowledge and skills but also in the six key competences of problem solving, reasoning, creativity and integration, communication, information processing, and attitude and practice. In particular, the three teachers appreciated the three overarching goals for middle-school mathematics, with the parts in italics below being the most important in realizing these goals in their classroom teaching:

(1) To develop an understanding of mathematical concepts, principles, and laws and their interrelationships and acquire mathematical skills through observing, analyzing, organizing, and expressing social and natural phenomena mathematically;
(2) To develop the ability to reason and communicate mathematically, understand social and natural phenomena, and solve problems in rational and creative ways based on creative and integrative thinking and the ability to process information;
(3) To develop awareness of the value of mathematics and cultivate interest and confidence and foster desirable attitudes and practical competencies as mathematics learners [24] (p. 5).

Unlike the three teachers, Kim noticed the beginning parts of the goals and said that the beginning part of the first goal, to develop an understanding of mathematical concepts, principles, and laws, is the most important but the least pursued in authentic classroom teaching. When asked the meaning of "authentic," he described how much he had been struggling to encourage students who are so diverse in academic achievement, linguistic ability, and attitude to learn mathematics. Then Kim read the presented learning objective regarding representative values, "to understand the meaning of median, mode, and mean and to be able to calculate them" [24] (p. 36). Kim said:

> I don't like this way of presenting learning objectives. It is like saying understand first and then calculate later. Isn't it hard to understand this objective itself? As I remember, Skemp differentiated relational understanding from instrumental understanding. There is a distinction between conceptual and procedural understanding too. Calculation should be integrated into the understanding, that is to say, understanding of the principle of calculation; a kind of procedural or instrumental understanding should be indicated in the curriculum

document. This way of presenting learning objectives by legitimate textbook authors gives the concept definition first and then moves to exercises. Definition doesn't guarantee understanding of meaning, and exercises do not necessarily enhance understanding of computation. Sometimes it would be better to present learning objectives using academic expressions such as procedural understanding.

This comment shows Kim has rich knowledge of learning theories that led him to make adaptations in the learning objectives to enhance students' understanding of representative values, which was codified as mathematical understanding (M1, Table 2). Although the point itself is presented in the 2015 Math Curriculum, he found a delicate discrepancy between one of the overarching goals and the learning objective discussed above and rephrased the learning objective as "students are able to develop a conceptual and procedural understanding of representative values". This adapted objective was used as an umbrella for his overall lesson planning and implementation.

Table 2. Kim's mathematical considerations.

Mathematical Competence	**Mathematical Consideration**	**Frequency (Percent) of Comments during Lesson Planning**		**Frequency (Percent) of Comments during Lesson Enacting**	
Conceptual learning	M1: Mathematical understanding	22 (19.3%)	53 (46.5%)	12 (11.0%)	41 (37.6%)
	M2: Defining representative values	18 (15.8%)		14 (12.8%)	
	M3: Multiplicity of representative values	13 (11.4%)		15 (13.8%)	
Mathematical reasoning	M4: Constraints of mean value as a representative value	16 (14.0%)	30 (26.3%)	6 (5.5%)	11 (10.1%)
	M5: Types of data	14 (12.3%)		5 (4.6%)	
Statistical processing	M6: Using technology	11 (9.6%)	31 (27.2%)	12 (11.0%)	57 (52.3%)
	M7: Graphic representation	8 (7.0%)		11 (10.1%)	
	M8: Comparing two data sets	7 (6.1%)		19 (17.4%)	
	M9: Adding and subtracting values	5 (4.4%)		15 (13.8%)	
Sum		114 (100%)		109 (100%)	

Kim critically analyzed the definition of representative values provided in the textbook and said, "There are definitions based on tautologies in the textbooks. The definition of representative values is one such example. Representative values are the values representing data? I think this kind of definition doesn't help students understand the meaning. I like Freudenthal's suggestion, defining instead of definition. Maybe I would include defining activities in one of the lessons" (2 February 2017, community discussion). This comment was labelled as defining representative values (M2, Table 2). We can see that Kim again moved away from the original approach to definition and employed an alternative approach by taking the researchers' suggestions.

There were many discussions about the multiplicity of representative values. Kim argued that this is related to the spirit of statistics, that is, the idea of modelling the world by going through iterative processes with multiple ideas, conjectures, and attempts. He wanted students to learn how to make the best choice among possible solutions using statistical reasoning instead of finding one correct solution (23 February 2017, interview), which was codified as multiplicity of representative values (M3,

Table 3). The three points M1, M2, and M3 enlarge the official curriculum in different ways to enhance conceptual learning. Kim's comments in this category of conceptual learning were almost half of the identified mathematics-related pedagogical comments (46.5%), as shown in Table 2.

Table 3. Kim's non-mathematical considerations.

General Competence	Non-Mathematical Consideration	Frequency (Percent) of Comments during Lesson Planning		Frequency (Percent) of Comments during Lesson Enacting	
Attitude and practice	N1: Vision for the society of the future	14 (15.7%)	48 (53.9%)	16 (15.7%)	82 (69.5%)
	N2: Critical understanding of real phenomena	12 (13.5%)		15 (12.7%)	
	N3: Equity in learning	12 (13.5%)		22 (18.6%)	
	N4: Interest in learning and affect	10 (11.2%)		29 (24.6%)	
Communication	N5: Natural language and gesture	13 (14.6%)	33 (37.1%)	8 (6.8%)	23 (19.5%)
	N6: Understanding and respecting others' arguments	11 (12.4%)		7 (5.9%)	
	N7: Productive discussion	9 (10.1%)		8 (6.8%)	
Information processing	N8: Digital literacy	8 (9.0%)	8 (9.0%)	13 (11.0%)	13 (11.0%)
Sum		89 (100%)		118 (100%)	

Mathematical reasoning is another key competence presented in the 2015 Math Curriculum, but Kim thought the vision was inappropriately veiled in the curricular material, including in the textbooks. He looked for pedagogical strategies that effectively engage students in going beyond "adding up" and "dividing by" when they look at data using mean values. The rest of the teachers agreed on this perspective but could not find an alternative pedagogy. Kim moved the discussion focus from the concept of arithmetic mean to the data it indicates. He believed if he encourages students to appreciate the nature of the bias introduced by an outlier, then they can develop reasoning about data using mean values. His many discussions about this point were codified as constraints of mean value as a representative value (M4, Table 2), and it developed into reasoning about data types (M5, Table 2). M4 and M5 have conceptual aspects in their nature, but Kim discussed them in association with their reasoning aspects, so we categorized these two as mathematical reasoning.

One of the key suggestions presented in the 2015 Math Curriculum is that teachers should make use of technological tools such as calculators, computers, and educational software to help students perform complicated computations (p. 39). Kim found that this suggestion is only moderately accepted in the textbook. His comments on this suggestion have four different pedagogical points: using technology (M6, Table 2), graphic representation (M7, Table 2), comparing two data sets (M8, Table 2), and adding and subtracting values (M9, Table 2). An example of a comment coded as M6 is "Statistics without using technology is nonsense. All the tasks should allow the use of technology" (2 February 2017, community discussion). M7 is identified in such comments as when Kim explained how graphic representations can help students to see data locally and globally so that they develop sound concept images of data distribution. M8 is related to integration of one of the higher order thinking skills, making comparisons, and the learning content of representation values. The rest of the teachers worried this would make students' learning more difficult, but Kim believed comparing two data sets would help students' learning and retained this strategy. M9 concerns hypothetical thinking that may strengthen students' understanding of variability in data. The four points M6 to M9 were discussed with the common assumption that technology should provide students with a great deal of autonomy and efficiency in statistical processing.

The mathematical considerations underlying Kim's alternative pedagogical decisions were categorized into three foci: conceptual learning, mathematical reasoning, and statistical processing (Table 2). The frequency of the comments in each category during lesson planning and implementation differs. He focused more on enhancing conceptual learning during lesson planning (46.5%) but focused more on statistical processing during lesson implementation (52.3%). This is related to how he promoted students' potential and actual MF development while considering mathematical issues, which will be discussed in the next two sections.

Of the six key competences presented in the 2015 Math Curriculum (see Section 4.1), the competence of attitude and practice was a strong focus in the community discussions and interviews with Kim. This competence is explained as the ability to recognize the value of mathematics and practice while having autonomous attitudes related to mathematics learning and democratic citizenship [24] (p. 4). Unlike other teachers, Kim interpreted this competence as something that should and could be explicitly facilitated in daily lessons. Furthermore, he claimed that "citizenship for the society of the future should be facilitated through providing opportunities to experience data creation and handling. We all are animals who are constantly creating data, and the society of the future is just around the corner. As a teacher, I hope my students to be well prepared for the society of the future" (6 April 2017, interview). We codified this pedagogical point as vision for the society of the future (N1, Table 3).

He also emphasized critical understanding of real phenomena around us as one of the most important components of democratic citizenship (N2, Table 3). This consideration was connected with the ending parts of the tasks designed to encourage students to practice. Another crucial consideration was equity in learning (N3, Table 3). Kim sought out pedagogical strategies aimed particularly at engaging those low performers in mathematical inquiry and giving them equal opportunities to enjoy in his classes. He then created several different ways of providing learning opportunities to them such as building requisite basics through supplementary activities, communication norms to distribute roles, and motivations with interesting stories, activities, and easy access to technology. Kim particularly mentioned he wants to lead students to experience flow as Csikszentmihalyi elaborated: "As I said many times, the majority of my students are lethargic in the mathematics classroom. Seeing them at other times and in other places, it feels totally different. They look so energetic. I want to get them to have energy in the mathematics lessons by experiencing flow" (23 February 2017, community discussion). After this comment, Kim continued to talk about how to promote intrinsic interest in learning and students' negative affect toward learning, which was codified as interest in learning and affect (N4, Table 3).

Kim's second concern was about students' weak competence in communication. Kim was in favor of using natural language and gesture (N5, Table 3) whenever possible, which was often objected to by other teachers because of possible confusion and time constraints. The discussions between Kim's wide language use to make sense of representative values and data in general and the rest of the teachers' conservative perspective in favor of rigorous use of mathematical terms and representations made Kim's lesson plans more realistic and helpful to students than his initial version while still having the potential to promote students' diverse use of language and informal representations. Kim and the other teachers worried that their students lacked experience in understanding and respecting others' arguments (N6). In particular, they discussed the tendency in mathematics classes for students to just follow and respect the contributions of high-achieving students and to often ignore low achievers' arguments even when they come up with new ideas or solution methods. Therefore, Kim considered understanding and respecting other students' arguments one of the key norms for classroom communication. Regarding the point of productive discussion (N7, Table 3), Kim and other teachers had no conflicts. They all agreed that it is challenging to find efficient strategies to engage low-performing students in productive discussion due to their lack of the knowledge and understanding necessary to make contributions.

The last point that Kim continued to consider during lesson planning was the competence of *information processing*. In particular he used the term digital literacy (N8, Table 3) to refer to an ability to find and compose information on various digital platforms. He said, "Students use their smart phones very well. They have created their own language to efficiently transfer their emotions, knowledge, and designs using mobile devices. Many teachers think mobile devices are distracting students' attention. I don't think so. I would use and develop students' digital literacy in my teaching because I feel that it can facilitate learning and interest in mathematics" (23 February 2017, interview).

The frequency of the non-mathematical considerations Kim focused on to adaptively use the official curriculum is summarized in Table 3. The category of attitude and practice (N1–N4) was the most frequently discussed (53.9%) during lesson planning and was focused on even more during lesson implementation (69.5%), with the communication (N5–N7) category being the second most frequent during lesson planning (37.1%) and implementation (19.5%). This tendency explains how Kim planned and enacted lessons designed to promote students' potential and actual MF development by considering non-mathematical issues.

4.2. Kim's PF and Students' Potential MF Chances

We found that Kim's PF can be described by his PS with the nine mathematical and eight non-mathematical considerations that he focused on while making negotiations with the official curriculum and the community members (Tables 2 and 3). Kim's PS with these considerations covered more extended pedagogical possibilities than are embedded in the official curriculum, especially in the textbooks and teacher's guides. We explored how Kim's extension of the original pedagogical possibilities are potentially related to students' MF development using exemplary cases.

4.2.1. Students' MF Envisioned by Mathematical and Mathematical (M-M) Relations during Lesson Planning

Kim explicitly intended to promote conceptual learning (M1–M3), mathematical reasoning (M4–M5), and statistical processing (M6–M9) in the 11 lesson plans. In particular, the prompts he used (Table 1) were for divergent thinking, which encourages students to think of something new. These include alternative definitions of representative value, other ways of representing certain data, and different calculation strategies. For example, he designed the prompts for Lesson 4 as "What types of data can have mode? What does mode mean and what does it not mean in representing data?" He explained:

> I remember there were students who said "dog" is the mode because it was the most frequent reply to the survey about a beloved pet in the past year's teaching. It was a moment I should have let students ponder over in order to help them deeply understand the concept of mode. This time, I have intentionally included chances for students to think about data type so that they can go back and forth between their conception and the definition of mode. This will hopefully help develop their conceptual and relational understanding about mode as a representative value. I am sure students can distinguish nominal data from numerical data and further develop their understanding of representative values according to data type. (23 February 2017, interview)

This comment can be viewed as relation-based pedagogy that reflects aspects M1–M5 and that was built and continuously evolved through different versions. Kim himself goes back and forth between the mathematical understanding issues (M1) and reasoning aspects of data types (M5). He intended to give students opportunities to recognize and compare data types so that they could develop in-depth understanding of representative values. Students' MF in this case is related to changing their conceptions of data from the colloquial sense, which is a synonym for information, to numerical data. This kind of change in conception can promote students' learning, in particular those students who confuse the meaning of mode as a representative value described in Kim's comment above. By enhancing conceptual understanding with divergent thinking, Kim was able to increase learning opportunities for those students who had previously been excluded in classroom learning by giving them access to a meta-level discussion rule [46], that is the rule for applying the concept of mode as a representative where only numerical data types are to be considered. Based on the M1–M5 relations in variation, Kim created multiple choices for pedagogy that can enhance students' conceptual learning by making shifts in their understanding and conceptions related to representative values. This example shows how Kim's PF in curriculum use during lesson planning increased students' potential MF chances by extending mathematics-related pedagogy.

4.2.2. Students' MF Envisioned by Mathematical and Non-Mathematical (M-N) Relations during Lesson Planning

The M-N relations in Kim's PS were observed when Kim planned lessons while connecting the mathematics-related issues on representative values to his extensive sense-making of the key competences presented in the 2015 Math Curriculum. An exemplary case is the reflective association between multiplicity of representative values (M3) and a vision for the society of the future (N1). When reading the official curriculum, Kim noticed that multiplicity of representative values is undervalued in the textbook in that each representative value is presented one by one rather than being discussed together as a multiple model. Kim claimed that multiplicity of mathematical objects is a very complex idea but can and should be understood. He believed that the topic of representative values provides students valuable chances to think about this multiplicity of mathematical objects. He argued, "The most striking feature of the society of the future involves multiplicity or diversity in many things, I think. We should teach students to get away from one and only one idea by appreciating multiple models for finding representation values" (23 February 2017, community discussion).

Kim particularly criticized how textbooks present each representative value as if it can be independently explained out of the context, like a dictionary definition. For example, the textbook presents this key learning content for median:

(a) The median is a representative value for data with extremely large and small values;
(b) The median can be found by picking the middle one after arranging all values in the data from lowest to highest. If there is an even number of values in the data, then the mean of the two middle values is the median.

Kim argued that this kind of overly rigid explanation makes students just accept and follow factual knowledge so that they develop mechanical learning habits. Kim additionally claimed that students can easily see whether there is one middle value or not based on the number of values in the data if they have a chance to reason about it. Other teachers in the research community agreed with him. However, T1 still doubted whether students would be able to notice the necessity of the median being a representative value for data with extremely large and small values and arranging numbers from lowest to highest to find it. This teacher thought that data with an even number of values could confuse students when thinking about the median. T1 asked: "Students can construct a partial meaning of median, but do you believe students can construct the entire procedure to find the median by making the distinction between data sets that have an odd number and an even number of values? Eventually, isn't it teacher's job to explain anyway?" Kim responded:

> If students have chances to search for data around them to report about, if they can create their own data to reason about, if they can freely inquire into their data, and if they can choose any values that they think represent particular data, then they become curious about and are able to find representative values, and then they can construct the necessary procedures. (23 February 2017, community discussion)

The outline of Kim's lesson plans below shows how M-N relations (multiple considerations of mathematics- and non-mathematics-related aspects) underlie his pedagogical design:

- Lesson enactment period: from 9 March–23 November 2017 (11 lessons spread over 9 months).
- Learning model: Long-term project-based learning.
- Project theme: inquiry about the world using data.
- Learning objectives for each lesson: understanding different types of data, multiplicity of representative values, and effective reasoning about data (see the key prompts in Table 2).
- Key activities: searching for data, collecting data, representing and summarizing data, analyzing and reasoning about data, communicating using data, and thinking about the world, real phenomena, and the society of the future.
- Classroom organization: heterogeneously grouping 26 students into five groups of five or six members according to students' background information of academic, affective, and career aspirations.

By making and noticing M-N relations, Kim and other teachers were convinced that the lesson series envisioned some ideological aspects of the 2015 Math Curriculum such as attitude and practice in his particular lessons to teach representative values. More importantly, Kim's students could have opportunities to create their own ideas and strategies for searching, collecting, summarizing, analyzing, and reporting diverse data. These plans offer students various kinds of substantial MF chances, which are not available in the textbook.

4.2.3. Students' MF Envisioned by Non-Mathematical and Non-Mathematical (N-N) Relations during Lesson Planning

By considering N-N relations, Kim pursued dynamic networking among general values such as future orientation, egalitarianism, and digital literacy in his lesson planning using adaptive curriculum use. There were two types of N-N relations: *within* and *between*. Within non-mathematical considerations such as equity issues (N3), Kim moved his focus from one to another by highlighting various aspects of possible learning disparities. He planned to start each lesson with a 1-min reporting from each group with the rule that all the group members take turns. In this way, Kim thought he could reduce one or two high achieving students dominating group activities. For those students with very low level of achievement and interest in mathematics who used to be excluded in classroom activities, Kim decided to use software called Easy Statistics, which was developed for school statistics education in Korea, so that those students without computation fluency could also handle data by trial and error. In the lesson plans, Kim presented detailed roles and anticipations about students' reactions when integrating the software. Kim particularly planned presentation sequences and content that give more low-achieving students successful experiences by using technology. Kim said, "Boys are better at using technology but not at giving presentations in front of the class and girls are the opposite. The most challenging for me is to build optimal learning opportunities for all students, whether they are high or low achieving, boys or girls, or like or dislike mathematics, and, no matter what their career aspirations are, to develop data literacy" (23 February 2017, interview). We recognized Kim's open and flexible dealing with equity issues (N3) as making N3–N3 relations that produced many chances for those students who were previously excluded. It allowed them to participate in small-group activities and whole-class discussions. This can increase opportunities to develop MF for not only a few but also the majority of students by opening up many kinds of construction chances.

N-N relations between two different non-mathematical considerations were observed in the lesson plans as well. In particular, the first non-mathematical consideration involving vision for the society of the future (N1) was frequently related to the other issues. Without connecting to other issues, N1 could be regarded as a kind of rhetorical rather than practical pedagogy. Kim related N1 to critical understanding of reality (N2) and the other way around. He said, "To survive in the society of the future, our students need to develop a critical eye for real phenomena. Otherwise they may easily get lost and be deceived by veiled hostility. Critical reasoning should be taught in mathematics classes. Learning how to make decisions using critical reasoning can help them become leaders in the society of the future" (2 February 2017, interview). After discussing the society of the future (N1), Kim also focused on equity issues (N3) and then moved his attention back to it: "We are running out of human resources because of the very low birth rate in Korea. Our population is declining rapidly. Each and every student is so important for our future. The society of the future will flourish if each and every student is educated equally well" (23 March 2017, community meeting). Likewise, with rich arguments, Kim planned lessons while moving focus back and forth between the eight non-mathematical considerations. This effort empowered his lesson plans by increasing possibilities for engaging students in divergent thinking about data and small-group activities and whole-class discussions. In this way, the majority of students' MF chances were potentially included in his lesson plans, in which we could identify where many actual MF cases in classroom teaching had their origins.

4.3. Kim's PF and Students' Actual MF Chances in Classroom Teaching

As described in the previous section, Kim planned the series of lessons for teaching representation values that spread out over 9 months. His students were very passive in the first lesson that was enacted on 9 March, not only because they were not ready for such divergent thinking about data and representative values but also because it was just the second week of new school year. From the second lesson, however, students gradually became more active in carrying out their group projects. We identified reflective associations within and between mathematical and non-mathematical considerations as planned and unplanned.

4.3.1. Actual MF Chances Resulting from M-M Relations in Classroom Teaching

Two types of M-M relation in classroom teaching occurred: as planned and as instantaneous pedagogy to unforeseen events. The former is what Kim intended to teach as a thought scheme that was discussed during lesson planning. It can be viewed as thoughtfully adaptive teaching in the sense that the teacher knew possible variations in the relations and made optimal choices among them. The latter is what Kim did to adaptively use his plans when facing unexpected occurrences. Not all M-M relations, either recognized in the lesson planning or classroom teaching, effectively supported students' MF chances. Management issues including time constraints (each lesson lasted 45 min) and difficulty in scaffolding students' divergent and convergent thinking were the main reasons. Nonetheless, Kim's PF with the M-M relations were useful in helping him notice students' particular behaviors that could be potentially related to the concepts and the procedures so that Kim was able to make timely decisions to support them. For example, Kim designed a task to compare two data sets (M8), which was not explicitly included in the textbook, by asking students to reason about the limits of mean value as a representative value (M4). In the lesson planning stage, he switched his focus between M8 and M4 flexibly, which also happened in the classroom teaching according to the interviews with him after the lessons. The two problems involving comparing two data sets are as follows:

The Adventure Problem

Min played a computer game called Adventure and found that the main character Hu is not able to swim. There are two ponds: Pond A and Pond B. The means of the measured depths found by selecting 10 points in each of the two ponds are the same: 58 cm. Min observed that Hu sometimes drowns while trying to go through Pond A but is able to successfully go through Pond B. Make your conjectures on the depth data of the two ponds and explain why.

The SNS Problem

Seong enjoys taking photos with funny poses and Jin likes taking photos of cats. Both posted their photos on a popular SNS. The means of the number of likes they received for all of the 10 photos they each posted over the past week were both 7. After seeing the data, Seong decided not to post any more photos and Jin decided to keep posting. Make your conjectures on the number of likes they received and explain why.

Because the contexts for the problems are closely related to students' daily lives, they successfully engaged most students in the initial inquiry process. While solving the problems, students derived the meaning and limits of mean value as a representative value and realized the necessity of alternative values for representing data. More importantly, we identified the moments when many students changed their view on data from seeing a set of individual measures (points view) to seeing a distribution (global view). The following is part of the related discussion in Group 4 during Lesson 3:

3L-4-004 S19: Teacher, is this problem asking to find the depth of the pond?
3L-4-005 Kim: Um ... I am not sure; you may further think about it.
3L-4-006 S18: Mean values for both ponds are 58 cm.
3L-4-007 Kim: Right.
3L-4-008 S19: Both mean values for the depth are 58 cm.
3L-4-009 Kim: Right.
3L-4-010 S20: But still one dies in one pond and doesn't in the other? Is there a monster living in the other pond? [Speaking to himself in a very small voice]
3L-4-011 S19: How about completing this table to get 58 for the mean value.

[Using the Easy Statistics software, Student S19 put 10 values in the table and changed them so that 58 was the mean value. She checked whether the resulting mean value was greater or less than 58 after adding two or three values. She concentrated on getting 58 for the mean value without solving the task.]

3L-4-012 S18: I got it! Only one deep place is enough to die. One outlier datum, say, one peak. Whatever the other measures are, say, 1, 1, 1... if we have one high value, then we have 58 for the mean value.
3L-4-013 S19: It is cool! Flat bottom, say, 1, 1, 1 and one peak.
3L-4-014 S18: Good idea. [Drawing a flat line and then a high point by hand in the air] Now I got it.
3L-4-015 S20: I see. Uh... In case we have one very deep place then Hu drowns.
3L-4-016 S19: One big value makes the trend of data different.
3L-4-017 S18: Yes, that's a good word, the trend.

S18 and S19 were very low achieving and not good at computation. Interacting with the software, however, both enjoyed learning and experienced many "aha" moments (see, for example, 3L-4-012 and 3L-4-013). S18 created an extreme case (1, 1, 1, and one high value) to solve the task, which is a generic example that could lead him to the idea of an outlier and its relationship with the distribution of data and mean value. This switch in perspective is the evidence of his MF. His peers understood S18's reasoning with hypothetical data right away so they did not raise any questions about whether or not places in the pond could realistically be 1 cm in depth. They learned to assume and reason mathematically when necessary rather than limit their thought to the physical conditions of the situation. Kim excitedly said, "Although I planned lessons with the task expecting such novel thinking, I didn't predict S18 could do it without any help from myself and his peers. I and S18, we both were literally and figuratively surprised. It was one of the most touching moments in my entire teaching career" (20 April 2017, interview). Without Kim's sentimental reflections, we would not recognize the moment was that surprising and meaningful, because while Kim's anticipation of students' learning included such reactions, he had predicted such reactions from high or middle performers.

The main contributor to students' MF described above was Kim's lesson design, which was largely supported by his PF involving comparing two data sets (M8) and mean value as a representative value (M4). In this way, Kim's PF in lesson planning could be partially mobilized to go along with students' thinking and become part of their MF. It is interesting to note that the students in Group

4 became excited and began referring to the example as the "1, 1, 1 method." This excitement was quickly transmitted to the other groups and further developed. Solving the second problem involving the number of likes in social media, the rest of the students quickly developed the "0, 0, 0 strategy," meaning three posts with no likes. In Group 5, one student suggested the "minus strategy" to use if they have negative values in the data. Their focus moved from the relationship between mean value and outlier to variations in the measures of data. By means of these shifts in their thinking, students were also able to realize that mean value has certain limitations as a representative value (M4), as they thought about the existence of outliers (3L-4-016, 3L-4-017) by comparing two data sets (M8) in a very creative way (the 1, 1, 1 method). This example shows Kim's PF involving M8-M4 relations in the lesson planning stage being successfully realized to promote students' MF in classroom teaching.

In unforeseen situations, the within M relations, in other words, M*i*-M*i* forms of thinking, were very helpful, based on what Kim said in the community meetings and the pre- and post-lesson interviews. Kim was able to rely on the multiple possibilities within a particular consideration to find optimal reaction strategies. For example, regarding the issue of type of data (M5), Kim designed lessons to encourage students to collect example data in their daily lives and inductively reason about what type of data they were. The students in Group 1 jumped in to make several conjectures about their data types using trials of differentiating data that came to their minds at that moment. Kim stood beside them and observed what was happening rather than stopping them from coming back to the original pathway of collecting information on diverse data. S3 started by differentiating numerical data from non-numerical by mentioning the number of filmgoers and data about favorite movies. S4 found this way of differentiating data interesting and continued by differentiating data with a mean value from data without a mean value. Then he said, "It is possible to distinguish data that cannot be sufficiently explained using mean value from data that can be described well using mean value." Other students also enjoyed classifying data in diverse ways. They joyfully threw ideas to each other like playing a game. Unlike Group 1, in other groups, all members began seriously searching and classifying data on a variety of websites (e.g., sports statistics, online game popularity rankings, and number of fans of idol groups) to make conclusions about data types. If Kim had directed Group 1 students to come back to his way of solving the task through web searching, they would not have gone on such interesting adventures in differentiating data using various standards. When asked why he did not steer Group 1 students back to the planned path, which is more inductive in the sense that it encourages students to search for various data and think about data types, Kim said, "Group 1's strategy is like hypothetical conjecturing in that they first took certain criteria then judged data types according to them. That was a chance for them to look at the data with a new purpose, to recognize any kinds of types. I didn't want them to lose that chance since it would be a more important and better approach to the task than just blindly searching for data. I actually worried that some students get lost in the ocean of data and become distracted by the story each data set delivers. Another benefit was that they could gradually improve their conjecturing to fit with my original purpose. With ownership, they paid attention to comparing, categorizing, and analyzing data" (20 April 2017, interview). Evidence of students' MF was observed at different moments and in diverse forms. As we can see from Kim's reflection on the lesson in the interview, he was able to notice the crucial moments and diverse forms with his reflective associations within M5, in other words, M5-M5 relations.

Lesson 4 aimed at teaching the meaning, definitions, and calculation methods of median and mode through inquiry using data from daily contexts. Kim retained the complex relations in the content as he introduced it rather than using a one-by-one approach. After learning the meaning and definitions of median and mode in daily-life situations, S2 presented his insight that people in different age groups hope to travel to different regions. According to the information he obtained from travel statistics, about 40% of people in their 20s preferred travelling to Europe. S2 asked a question: "This means that 40 out of 100 people gave 'Europe' as their answer. Then, what is the mode? Europe? Forty percent?" This reaction was exactly the same as the one Kim reported receiving from his students when teaching this content in the previous year. This time, Kim did not have to create any additional pedagogy, since the hypothetical creation of data for number of likes on SNS covered in Lesson 3 was helpful in promoting S2's swift thinking. S2 said, "1, 1, 1 strategy," then described SNS contexts using the numerical data consisting of 1, 1, 1, 1, and 5. S2 said that "1" is the mode rather than "like." He argued even further that "80% of the data points are 1, since four of the five are 1, but the mode is still 1 instead of 80%. Many numbers are in the data, so we need to carefully decide which numbers matter" (4 May 2017, Lesson 4).

Students' MF experiences described above can be viewed as one evidence of their meta-level learning [46], in that they adopted the rules that govern discourse on how to conceptualize data and representative values rather than just accepting the concept definitions and procedures related to representative values. The various relations and shifts in foci between the meaning of the representative value, the characteristics of the data, and data types are viewed as students' MF development. This case is an unforeseen event where Kim was able to wait optimistically for students to return to their own learning paths. It was helpful that Kim's extended PS with multiple considerations included his expectation of multiple possibilities in students' and his reactions to the planned and unplanned situations; as a result, he was able to maintain his optimism and effectiveness in promoting students' MF.

4.3.2. Actual MF Chances Resulting from M-N Relations in Classroom Teaching

The M-N relations in the classroom teaching we identified in Kim's comments in the community discussions and the interviews were relatively less frequent than M-M and N-N relations. Kim explained that this was because his pedagogical decisions during classroom teaching should be quick and directly connected to either mathematics- or non-mathematics-related considerations. He said, "Sometimes I unconsciously make a choice among variations in those considerations, but many cases turned out to be something I had already thought of. Thus, it might be an M-N relation deep in my mind, but from the outside it appears to be an M-M, N-N, or even an M or N consideration" (18 January 2018, interview).

The cases of M-N relations found during Kim's discussions about his lesson implementation were mainly between the statistical processing issues (M6–M9) and eight non-mathematical considerations (N1–N8). Kim's pedagogy had a basic emphasis on divergent thinking, but the rest of the teachers used it only moderately. During planning and implementing the 11 lessons, Kim put a great deal of emphasis on finding new ideas and broader meanings, while the rest of the teachers worried about the trade-offs involved in divergent thinking, such as possible loss of clear definitions and procedural knowledge. These worries were maximized when discussing Lesson 7, where students analyzed data they had collected from 30 friends during summer vacation. Many students still tended to wait to find out what Kim wanted them to do in terms of data analysis and reporting. Kim still decided not to give explicit procedures, asked students to think divergently using negotiation, and invited the students who were in charge of designing their posters in each group to come to the front and discuss an effective reporting format together. Kim said, "As S16 mentioned, it may not be a good idea to present all the details. I support S4's suggestion to use an appropriate amount of graphic representation. You can examine other reports on the internet and get hints for your presentation strategies." Kim assigned the role of data surveyor to the students with relatively low grades in each group in order to engage them in putting data into the software and looking at the various representations they obtained by trial and error using the computer. These students were then responsible for choosing the representation strategies. This approach was considered very innovative by the other teachers in the community discussion but sometimes slowed down all the small-group activities since the other students needed to wait for the results to come out on the screen. However, as an approach to the later lessons, creative ideas were made by collaborative work among students. One such example discussion of Group 2 is as follows:

7L-2-089 S6: I want to graph this. What graph can we make?

7L-2-090 S7: Bar graph, line graph, and so on.

[All silently think about graphs.]

7L-2-091 S7: And what?

7L-2-092 S8: [Drawing a horizontal axis in the air with his finger] What should we put on the horizontal axis?

7L-2-093 S7: The number of likes, 2, 5, 8 . . .

7L-2-094 S6: Ok. Let's put 2 here.

7L-2-095 S7: What comes out when you draw like that? What does that mean? Let's draw first and think later. [Puts the data into the software and produces the graph in Figure 3.]

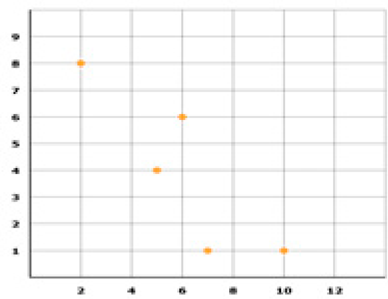

Figure 3. Frequency of number of likes.

Figure 3 shows a graph with data where the frequency is eight 2s, four 5s, six 6s, one 7, and one 10. Students argued that this graph has the advantage of being able to immediately confirm what the mode is (the number with frequency eight is 2). However, to obtain the median or mode, students have no choice but to list or calculate the data, and therefore, they thought that this graph was not important.

7L-2-145 S7: (reading the graph) 2, 2, 2, 2, 2, 2, 2, 2, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 7, 10.

7L-2-146 S8: 8 + 4 + 6 + 1 + 1. So, how much? The median is between 5 and 5, so 5! 5!

Unlike S7, S8 did not list the values. After determining the place of the median by comparing and matching both ends of the frequency of each value ("6 + 1 + 1 is 8"), S8 considered the median to be 5, which is the middle of 5 and 5. The students confirmed by calculating that the mean was 6. After modifying the data, S9 drew Figure 4 and said, "In this case, the mean, median, and mode are all 6." Other students, who immediately noticed how S9 transformed the data, also showed great interest in and attention to the relationship between the data and the graph and the relationship between mean, median, and mode. The atmosphere of Group 2 was as if they were at a festival, and even after the class, the members had conversations through SNS until late night to continue their discussion on the relationships among the three representative values. They found various data where the median is less than the mean, the mode is greater than the mean, and the mean is greater than the mode.

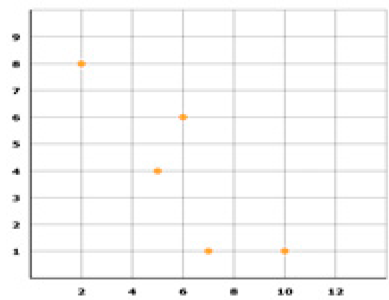

Figure 4. The graph by S9.

Group 2's interesting journey into graphic representations of data and possibilities of the relationships between the three representative values revealed how their MF was facilitated and used to create meaningful learning chances. Although these chances were not explicitly prepared by Kim during lesson planning, we could see a link between Kim's PF with M7, which related diverse and sophisticated graphic representations in reporting their survey results, and students' MF, which related to creative use of representation tools. Other teachers were quite surprised at Group 2's inquiry experiences. The teachers in the community meetings who worried about students' learning possibilities were impressed by this story and they highly valued students' persistence in inquiry regardless of their productions. T2 said, "Brilliant. This is an in-depth exploration in data distribution! They noticed the concept of skewness of data. It is amazing" (21 September 2017, community discussion). However, there was a controversy over how far a teacher should allow students to explore when they go beyond the curricular materials. Kim said about this problem:

> I think it would be better to open the ceiling. If we ban students from thinking about something new and unusual and going beyond what we know, they will lose interest in learning. Rather, I think that students can develop motivation to learn mathematics when freely crossing boundaries. Visual and gestural representations are really good tools to go beyond the already known. (21 September 2017, community discussion)

The interest and attention (N4) and consideration about representation (M7) issues interplayed well in Kim's discussion as described above. He talked about N4 relating to M7 and the other way around. This kind of two-way relationship of M7 and N4 was again mobilized in the form of shifts between students' representation activities and excitement in learning.

4.3.3. Actual MF Chances Resulting from N-N Relations in the Classroom Teaching

The eight non-mathematical considerations are derived from Kim's flexible interpretation of the 2015 Math Curriculum in the direction of building good learning practices and environments, as discussed earlier. The more Kim discussed specific issues, the more changes he made to the curricular material or his plan. The more changes he made according to in-the-moment decisions, the more PF cases that supported more students' learning were observed. Actually, during lesson planning, Kim commented on vision for the society of the future (N1) most frequently (15.7%) and changed the curricular materials based on vision for the society of the future accordingly. Moreover, Kim made in-the-moment decisions on interest in learning and affect (N4), which was most frequently discussed (24.6%) when talking about lesson implementation. Furthermore, this changed frequency implies that Kim was more concerned about students' interest and affect during lesson implementation than planning.

Non-mathematics-related considerations appear to be secondary and classroom managing issues, but they played significant roles in promoting change in students' mathematical conceptions and representations by supporting students' motivation and competences for learning mathematics. According to Kim, his students were rather passive in learning and rarely asked inquiry questions. Nonetheless, he made a great deal of effort to involve students in critical understanding of real phenomena (N2). The key prompts for Lesson 6 that reflect his intent to promote students' critical understanding of real phenomena were, "Why are people not satisfied with the government report that the salary mean has been increased? How can we critically read newspaper articles that include representative value information?" As he had thought, his students waited for Kim to give directions to tackle the prompts. Presenting newspaper clips, Kim asked students to think about the reason why it is important to critically understand government reports rather than just accept them. He walked around the classroom to support each group to have diverse conversations on the problem context, which helped students to be ready to tackle the main prompt of evaluating the government report on the salary increase. His comments on each group's diverse conversation revealed he had variations in pedagogy for dealing with N2 issues according to students' diverse understandings of the context.

His intent for the activity had not been the pursuit of any fixed goals. Rather, he wanted students to be developing critical minds at their own speeds and abilities. Kim then asked students to move on to making justifications of their argumentations on the report. Kim's decision to engage students in discussing the problem context before tackling it (N2) may look like a detour, but it was significant in developing students' understanding of the fact that representative values are used in real phenomena by changing their perspectives in reading the government reports, applying representative values, and making their own decisions. This case shows how Kim's various considerations of N2 issues promoted many aspects of students' MF experiences.

We also observed PF with N-N relations between different non-mathematics-related pedagogy. The representative case is the relation between vision for the society of the future (N1) and digital literacy (N8). As earlier described, Kim assigned the lowest performer in every group to play a role of data surveyor, asking them to put data in the computer and try different representation tools to give an effective summary of the data. Kim moved his attention back and forth between N1 and N8 and said that even students who fail to achieve specific learning goals should develop the ability to read and represent data using digital tools for their future lives. Kim mentioned the reason for this:

> In the future, digital tools will be more important than now. More students need to be able to fluently use them; however, they are currently not given much experience related to these in mathematics classes. There are many students who do not like calculation but are interest in using digital tools. They may come back to learning mathematics if they succeed in doing something in mathematics lessons. Actually, I realized that most students are better than I am in using new digital instruments. I flatter them and ask them to help me teach how to use them, which makes students feel proud of themselves and holds their interest in mathematics. This is how I, not fully understanding the society of the future and lacking digital skills, can still educate students. (15 June 2017, interview)

Kim not only emphasized digital literacy as skills in using tools but also related it to various aspects of the society of the future. Moreover, rather than highlighting only the complexity and unpredictability of the society of the future, Kim created pedagogy to prepare for it by developing digital literacy. Students' MF experiences were parallels of these shifts in perspective. The data surveyors in each group made quite a significant contribution to the small-group activities by handling data and having positive attitudes towards using the software. In this way, Kim's N-N relations supported many students to immerse themselves in learning by contributing new ideas (for instance S18's "1, 1, 1 strategy" described in Section 4.3.1), creative representation methods (S7's graphic expression in lesson 7 discussed in Section 4.3.2), and diverse interpretations of representation values in various contexts.

5. Discussion and Concluding Remarks

The present study explored teachers' PF in curriculum use and its impact on students' MF experiences. A few previous studies have highlighted the significance of PF using concepts similar to PF such as interpretative flexibility [15], teacher flexibility [16], in-the-moment decision making [34], and contingent teaching [34]. To make it more operational, we developed a conceptual framework for PF based on Krutetskii's reversibility framework [27]. A core idea of this framework is that a teacher's flexible decisions are guided by relations within and between considerations in PS. If a pedagogical consideration is isolated in space, it easily loses its meanings and values and falls into a plausible point that lacks specific practical orientation. To describe meaningful PF in curriculum use, we characterized pedagogical consideration as mathematical or non-mathematical and classified their two-way relations into M-M, M-N, and N-N. This framework aligns with participatory views on teachers' curriculum use [11,12,32,33], but adds on descriptions of reflective and associative aspects of pedagogical reasoning that are the essence of flexible thinking [27]. Our research findings illustrate that enlargement and reconstruction of PS, which enables PF, can promote students' MF experiences by relating two different mathematical ideas, sustaining inquiry with their own questions, and engaging in meta-level learning. Below, we discuss specific results of the analysis and situate our framework in the literature.

Analysis of Kim's lesson planning and implementation showed that his enlarged PS created opportunities for students' MF experiences. Extension of PS involves the effects of overcoming close adherence to an official curriculum or lesson plan, which is a significant indication of flexibility [14]. Kim was able to equip himself with multiple pedagogical options and follow anticipation as he exceeded PS embedded in the official curriculum. Also, multiple options invited students to develop creative representation of data and to reach meta-level discourse in the classrooms, at their own levels of interest and proficiency. Even low-achieving and low-interest students were able to experience MF with personalized and meta-level learning. This result indicates that an indicator of Kim's high PF seemed to be the size of his PS. Brown [11] proposed a design metaphor that emphasized that teachers can adaptively create their own instruction rather than offload curriculum materials. Our spatial metaphor in the concept of PS adds more description: that the design work involves creation of pedagogical options that can be extended or confined in a fixed way. Schoenfeld [34] similarly described teachers' in-the-moment decisions as actually being the selection among several pedagogical options according to their resources, orientations, and goals. In his study, more interest was offered to teachers' prioritization among different choices, which often push teachers into conflicting situations. On the other hand, the concepts of PS and PF shed more light on illustrating how the multiple choices are constructed and actualized according to different associations among pedagogical considerations.

As mentioned above, size of PS does not solely determine PF. Associations of its elements are no less substantial. We classified associations into M-M, M-N, and N-N before analysis, and our analysis additionally revealed that Kim's flexible relations were established either within or between pedagogical considerations. Kim's M-M relation within a consideration (M*i*-M*i* type) helped him to find optimal pedagogical decisions among diverse choices in unexpected teaching situations. In this vein, M*i*-M*i* relations are substantial for contingent teaching, which involves readiness to respond and deviate in a timely manner from a prepared agenda [35,47]. The significance of M*i*-M*i* relations also aligns with Foster's [35] suggestion that "we should not underestimate the importance of preparing for the unexpected and helping the teacher to draw on what they already know" (p. 1085). Kim's M-M relation between two considerations (M*i*-M*j* type) presented another role of PF. We found that Kim's M*i*-M*j* relations were reflected in task design, and these tasks facilitated his students' conceptual learning about representative values as they relate to several aspects, such as data type, definition, and their representation. This result implies that the M*i*-M*j* relation of teachers can amplify students' conceptual understanding through flexible thinking. Although pedagogical affordances of PF have rarely been observed in the literature except for problem-solving contexts (e.g., [16]), this study offers evidence that activation of PF can develop flexible understanding of a mathematical concept.

Kim's N-N relation took the role of supporting students' motivation and sustaining their flexible thinking. A notable result was that Kim's N-N relation was frequently associated with his considerations about the society of the future (N1). Kim's curriculum use and noticing was not limited to interpreting curricular materials and expecting and understanding individual students, as has often been suggested in prior studies [11,12,32]. This result expands our understanding of curriculum use to include bridging between past and future and between school and society. These comprehensive noticing and relating of non-mathematical issues are another indicator of his activation of PF.

The M-N relation was less common than other types of relations, even though it was also crucial for promoting MF. Kim moved back and forth between mathematical (M7) and non-mathematical points (N4) interchangeably and managed a long learning trajectory of a group of students. Interestingly, some teachers in the lesson study group undervalued Kim's M-N relation in that it tends to ignore pre-established learning goals in the official curriculum. This controversy parallels well-documented didactic tensions between developing students' personal mathematical meaning and fulfilment of lesson goals [48,49]. Although pursuing students' personal mathematical meaning might engender responses that the mathematics community might find inappropriate, Kim's lessons indicate that the accumulation of students' own meanings can eventually be connected to culturally valuable mathematical knowledge. In that vein, M-N relations are pedagogical thinking that is crucial to sustaining students' own sense-making. Students' meaning can develop as they flexibly represent and negotiate their own ideas with the help of PF.

Another interesting finding was the discrepancy between the degrees of Kim's attention to pedagogical points in lesson planning and implementation. Specifically, considerations of statistical processing and attitude and practice were far more distinct in lesson implementation than in planning. This results are similar to Roth McDuffie et al. [32] finding of inconsistencies between teachers' awareness of intended and enacted curriculum, where teachers' noticing sometimes became more distant from their orientations, particularly when they encountered students' cognitive challenges. Our analysis complements different aspects of changes in teachers' noticing by incorporating students' affect. In Kim's lesson implementation, the issue of students' interest and affect became more important, as Kim observed his students' passive attitude and timidity against the divergent thinking tasks that Kim had developed. Hence, Kim's PF in lesson implementation was activated in order to deal with considerations of students' attitudes and practice, and these considerations were often connected with considerations of equity (non-mathematical) and statistical processing (mathematical). Eventually, Kim's PF in lesson implementation was effective in stimulating students' MF at the levels of their own interest and statistical proficiency. This case suggests that enacting a lesson aimed at providing MF experience demands teachers to exert high PF that can incorporate not only mathematical but also non-mathematical considerations.

Our PF framework has some similarities to and differences from frameworks in prior research. Similar to our framework, the concepts of interpretative flexibility, [15] teacher flexibility [16], responsive and contingent teaching [35] all concern the adaptive capability of teachers in their goal-directed activities. However, unlike our study, all these concepts have mostly focused on describing teachers' instructional flexibility in classroom teaching rather than its relation to PF exerted in the lesson planning stage. In particular, the concept of teacher flexibility has been used to describe how a teacher flexibly deals with unexpected situations in the classroom. On the other hand, using the PF framework, we found that PF in classroom teaching can be viewed not only as instantaneous pedagogy but also as planned pedagogy. Some pedagogical decisions in Kim's classroom teaching involved following his lesson plan, but were flexible enough in that Kim's consideration exceeded the PS embedded in the official curriculum materials. Another distinct characteristic of our framework is its attention to flexible thinking. Leikin and Dinur's [16] framework of teacher flexibility is much more focused on product (actualized learning trajectory) than process (e.g., thinking and attitude). It seems natural that they described patterns of teacher flexibility according to students' flexible outcomes in problem-solving activity. Since our description of flexibility is based on the concept of reversibility from Krutetskii [27], our framework has strength in characterizing teachers' associative and relational thinking. Furthermore, the focus of thinking makes it easier to observe relations between teachers' and students' flexible thinking.

In summary, application of Krutetskii's reversibility framework to teachers' curriculum use gained more insight into their flexible thinking behind adaptive curriculum use [11,12,36]. Moreover, the concept of PS and classification of relations between and within pedagogical considerations make it operational to observe teachers' PF and its impact on students' MF, which complement prior frameworks related to PF [10,15,16,34,35]. However, our study can still be complemented by future studies. Future studies are likely to investigate practical potentials of our framework for teacher education. For example, how teachers who intend to develop students' MF can use this framework as a reflection tool can be explored. Furthermore, future work can replicate our study in other pedagogical contexts, such as mathematical modeling activities, in order to expand our understanding of PF and its relations with MF.

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