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Characterizations, Potential, and an Implementation of the Shapley-Solidarity Value

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Abstract: In this paper, we provide cooperative and non-cooperative interpretations of the Shapley–Solidarity value for cooperative games with coalition structure. Firstly, we present two new characterizations of this value based on intracoalitional quasi-balanced contributions property. Secondly, we study a potential function of the Shapley–Solidarity value. Finally, we propose a new bidding mechanism for the Solidarity value and then apply the result to the Shapley–Solidarity value.

Keywords: Shapley-Solidarity value; coalition structure; potential; bidding mechanism

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1. Introduction

In economic situations, players usually form a coalition with the aim of obtaining more profits. It is an important issue to distribute the surplus of cooperation among the players. Game theory provides general mathematical techniques to analyze such distribution problems. The solution part of cooperative game theory deals with the allocation problem of how to divide the overall earnings among the players in the game. There are various solution concepts in the field of cooperative games.

Probably, the Shapley value [1] and the Solidarity value [2] are the two most well-known solutions in cooperative game theory. They have similar formula, but a different way to value players' contributions to the coalitions. The Shapley value is a marginalist value because it considers the pure marginal contribution of every player, while the Solidarity value behaves more equalitarian than the Shapley value, since it considers the average marginal contribution of every player.

Various authors have proposed many characterizations of these two solutions. Shapley [1] initially proposed an axiomatization of the Shapley value by means of efficiency, additivity, symmetry and the null player property. Among the latter ones, Young's axiomatization [3] stands out by the elegance of its marginality axiom, which replaces additivity and the null player property in the initial characterization of the Shapley value. Myerson [4] proposed an axiom of the balanced contributions property in order to characterize the Shapley value. van den Brink [5] interpreted the Shapley value as the unique solution satisfying fairness, efficiency, and the null player property. Mcquillin and Sugden [6] characterized the Shapley value while using a condition of 'undominated merge-externalities'. Calleja and Llerena [7] provided an axiomatization of the Shapley value by weak self consistency, weak fairness, and the dummy player property. Recently, Alonso-Meijide et al. [8] introduced new properties by considering three special kinds of agents: null, nullifying, and necessary agents, and then obtained new characterizations of the Shapley value. van den Brink et al. [9] discussed two solutions for cooperative transferable utility games, the Shapley value and the proper Shapley value and characterized them by the property of affine invariance.

Moreover, Yokote et al. [10] introduced the balanced contributions property for equal contributors to the γ-egalitarian Shapley values which include many variants of the Shapley value, such as the egalitarian Shapley values and the discounted Shapley values. Casajus [11] proposed the relaxations of symmetry, weak sign symmetry, and characterized the class of the weighted Shapley values, together with efficiency, additivity, and the null player property. Abe and Nakada [12] proposed a new class of allocation rules for TU-games, named the weighted-egalitarian Shapley values, where each rule in this class takes into account each player's contributions and heterogeneity among players in order to determine each player's allocation. They provided an axiomatic foundation for the rules. Choudhury et al. [13] defined the generalized egalitarian Shapley value that gives the planner more flexibility in choosing the level of marginality based on the coalition size, and provided two characterizations of the generalized egalitarian Shapley value. An overview of study that was associated with the Shapley value can be found in literature [14].

Nowak and Radzik [2] characterized the Solidarity value by introducing the A-null player property instead of the null player property in the axiomatization of the Shaplay value [1]. Casajus [15] proposed a differential version of Young's marginality axiom. Differential marginality, together with efficiency and the (A-)null player property, allows for a direct proof of additivity, entailing characterizations of the Shapley value and the Solidarity value. Gutiérrez-López [16] defined the class of all egalitarian solidarity values, which are convex combinations of the solidarity value and the equal division solution, then provided two alternative characterizations of the corresponding values. There are also more various axiomatizations of the Shapley value and the Solidarity value in literature [17–19].

The coalition structure is used to model the situation where the players form groups for bargaining payoffs in cooperative games. The Owen value [20] is a well-known value for games with coalition structure. The Owen value adopts the Shapley value both inside the unions and among the unions. It behaves as a pure marginalist solution even inside the union, where the player's behaviour should be united. Alonso-Meijide et al. [21] extended the equal division and the equal surplus division values to the more general setup of cooperative games with a coalition structure, and provided axiomatic characterizations of the values. Hu [22] studied the weighted Shapley-egalitarian value and the collective value, and parallel axiomatizations of them were proposed by replacing the collective balanced contributions axiom with two intuitive axioms. Zou et al. [23] defined an extended Shapley value for generalized cooperative games under precedence constraints and provided two axiomatic characterizations of this value.

Calvo and Gutiérrez [24] combined the Shapley value and the Solidarity value together and defined a value named the Shapley–Solidarity value for games with coalition structure. They thought that the players within a union were more willing to show their solidarity and each union was more inclined to protect its revenue. Therefore, the Solidarity value is applied in order to obtain the payoffs of the players inside each union while the Shapley value is applied in order to compute the total payoffs of each union.

Calvo and Gutiérrez [24] characterized the Shapley-Solidarity value using efficiency, coalitional balanced contributions and intracoalitional equal averaged gains. Intracoalitional equal averaged gains is a coalitional version of the equal averaged gains property, which is used to characterize the Solidarity value [24]. Recently, Hu and Li [25] gave another characterization of the Shapley-Solidarity value by five axioms, including efficiency, additivity, intracoalitional symmetry, coalitional symmetry, and the partial A-null player property. The partial A-null player property states that a partial A-null player should receive zero in the game. The partial A-null player property is an extension of the A-null player property [2].

The intracoalitional balanced contributions property was introduced by Lorenzo–Freire [26,27] to characterize the Owen value and the Banzhaf-Owen value. In this paper, we propose a similar axiom, the intracoalitional quasi-balanced contributions property, in order to characterize the Shapley-Solidarity value.

Hart and Mas-Colell [28] introduced a way of potential function to characterize the Shapley value. They defined a potential function assigning to every cooperative game a real number and proved that the marginal contribution vector of the function coincides with the Shapley value. Winter [29] extended this way of characterization to games with coalition structure where the Owen value [20], the AD value [30] were considered. Xu et al. [31] adjusted the original potential function for the Shapley value, so that they got a potential function for the Solidarity value. For the Shapley–Solidarity value, we can obtain the potential function from the potential functions of the Owen and Solidarity value.

Another approach to study a cooperative game solution is mechanism design which can be viewed as a part of the Nash program [32] to implement cooperative game solutions through non-cooperative game theory. Bidding mechanism, first introduced by Pérez-Castrillo and Wettstein [33], is used to implement the Shapley value. Vidal-Puga and Bergantiños [34] implemented the Owen value by a two-rounds coalitional bidding mechanism. Sikker [35] studied the non-cooperative foundations of network allocation rules, including several classical solutions, such as the Myerson value and the position value, which are implemented by bidding mechanism. Ju and Wettstein [36] introduced a generalized bidding approach based on the original bidding approach in Pérez-Castrillo and Wettstein. Addtitionally, the Shapley, consensus [37] and equal surplus values [38] are implemented. van den Brink et al. [39] implemented the egalitarian Shapley value based on Pérez-Castrillo and Wettstein's bidding approach. For the discounted Shapley value, van den Brink and Funaki [40] implemented it in a similar approach.

There is no present research for studying the non-cooperative implementation of the Shapley–Solidarity value. If we want to implement the Shapley-Solidarity value we should replace the first round in the coalitional bidding mechanism in Vidal-Puga and Bergantiños [34] with a mechanism that can implement the Solidarity value.

In this paper, we will study the Shapley–Solidarity value from three aspects. First of all, we give axiomatic characterizations of the Shapley–Solidarity value mainly using intracoalitional quasi-balanced contributions, which is an extended version of quasi-balanced contributions [31]. Subsequently, we obtain an adjusted potential function with respect to the Shapley–Solidarity value. At last, we give a new implementation of the Solidarity value using bidding mechanism. Additionally, we apply this result to the games with coalition structure and implement the Shapley-Solidarity value.

The paper is organized, as follows. In Section 2, we introduce some basic definitions and notations. Axiomatic characterizations of the Shapley-Solidarity value will be presented in Section 3. Subsequently, we study the potential function for the Shapley–Solidarity value in Section 4. Finally, in Section 5, we provide a non-cooperative implementation of the Shapley–Solidarity value.

2. Preliminaries

A cooperative game with transferable utility or simply a game is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ is a player set and $v : 2^n \to \mathbb{R}$ with $v(\emptyset) = 0$ is a characteristic function which assigns to each coalition $S \in 2^n$ the worth $v(S)$. Denote the family of all cooperative games by $\mathcal{G}$, and then denote the family of all cooperative games with player set N by $\mathcal{G}^N$. A game $(N, v) \in \mathcal{G}^N$ is zero-monotonic if $v(S) + v(\{i\}) \leq v(S \cup \{i\})$ for all $S \subseteq N$ with $i \notin S$, and is monotonic if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.

A value on $\mathcal{G}^N$ is a function φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^n$ to every games $(N, v) \in \mathcal{G}^N$. The Shapley value [1], which is denoted by Sh, assigns to every player the expectation of his marginal contributions with respect to all coalitions. For any $(N, v) \in \mathcal{G}^N$ and $i \in N$,

$$Sh_i(N, v) = \sum_{i \in S, S \subseteq N} \frac{(|N| - |S|)!(|S| - 1)}{|N|!} [v(S) - v(S \backslash \{i\})],$$

where $|S|$ is the cardinality of player set S. The Solidarity value [2] is similar to the Shapley value, but it behaves more equalitarian, which assigns to every player the expectation of his average marginal contributions with respect to all coalitions. The Solidarity value $Sol(N,v) \in \mathbb{R}^n$ is defined as, for any $(N,v) \in \mathcal{G}^N$ and $i \in N$,

$$Sol_i(N,v) = \sum_{i \in S, S \subseteq N} \frac{(|N|-|S|)!(|S|-1)}{|N|!} \frac{1}{|S|} \sum_{i \in S} [v(S) - v(S \backslash \{i\})].$$

For a player set N, a coalition structure over N is a partition of N, i.e., $C = \{C_1, C_2, ..., C_m\}$, satisfying $\bigcup_{h \in M} C_h = N$ and $C_h \cap C_r = \varnothing$ when $h \neq r$, where $M = \{1, 2, ..., m\}$. The set $C_h \in C$ are called unions. Denote the set of all coalition structures over N by $\mathcal{C}(N)$. A game (N,v) with coalition structure $C \in \mathcal{C}(N)$ is denoted by (N,v,C). Denote the family of all the games with coalition structure with player set N by $\mathcal{CG}^N$.

For all games with coalition structure, the game defined between unions is called quotient game. Formally, for all game $(N,v,C) \in \mathcal{CG}^N$, with $C = \{C_1, C_2, ..., C_m\}$, the quotient game is denoted by $(M, v_C) \in \mathcal{G}^N$, where $M = \{1, 2, ..., m\}$ and $v_C(T) = v(\bigcup_{i \in T} C_i)$ for all $T \subseteq M$. For all $k \in M$ and all $S \subseteq C_k$, denote, by $C|_S$, the new coalition structure $(\bigcup_{j \neq k} C_j) \bigcup S$, which means that the union C_k is replaced by coalition S in the original coalition structure. The internal game (C_k, v_k) is defined in Owen [20] where $v_k(S) = Sh_k(M, v_{C|_S})$.

A coalitional value f on $\mathcal{CG}^N$ is a function assigning a vector $f(N,v,C) \in \mathbb{R}^N$ to each game with coalition structure $(N,v,C) \in \mathcal{CG}^N$. The Owen value [20] of the game (N,v,C) is the coalitional value, defined as

$$Ow_i(N,v,C) = Sh_i(C_h, v_h), \text{ for all } h \in M \text{ and all } i \in C_h.$$

Another interesting coalitional value, called the Shapley–Solidarity value [24], was introduced and characterized by Calvo and Gutiérrez. They stuck with the Shapley value among unions and applied the Solidarity value between players within the same union. Formally, for a game $(N,v,C) \in \mathcal{CG}^N$ the Shapley–Solidarity value is defined as

$$SS_i(N,v,C) = Sol_i(C_h, v_h), \text{ for all } h \in M \text{ and all } i \in C_h. \tag{1}$$

3. New Characterizations for the Shapley-Solidarity Value

Axiomatization is one of the main ways to characterize the reasonability of solutions with a set of properties in cooperative games. In this section, we characterize the Shapley–Solidarity value by introducing the intracoalitional quasi-balanced contributions property, inspired from the balanced contributions property.

The balanced contributions property, as introduced by Myerson [4], indicates that, for any pair of players, the influence of a player who leaves the grand coalition on the other player is the same as the impact of the other player's departure on it. Myerson [4] characterized the Shapley value by the balanced contributions property and efficiency.

Balanced contributions **(BC)**: a solution φ on $\mathcal{G}^N$ satisfies the balanced contributions property if for any $(N,v) \in \mathcal{G}^N$, $i, j \in N$ with $i \neq j$,

$$\varphi_i(N,v) - \varphi_i(N \backslash \{j\}, v) = \varphi_j(N,v) - \varphi_j(N \backslash \{i\}, v).$$

We first introduce two axioms, inspired by the balanced contributions property, and both of them are used in order to characterize the Solidarity value. In Xu et al. [31], the Solidarity value is characterized by the quasi-balanced contributions property with efficiency.

Quasi-balanced contributions **(QBC)**: a solution φ on $\mathcal{G}^N$ satisfies the quasi-balanced contributions property if for any $(N,v) \in \mathcal{G}^N$, $i, j \in N$ with $i \neq j$,

$$\varphi_i(N,v) - \varphi_i(N \backslash \{j\}, v) + \frac{1}{n} v(N \backslash \{j\}) = \varphi_j(N,v) - \varphi_j(N \backslash \{i\}, v) + \frac{1}{n} v(N \backslash \{i\}).$$

In Calvo and Gutiérrez [24], the Solidarity value is also characterized by the equal averaged gains property together with efficiency.

Equal averaged gains **(EAG)**: a solution φ on $\mathcal{G}^N$ satisfies the equal average gains property if for any $(N, v) \in \mathcal{G}^N$, $i, j \in N$ with $i \neq j$,

$$\frac{1}{n}\sum_{k\in N}[\varphi_i(N,v) - \varphi_i(N\backslash\{k\},v)] = \frac{1}{n}\sum_{k\in N}[\varphi_j(N,v) - \varphi_j(N\backslash\{k\},v)].$$

The previous two properties have been both used for characterizing the Solidarity value separately. However, one can easily check that these two properties are independent with each other.

Calvo et al. [41] introduced the property of intracoalitional balanced contributions to characterize the level structure value. It states that, for any two players in one union, the influence of a player who leaves the union on the other player is the same as the impact of the other player's departure on it.

Intracoalitional balanced contributions **(IBC)**: For all $(N, v, C) \in \mathcal{CG}^N$ and all $i, j \in C_h \in C$, $i \neq j$,

$$f_i(N,v,C) - f(N\backslash\{j\},v,C_{N\backslash\{j\}}) = f_j(N,v,C) - f(N\backslash\{i\},v,C_{N\backslash\{i\}}). \tag{2}$$

In Lorenzo-Freire [27], it is proved that, if a coalitional value satisfies the intracoalitional balanced contributions property, then it can be computed by means of the Shapley value. Next, we will define the intracoalitional quasi-balanced contributions property, and we will prove that if a coalitional value satisfies the intracoalitional quasi-balanced contributions property, it can be computed by means of the Solidarity value.

Intracoalitional quasi-balanced contributions **(IQBC)**: For all $(N, v, C) \in \mathcal{CG}^N$ and all $i, j \in C_h \in C$, $i \neq j$,

$$\begin{aligned} &f_i(N,v,C) - f_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) + \frac{v^{f,C}(C_h\backslash\{j\})}{|C_h|} \\ =&f_j(N,v,C) - f_j(N\backslash\{i\},v,C_{N\backslash\{i\}}) + \frac{v^{f,C}(C_h\backslash\{i\})}{|C_h|}, \end{aligned} \tag{3}$$

where the game $(C_h, v^{f,C})$ [27] is defined as

$$v^{f,C}(T) = \sum_{i\in T} f_i((N\backslash\{C_h\})\cup T, v, (C\backslash\{C_h\})\cup T), \tag{4}$$

for all $T \subseteq C_h$, $T \neq \emptyset$. This game means that the payoff for each subset of players in the union is the sum of the payoffs of these players given by the coalitional value when the union is replaced by the subset. Obviously, this property is directly extended from the quasi-balanced contributions property.

Proposition 1. *A coalitional value f satisfies IQBC if and only if for all $(N, v, C) \in \mathcal{CG}^N$ and all $i \in C_h$ with $C_h \in C$,*

$$f_i(N,v,C) = \frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\})] + \frac{1}{|C_h|}\sum_{j\in C_h\backslash\{i\}} f_i(N\backslash\{j\},v,C_{N\backslash\{j\}}). \tag{5}$$

Proof. Given $i \in C_h$, if a coalitional value f satisfies IQBC, then for all $j \in C_h\backslash\{i\}$,

$$\begin{aligned} &f_i(N,v,C) - f_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) + \frac{v^{f,C}(C_h\backslash\{j\})}{|C_h|} \\ =&f_j(N,v,C) - f_j(N\backslash\{i\},v,C_{N\backslash\{i\}}) + \frac{v^{f,C}(C_h\backslash\{i\})}{|C_h|}. \end{aligned}$$

Summing over $j \in C_h\backslash\{i\}$, we obtain that

$$
\begin{aligned}
&(|C_h|-1)f_i(N,v,C)-\sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})+\frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{j\})\\
=&\sum_{j\in C_h\setminus\{i\}} f_j(N,v,C)-\sum_{j\in C_h\setminus\{i\}} f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})+\frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{i\}).
\end{aligned}
$$

Consider the definition of the game $(C_h, v^{f,C})$, then we have

$$
\begin{aligned}
&(|C_h|-1)f_i(N,v,C)\\
=&\sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})-\frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{j\})\\
&-v^{f,C}(C_h\setminus\{i\})+\frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{i\})+v^{f,C}(C_h)-f_i(N,v,C)\\
=&\sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})-\frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} v^{f,C}(C_h\setminus\{j\})\\
&-\frac{1}{|C_h|}v^{f,C}(C_h\setminus\{i\})+v^{f,C}(C_h)-f_i(N,v,C)\\
=&\sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})-\frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\setminus\{j\})+v^{f,C}(C_h)-f_i(N,v,C).
\end{aligned}
$$

Thus,

$$
f_i(N,v,C)=\frac{1}{|C_h|}[v^{f,C}(C_h)-\frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\setminus\{j\})]+\frac{1}{|C_h|}\sum_{j\in C_h\setminus\{i\}} f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}).
$$

Now, it remains to proven the reverse part. The proof will be done by induction on $|C_h|$. Suppose that $C_h=\{i,j\}$, then we have

$$
\begin{aligned}
f_i(N,v,C)=&\frac{1}{2}[v^{f,C}(C_h)-\frac{1}{2}v^{f,C}(C_h\setminus\{j\})-\frac{1}{2}v^{f,C}(C_h\setminus\{i\})]+\frac{1}{2}f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})\\
=&\frac{1}{2}[f_i(N,v,C)+f_j(N,v,C)-\frac{1}{2}v^{f,C}(C_h\setminus\{j\})+\frac{1}{2}v^{f,C}(C_h\setminus\{i\})]\\
&-\frac{1}{2}f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})+\frac{1}{2}f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}).
\end{aligned}
$$

Accordingly, we have

$$
\begin{aligned}
&f_i(N,v,C)-f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})\\
=&f_j(N,v,C)-f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})+\frac{v^{f,C}(C_h\setminus\{i\})}{2}-\frac{v^{f,C}(C_h\setminus\{j\})}{2}.
\end{aligned}
$$

Suppose that $|C_h|>2$. For any $i,j\in C_h$, we have

$$
\begin{aligned}
&|C_h|[f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})-f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})]\\
=&(|C_h|-1)[f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})-f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})]\\
&+[f_i(N\setminus\{j\},v,C_{N\setminus\{j\}})-f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})]\\
=&v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\})-\frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{j\}} v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j,k\})\\
&+\sum_{k\in C_h\setminus\{i,j\}} f_i(N\setminus\{j,k\},v,C_{N\setminus\{j,k\}})
\end{aligned}
$$

$$
\begin{aligned}
&- [v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\}) - \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{i\}} v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i,k\}) \\
&+ \sum_{k\in C_h\setminus\{i,j\}} f_j(N\setminus\{i,k\},v,C_{N\setminus\{i,k\}})] \\
&+ [f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})] \\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{i,j\}} [v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j,k\}) \\
&- v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i,k\})] - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\}) \\
&+ \sum_{k\in C_h\setminus\{i,j\}} [f_i(N\setminus\{j,k\},v,C_{N\setminus\{j,k\}}) - f_j(N\setminus\{i,k\},v,C_{N\setminus\{i,k\}})] \\
&+ [f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})].
\end{aligned}
$$

By the induction hypothesis, the following result is true for all $k \in C_h\setminus\{i,j\}$,

$$
\begin{aligned}
& f_i(N\setminus\{j,k\},v,C_{N\setminus\{j,k\}}) - f_j(N\setminus\{i,k\},v,C_{N\setminus\{i,k\}}) \\
=& f_i(N\setminus\{k\},v,C_{N\setminus\{k\}}) - f_j(N\setminus\{k\},v,C_{N\setminus\{k\}}) + \frac{1}{|C_h|-1} v^{f,C_{N\setminus\{k\}}}(C_h\setminus\{k,j\}) \\
&- \frac{1}{|C_h|-1} v^{f,C_{N\setminus\{k\}}}(C_h\setminus\{k,i\}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& |C_h|[f_i(N\setminus\{j\},v,C_{N\setminus\{j\}}) - f_j(N\setminus\{i\},v,C_{N\setminus\{i\}})] \\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\}) \\
&- \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{i,j\}} [v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j,k\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i,k\})] \\
&+ \sum_{k\in C_h\setminus\{i\}} f_i(N\setminus\{k\},v,C_{N\setminus\{k\}}) - \sum_{k\in C_h\setminus\{j\}} f_j(N\setminus\{k\},v,C_{N\setminus\{k\}}) \\
&+ \frac{1}{|C_h|-1}\sum_{k\in C_h\setminus\{i,j\}} [v^{f,C_{N\setminus\{k\}}}(C_h\setminus\{k,j\}) - v^{f,C_{N\setminus\{k\}}}(C_h\setminus\{k,i\})] \\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\}) \\
&+ \sum_{k\in C_h\setminus\{i\}} f_i(N\setminus\{k\},v,C_{N\setminus\{k\}}) - \sum_{k\in C_h\setminus\{j\}} f_j(N\setminus\{k\},v,C_{N\setminus\{k\}}) \\
=& v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\}) \\
&+ [v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{k\in C_h} v^{f,C}(C_h\setminus\{k\}) + \sum_{k\in C_h\setminus\{i\}} f_i(N\setminus\{k\},v,C_{N\setminus\{k\}})] \\
&- [v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{k\in C_h} v^{f,C}(C_h\setminus\{k\}) + \sum_{k\in C_h\setminus\{j\}} f_j(N\setminus\{k\},v,C_{N\setminus\{k\}})] \\
=& |C_h|[f_i(N,v,C) - f_j(N,v,C) + \frac{1}{|C_h|}(v^{f,C_{N\setminus\{j\}}}(C_h\setminus\{j\}) - v^{f,C_{N\setminus\{i\}}}(C_h\setminus\{i\}))]. \quad \square
\end{aligned}
$$

Proposition 2. *A coalitional value f satisfies IQBC if and only if for all $(N,v,C) \in \mathcal{CG}^N$ and all $i \in C_h$ with $C_h \in C$, $f_i(N,v,C) = Sol_i(C_h, v^{f,C})$.*

Proof. Because the Solidarity value satisfies the quasi-balanced contributions, we have that for all i, $j \in C_h$,

$$
\begin{aligned}
& f_i(N,v,C) - f_i(N\backslash\{j\},v,C_h\backslash\{j\}) + \frac{v^{f,C}(C_h\backslash\{j\})}{|C_h|} \\
=& Sol_i(C_h,v^{f,C}) - Sol_i(C_h\backslash\{j\},v^{f,C_{N\backslash\{j\}}}) + \frac{v^{f,C}(C_h\backslash\{j\})}{|C_h|} \\
=& Sol_j(C_h,v^{f,C}) - Sol_j(C_h\backslash\{i\},v^{f,C_{N\backslash\{i\}}}) + \frac{v^{f,C}(C_h\backslash\{i\})}{|C_h|} \\
=& f_j(N,v,C) - f_j(N\backslash\{i\},v,C_h\backslash\{i\}) + \frac{v^{f,C}(C_h\backslash\{i\})}{|C_h|}.
\end{aligned}
$$

To prove the counterpart, we will use the induction on $|C_h|$.

For $C_h = \{i\}$, $Sol_i(C_h,v^{f,C}) = Sol_i(\{i\},v^{f,C}) = v^{f,C}(\{i\}) = f_i(N,v,C)$. Suppose that the result is true for $|C_h| < m$ where $m \in \mathbb{R}$. For $|C_h| = m$, by previous proposition, we have

$$
\begin{aligned}
& |C_h| f_i(N,v,C) \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) + \sum_{j\in C_h\backslash\{i\}} f_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) + \sum_{j\in C_h\backslash\{i\}} Sol_i(C_h\backslash\{j\},v^{f,C_{N\backslash\{j\}}}) \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) \\
& + \sum_{j\in C_h\backslash\{i\}} \sum_{T\subseteq C_h\backslash\{i,j\}} \frac{t!(m-t-2)!}{(m-1)!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C_{N\backslash\{j\}}}(T\cup\{k\}) - v^{f,C_{N\backslash\{j\}}}(T)] \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) \\
& + \sum_{j\in C_h\backslash\{i\}} \sum_{T\subseteq C_h\backslash\{i,j\}} \frac{t!(m-t-2)!}{(m-1)!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C}(T\cup\{k\}) - v^{f,C}(T)] \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) \\
& + \sum_{T\subsetneq C_h\backslash\{i\}} \sum_{j\in C_h\backslash(T\cup\{i\})} \frac{t!(m-t-2)!}{(m-1)!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C}(T\cup\{k\}) - v^{f,C}(T)] \\
=& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j\in C_h} v^{f,C}(C_h\backslash\{j\}) \\
& + \sum_{T\subsetneq C_h\backslash\{i\}} \frac{t!(m-t-1)!}{(m-1)!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C}(T\cup\{k\}) - v^{f,C}(T)],
\end{aligned}
$$

where t denotes the cardinality of the coalition T. Thus, we have

$$
f_i(N,v,C) = \sum_{T\subsetneq C_h\backslash\{i\}} \frac{t!(m-t-1)!}{c_h!} \sum_{k\in T\cup\{i\}} \frac{1}{t+1}[v^{f,C}(T\cup\{k\}) - v^{f,C}(T)] = Sol_i(C_h,v^{f,C}). \quad \square
$$

Calvo and Gutiérrez [24] extended the equal averaged gains property to games with coalition structure and they introduced the following property.

Intracoalitional equal averaged gains **(IEAG)**: For all $(N,v,C) \in \mathcal{CG}^N$, all $h \in M$ and all $i,j \in C_h$,

$$
\frac{1}{|C_h|}\sum_{k\in C_h}[f_i(N,v,C) - f_i(N\backslash\{k\},v,C_{N\backslash\{k\}})] = \frac{1}{|C_h|}\sum_{k\in C_h}[f_j(N,v,C) - f_j(N\backslash\{k\},v,C_{N\backslash\{k\}})].
$$

Proposition 3. *A coalitional value f satisfies IEAG if and only if for all $(N, v, C) \in \mathcal{CG}^N$ and all $i \in C_h$ with $C_h \in C$,*

$$f_i(N, v, C) = \frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j \in C_h} v^{f,C}(C_h \backslash \{j\})] + \frac{1}{|C_h|}\sum_{j \in C_h \backslash \{i\}} f_i(N \backslash \{j\}, v, C_{N \backslash \{j\}}). \tag{6}$$

Proof. If the coalitional value f satisfies IEAG then for all $(N, v, C) \in \mathcal{CG}^N$, all $h \in M$ and all $i, j \in C_h$,

$$\frac{1}{|C_h|}\sum_{k \in C_h}[f_i(N, v, C) - f_i(N \backslash \{k\}, v, C_{N \backslash \{k\}})] = \frac{1}{|C_h|}\sum_{k \in C_h}[f_j(N, v, C) - f_j(N \backslash \{k\}, v, C_{N \backslash \{k\}})].$$

Fixing i and summing over $j \in C_h$, we have

$$\begin{aligned} &|C_h| f_i(N, v, C) - \frac{1}{|C_h|}\sum_{j \in C_h}\sum_{k \in C_h} f_i(N \backslash \{k\}, v, C_{N \backslash \{k\}}) \\ =& v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{j \in C_h}\sum_{k \in C_h} f_j(N \backslash \{k\}, v, C_{N \backslash \{k\}}). \end{aligned}$$

i.e.,

$$|C_h| f_i(N, v, C) - \sum_{k \in C_h \backslash \{i\}} f_i(N \backslash \{k\}, v, C_{N \backslash \{k\}}) = v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{k \in C_h} v^{f,C}(C_h \backslash \{k\}).$$

It remains to prove the counterpart,

$$\begin{aligned} &\frac{1}{|C_h|}\sum_{k \in C_h}[f_i(N, v, C) - f_i(N \backslash \{k\}, v, C_{N \backslash \{k\}})] \\ =& f_i(N, v, C) - \frac{1}{|C_h|}\sum_{k \in C_h \backslash \{i\}} f_i(N \backslash \{k\}, v, C_{N \backslash \{k\}}) \\ =& \frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{l \in C_h} v^{f,C}(C_h \backslash \{l\})] + \frac{1}{|C_h|}\sum_{l \in C_h \backslash \{i\}} f_i(N \backslash \{l\}, v, C_{N \backslash \{l\}}) \\ &- \frac{1}{|C_h|}\sum_{k \in C_h \backslash \{i\}} f_i(N \backslash \{k\}, v, C_{N \backslash \{k\}}) \\ =& \frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{l \in C_h} v^{f,C}(C_h \backslash \{l\})] \\ =& \frac{1}{|C_h|}[v^{f,C}(C_h) - \frac{1}{|C_h|}\sum_{l \in C_h} v^{f,C}(C_h \backslash \{l\})] + \frac{1}{|C_h|}\sum_{l \in C_h \backslash \{j\}} f_j(N \backslash \{l\}, v, C_{N \backslash \{l\}}) \\ &- \frac{1}{|C_h|}\sum_{k \in C_h \backslash \{j\}} f_j(N \backslash \{k\}, v, C_{N \backslash \{k\}}) \\ =& f_j(N, v, C) - \frac{1}{|C_h|}\sum_{k \in C_h \backslash \{j\}} f_j(N \backslash \{k\}, v, C_{N \backslash \{k\}}). \quad \square \end{aligned}$$

Accordingly, we have following corollaries.

Corollary 1. *A coalitional value f satisfies IQBC if and only if it satisfies IEAG.*

Corollary 2. *A coalitional value f satisfies IEAG if and only if for all $(N, v, C) \in \mathcal{CG}^N$ and all $i \in C_h$ with $C_h \in C$, $f_i(N, v, C) = Sol_i(C_h, v^{f,C})$.*

Now, we introduce some properties that will be used in the characterizations of the Shapley–Solidarity value.

A coalitional value f on $\mathcal{CG}^N$ satisfies:

- *Efficiency* **(EFF)**: if for all $(N,v,C) \in \mathcal{CG}^N$, $\sum_{i\in N} f_i(N,v,C) = v(N)$.
- *Additivity* **(ADD)**: if for any (N,v_1,C), $(N,v_2,C) \in \mathcal{CG}^N$ and $i \in N$, $f_i(N,v_1+v_2,C) = f_i(N,v_1,C) + f_i(N,v_2,C)$ where $(v_1+v_2)(S) = v_1(S) + v_2(S)$ for any $S \subseteq N$.
- *Coalitional Symmetry* **(CSY)**: if for all $(N,v,C) \in \mathcal{CG}^N$ and all symmetry coalitions C_h, $C_r \in C$, $\sum_{i\in C_h} f_i(N,v,C) = \sum_{i\in C_r} f_i(N,v,C)$, where C_h and C_r are symmetry if $v(C_h \cup (\bigcup_{k\in T} C_k)) = v(C_r \cup (\bigcup_{k\in T} C_k))$ for all $T \subseteq M\backslash\{h,r\}$.
- *Coalitional Strong Marginality* **(CSM)**: if for all (N,v,C), $(N,v',C) \in \mathcal{CG}^N$, $v(S\cup C_h) - v(S) \geq v'(S\cup C_h) - v'(S)$ for all $S \in N\backslash C_h$, then $\sum_{i\in C_h} f_i(N,v,C) \geq \sum_{i\in C_h} f_i(N,v',C)$.
- *Null Union* **(NU)**: if for all $(N,v,C) \in \mathcal{CG}^N$ and $C_h \in C$, C_h is a null union, then $\sum_{i\in C_h} f_i(N,v,C) = 0$, where C_h is a null union if $v(C_h \cup (\bigcup_{k\in T} C_k)) = v(\bigcup_{k\in T} C_k)$ for all $T \subseteq M\backslash\{h\}$.

Lorenzo-Freire [26] introduced the following proposition.

Proposition 4. *For all $C_h \in C$, if a coalitional value f satisfies EFF, CSY and CSM, then $\sum_{i\in C_h} f_i(N,v,C) = Sh_h(M,v_C)$.*

Therefore, we have the following characterizations of the Shapley-Solidarity value.

Theorem 1.

(a) The Shapley–Solidarity value is the only coalitional value that satisfies EFF, IQBC/IEAG, CSY, and CSM.
(b) The Shapley-Solidarity value is the only coalitional value that satisfies EFF, IQBC/IEAG, ADD, CSY, and NU.

Proof. Because the property of IQBC and IEAG are equivalent, we only consider IEAG in the following proof.

(a) We first check that the Shapley–Solidarity value satisfies these properties. Calvo and Gutiérrez [24] have already proved the Shapley-Solidarity value satisfies EFF and IEAG. Because the Shapley-Solidarity value satisfies quotient game property i.e., $\sum_{i\in C_h} SS_i(N,v,C) = Sh_h(M,v_C)$, and the Shapley value satisfies symmetry and strong marginality, the Shapley–Solidarity value satisfies CSY and CSM.

Next, we will prove these properties can identify the Shapley-Solidarity value uniquely. Suppose the coalitional value f satisfies EFF, IEAG, CSY, and CSM. From former propositions, we have proved that if a coalitional value f satisfies IEAG, then for all $i \in C_h$ and $C_h \in C$ we have $f_i(N,v,C) = Sol_i(C_h, v^{f,C})$, where $v^{f,C}(T) = \sum_{i\in T} f_i((N\backslash\{C_h\}) \cup T, v, (C\backslash\{C_h\}) \cup T)$, $T \subseteq C_h$.

By Proposition 4, if f satisfies EFF, CSY, and CSM, then for all $C_h \in C$

$$\sum_{i\in C_h} f_i(N,v,C) = Sh_h(M,v_C).$$

Thus,

$$v^{f,C}(T) = \sum_{i\in T} f_i((N\backslash\{C_h\}) \cup T, v, (C\backslash\{C_h\}) \cup T) = Sh_h(M, v_{C_{(N\backslash C_h)\cup T}}).$$

From the definition of the Shapley–Solidarity value (1), we have $f(N,v,C) = SS_i(N,v,C)$.

(b) Because ADD and NU can imply CSM [26], the proof of this part is similar to the proof of **(a)**, and we omit it. □

Lemma 1. *Note that Calvo [24] characterized the Shapley–Solidarity value by EFF, IEAG and CBC, where CBC is coalitional balanced contributions property which is shown as follows.*

Coalitional balanced contributions ***(CBC)****: for all* $(N, v, C) \in \mathcal{CG}$ *and all* C_i*,* $C_j \in C$*,*

$$\begin{aligned}&\sum_{\alpha \in C_i} f_\alpha(N, v, C) - \sum_{\alpha \in C_i} f_\alpha(N \backslash \{C_j\}, v, C \backslash \{C_j\}) \\ =& \sum_{\alpha \in C_j} f_\alpha(N, v, C) - \sum_{\alpha \in C_j} f_\alpha(N \backslash \{C_i\}, v, C \backslash \{C_i\}).\end{aligned} \tag{7}$$

Obviously, the CBC property is extended from the balanced contributions in Myerson [4]. It states that, for all C_i*,* $C_j \in C$*, the influence of the members in* C_i *who leave the grand coalition on* C_j *is the same as the impact of the members's departure in* C_i *on* C_j*.*

4. The Potential Function for the Shapley-Solidarity Value

Hart and Mas-Colell [28] characterized the Shapley value by means of consistency property and standard for two-person games. An interesting concept, called potential function, is used for the characterization.

A function $P : \mathcal{G} \to \mathbb{R}$ with $P(\emptyset, v) = 0$ is called a potential function if it satisfies for any $(N, v) \in \mathcal{G}$,

$$\sum_{i \in N} D^i P(N, v) = v(N), \tag{8}$$

where $D^i P(N, v)$ represents the marginal contributions of a player $i \in N$ with respect to the potential function, which is defined by

$$D^i P(N, v) = P(N, v) - P(N \backslash \{i\}, v). \tag{9}$$

In a similar way, Xu et al. [31] defined $A^i P^*(N, v) = D^i P(N, v) + \frac{1}{n} v(N \backslash \{i\})$ as the adjusted marginal contribution for player i and they obtained an A-potential function. They also proved that the vector of the adjusted marginal contributions coincides with the Solidarity value.

For games with coalition structure, Winter [29] first extended the concept of potential function. Our potential function for the Shapley–Solidarity value is inspired by the potential function of the Owen value introduced in Winter [29] and use the similar way of adjustment in Xu et al. [31].

Now, we define a potential function for games with coalition structure.

Definition 1. *Let* P *be a function defined on* $\mathcal{CG}$ *s.t.* $P(N, v, C) \in \mathbb{R}^m$*, where* $C = (C_1, C_2, \cdots, C_m)$ *and* $P^j(N, v, C) = 0$ *when* $C_j \cap N = \emptyset$*. The marginal contribution of player* $i \in C_j$ *to* (N, v, C) *is*

$$D^i P(N, v, C) = P^j(N, v, C) - P^j(N \backslash \{i\}, v, C_{N \backslash \{i\}}). \tag{10}$$

The function P *is said to be an adjusted potential function for games with coalition structure if for all* $S \in C$*,*

$$\sum_{i \in S} [D^i P(N, v, C) + \frac{1}{|S|} D^{[S \backslash \{i\}]} P([C|_{S \backslash \{i\}}], v_C, [C|_{S \backslash \{i\}}])] = D^{[S]} P([C], v_C, [C]), \tag{11}$$

and

$$\sum_{i \in N} [D^i P(N, v, C) + \frac{1}{|S|} D^{[S \backslash \{i\}]} P([C|_{S \backslash \{i\}}], v_C, [C|_{S \backslash \{i\}}])] = v(N). \tag{12}$$

Denote, by $[S]$*, the coalition* $S \in C$ *when considered as a player, and the set of players in* v_C *by* $[C]$*.* $C|_{S \backslash \{i\}}$ *is the coalition structure that player* i *is deleted from* S*.*

For simplicity, we denote $D^iP(N,v,C) + \frac{1}{|S|}D^{[S\backslash\{i\}]}P([C|_{S\backslash\{i\}}], v_C, [C|_{S\backslash\{i\}}])$ the $A^iP(N,v,C)$ and call it the adjusted marginal contributions of player i in the rest of the paper.

Expression (11) says that the sum of the adjusted marginal contributions of players $i \in S \in C$ is the marginal contribution of the player $[S]$ to the $([C], v_C, [C])$, where the player set is $C = (C_1, C_2, \cdots, C_m)$, the game is v_C. So, combining with (12), $P(\emptyset, v, C) = 0$ and the definition of potential function for the Shapley value in Hart and Mas-Colell [28], we have

$$D^{[S]}P([C], v_C, [C]) = Sh^{[S]}([C], v_C), \text{ for all } S \in C. \tag{13}$$

Theorem 2. *There exists a unique adjusted potential function P for games with coalition structure. Moreover, for all $(N,v,C) \in \mathcal{CG}$ and $i \in N$, $A^iP(N,v,C) = SS_i(N,v,C)$.*

Proof. First, we show uniqueness. for any $(N,v,C) \in \mathcal{CG}$ and $S \in C$, from (10) and (11), we have following recursive form of the potential function P.

$$\begin{aligned} P(N,v,C) =& \frac{1}{|S|}[D^{[S]}P([C], v_C, [C]) - \frac{1}{|S|}\sum_{i \in S} D^{[S\backslash\{i\}]}P([C|_{S\backslash\{i\}}], v_C, [C|_{S\backslash\{i\}}]) \\ &+ \sum_{i \in S} P(N\backslash\{i\}, v, C_{N\backslash\{i\}})]. \end{aligned}$$

Together with $P(\emptyset, v, \emptyset) = 0$, $P(N,v,C)$ can be obtained by recursion and can be uniquely determined.

Now, we show that for all $(N,v,C) \in \mathcal{CG}$ and $i \in N$, $A^iP(N,v,C) = SS_i(N,v,C)$. We know that the Shapley–Solidarity value is characterized by EFF, IQBC, and CBC from the Corollary 1 and Calvo [24]. If we can prove that $A^iP(N,v,C)$ satisfies these three properties, we can claim that $A^iP(N,v,C)$ coincides with the Shapley–Solidarity value.

The EFF is obvious from the definition of the potential function. For any game with coalition structure $(N,v,C) \in \mathcal{CG}$, $h \in M$, and $i,j \in C_h$, if we want to prove that $A^iP(N,v,C)$ satisfies IQBC, we should prove the following equation.

$$\begin{aligned} &A^iP(N,v,C) - A^iP(N\backslash\{j\}, v, C_{N\backslash\{j\}}) + \frac{v^{AP,C}(C_h\backslash\{j\})}{|C_h|} \\ =&A^jP(N,v,C) - A^jP(N\backslash\{i\}, v, C_{N\backslash\{i\}}) + \frac{v^{AP,C}(C_h\backslash\{i\})}{|C_h|}, \end{aligned}$$

where AP denotes $AP(N,v,C)$. From the definition of the game $(C_h, v^{f,C})$ (4) and (11), we have $v^{AP,C}(C_h\backslash\{j\}) = D^{[C_h\backslash\{j\}]}P([C|_{C_h\backslash\{j\}}], v_C, [C|_{C_h\backslash\{j\}}])$. Thus,

$$\begin{aligned} &A^iP(N,v,C) - A^iP(N\backslash\{j\}, v, C_{N\backslash\{j\}}) + \frac{v^{AP,C}(C_h\backslash\{j\})}{|C_h|} \\ =&D^iP(N,v,C) + \frac{1}{|C_h|}D^{[C_h\backslash\{i\}]}P([C|_{C_h\backslash\{i\}}], v_C, [C|_{C_h\backslash\{i\}}]) \\ &- D^iP(N\backslash\{j\}, v, C_{N\backslash\{j\}}) - \frac{1}{|C_h|-1}D^{[C_h\backslash\{i,j\}]}P([C|_{C_h\backslash\{i,j\}}], v_C, [C|_{C_h\backslash\{i,j\}}]) \\ &+ \frac{1}{|C_h|}D^{[C_h\backslash\{j\}]}P([C|_{C_h\backslash\{j\}}], v_C, [C|_{C_h\backslash\{j\}}]) \\ =&P(N,v,C) - P(N\backslash\{i\}, v, C_{N\backslash\{i\}}) + \frac{1}{|C_h|}D^{[C_h\backslash\{i\}]}P([C|_{C_h\backslash\{i\}}], v_C, [C|_{C_h\backslash\{i\}}]) \\ &- P(N\backslash\{j\}, v, C_{N\backslash\{j\}}) + P(N\backslash\{i,j\}, v, C_{N\backslash\{i,j\}}) \\ &- \frac{1}{|C_h|-1}D^{[C_h\backslash\{i,j\}]}P([C|_{C_h\backslash\{i,j\}}], v_C, [C|_{C_h\backslash\{i,j\}}]) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|C_h|}D^{[C_h\setminus\{j\}]}P([C|_{C_h\setminus\{j\}}],v_C,[C|_{C_h\setminus\{j\}}])\\
=&D^jP(N,v,C)+\frac{1}{|C_h|}D^{[C_h\setminus\{i\}]}P([C|_{C_h\setminus\{i\}}],v_C,[C|_{C_h\setminus\{i\}}])\\
&-D^jP(N\setminus\{i\},v,C_{N\setminus\{i\}})-\frac{1}{|C_h|-1}D^{[C_h\setminus\{i,j\}]}P([C|_{C_h\setminus\{i,j\}}],v_C,[C|_{C_h\setminus\{i,j\}}])\\
&+\frac{1}{|C_h|}D^{[C_h\setminus\{j\}]}P([C|_{C_h\setminus\{j\}}],v_C,[C|_{C_h\setminus\{j\}}])\\
=&A^jP(N,v,C)-A^jP(N\setminus\{i\},v,C_{N\setminus\{i\}})+\frac{v^{AP,C}(C_h\setminus\{i\})}{|C_h|}.
\end{aligned}
$$

It verifies that $A^iP(N,v,C)$ satisfies IQBC.

It remains to prove that $A^iP(N,v,C)$ satisfies CBC. From (13), we know that $\sum_{i\in S}A^iP(N,v,C)=Sh^S([C],v_C)$. Because the Shapley value satisfies balanced contributions [4]. For any C_i, $C_j\in C$, we have

$$Sh^{C_{k_1}}([C],v_C)-Sh^{C_{k_1}}([C]\setminus\{C_{k_2}\},v_C)=Sh^{C_{k_2}}([C],v_C)-Sh^{C_{k_2}}([C]\setminus\{C_{k_1}\},v_C).$$

Subsequently, we have

$$
\begin{aligned}
&D^{C_{k_1}}P([C],v_C,[C])-D^{C_{k_1}}P([C]\setminus\{C_{k_2}\},v_C,[C])\\
=&D^{C_{k_2}}P([C],v_C,[C])-D^{C_{k_2}}P([C]\setminus\{C_{k_1}\},v_C,[C]).
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{\alpha\in C_{k_1}}A^\alpha P(N,v,C)-\sum_{\alpha\in C_{k_1}}A^\alpha P(N\setminus\{C_{k_2}\},v,C\setminus\{C_{k_2}\})\\
=&\sum_{\beta\in C_{k_2}}A^\beta P(N,v,C)-\sum_{\beta\in C_{k_2}}A^\beta P(N\setminus\{C_{k_1}\},v,C\setminus\{C_{k_1}\}).
\end{aligned}
$$

Thus, the Shapley-Solidarity value satisfies CBC. And we have that $A^iP(N,v,C)=SS_i(N,v,C)$. □

5. The Coalitional Bidding Mechanism for the Shapley-Solidarity Value

Mechanism design can be seen as the part of Nash program to bridge the gap between cooperative and non-cooperative game theory. It is a significant approach to characterize the rationality of solutions for cooperative games, and it has been widely studied in the field of cooperative games. Pérez-Castrillo and Wettstein [33] firstly proposed a bidding mechanism to give rise to the Shapley value as the result of equilibrium behavior. Our coalitional bidding mechanism is similar to the coalition bidding mechanism for the Owen value [34], which is extended from the bidding mechanism for the Shapley value [33]. Both of the smechanisms have two round and the second round is the same while the difference between our mechanism for the Shapley–Solidarity value and the one for the Owen value lies in the first round. We first give a new bidding mechanism for the Solidarity value, and we will restrict the underlying game to monotonic games.

5.1. A New Bidding Mechanism for the Solidarity Value

Our bidding mechanism is inspired from Pérez-Castrillo and Wettestein's for the Shapley value. The Solidarity value and Shapley value are the excepted values with respect to different considerations of the marginal contributions. The Shapley value only takes care of every individual himself, and it uses the pure marginal contributions, while the Solidarity value uses the average marginal contributions which behaves as the property of solidarity. When a player joins or leaves a coalition, he will share their gains or losses with other players in the coalition rather than himself only. In our bidding mechanism, we will consider the property of solidarity in the Solidarity value.

Now, we describe the bidding mechanism for the Solidarity value. When there is only one player i in the game, the player i receives $v(\{i\})$. Suppose that the rules of the bidding mechanism is known when there are at most $n-1$ players. The procedure of the bidding mechanism for a set of players $N = \{1, 2, ..., n\}$ is as follows.

- **Stage 1:** the players bid for each other for electing a proposer, i.e., each player $i \in N$ makes bid $b^i_j \in \mathbb{R}$ for every $j \in N \backslash \{i\}$. Let $B^i = \sum_{j \in N\backslash\{i\}} b^i_j - \sum_{j\in N\backslash\{i\}} b^j_i$ denote the net bid of player $i \in N$. Find a player α be the proposer whose net bid is max among all the players. If there is more than one maximizer, then randomly choose a player from the maximizers.
- **Stage 2:** the proposer α makes an offer $x^\alpha_j \in \mathbb{R}$ to every player $j \in N\backslash\{\alpha\}$.
- **Stage 3:** all other players, except proposer sequentially accept or reject the offer. If all other players agree the proposer, we say the offer is accepted otherwise the offer is rejected.
  If the offer is accepted, player $j \in N\backslash\{\alpha\}$ receives x^α_j, and the proposer receives $v(N) - \sum_{j\in N\backslash\{\alpha\}} x^\alpha_j$. Consider the bids in stage 1, the player $j \in N\backslash\{\alpha\}$ totally receives $x^\alpha_j + b^\alpha_j$ and the proposer eventually receives $v(N) - \sum_{j\in N\backslash\{\alpha\}}(b^\alpha_j + x^\alpha_j)$.
  If the offer is rejected, a player β randomly chosen from N with probability $\frac{1}{n}$ leaves the game and gets nothing. Note that the proposer and other players have same probability to be player β. Other players proceed with the game in same rules with player set $N\backslash\{\beta\}$. In this case, when we consider the bids in stage 1, every player j other than proposer α gets b^α_j and proposer α loses $\sum_{j\in N\backslash\{\alpha\}} b^\alpha_j$. The randomly chosen player β will be excluded in the rest of the games and his bid in stage 1 will not be changed.

Theorem 3. *For any monotonic game $(N, v) \in \mathcal{G}$, the above mechanism implements the Solidarity value in any subgame perfect equilibrium (SPE).*

Proof. We prove this result by induction on the number of players. Obviously, the theorem is true if there is only one player in the game. We assume that it holds for all player set number $m \le n-1$. Now, we show that it is true for $m = n$.

Firstly, we construct a subgame perfect equilibrium whose outcome is the Solidarity value of game (N, v). Consider the following strategy profiles.

At stage 1, each player $i \in N$ makes bids $b^i_j = Sol_j(N, v) - \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$ to every player $j \in N\backslash\{i\}$.

At stage 2, the proposer α offers $x^\alpha_j = \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$.

At stage 3, each player $j \in N\backslash\{\alpha\}$ will accept the offer if $x^\alpha_j \ge \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$, otherwise the offer is rejected.

It is easy to check that every player other than the proposer obtains their Solidarity value. Because the Solidarity value is an efficient solution, the proposer also gets his Solidarity value.

In these strategy profiles, the net bids B^i of every player $i \in N$ is equal to zero by the equal averaged gains property of the Solidarity value.

$$
\begin{aligned}
B^i &= \sum_{j\in N\backslash\{i\}} (b^i_j - b^j_i) \\
&= \sum_{j\in N\backslash\{i\}} [Sol_j(N, v) - \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) - (Sol_i(N, v) - \frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v))] \\
&= 0.
\end{aligned}
$$

Now, we prove that the above strategies constitute a SPE. At stage 3, in the case of rejection, one of the players, including the proposer, is randomly chosen and he/she will be excluded in the next round. The players who are left in the game will play the subgame according to the same rules. Subsequently, we can obtain the Solidarity value as the outcome of those games by the induction hypothesis. If any player rejects the proposal, then the expected payoff for any player $i \in N$ in the subgame is $\frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\},v)$. Thus, each player $i \in N\backslash\{\alpha\}$ accepts any offer $x_i^\alpha \geq \frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\},v)$, and rejects any offer $x_i^\alpha < \frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\},v)$. Therefore, the strategies are best responses at stage 3.

At stage 2, the strategies are also best responses. According to these strategies, the payoff of α starting from stage 2 is

$$
\begin{aligned}
v(N) - \sum_{j\in N\backslash\{\alpha\}} x_j^\alpha =& v(N) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n} \sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v) \\
=& v(N) - \frac{1}{n}\sum_{k\in N} v(N\backslash\{k\}) + \frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\},v) \\
\geq& \frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\},v),
\end{aligned} \tag{14}
$$

where the last inequation holds from the fact that the game is a monotonic game. If he offers some player j the payoff x_j^α less than $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$, the proposal is rejected and the expected payoff he will obtain is $\frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\},v)$. If he offers some player j the value x_j^α higher than $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$, the proposal is accepted, but his payoff is strictly worse off.

Now, consider the strategies at stage 1. Obviously, if a player increases his total bid, then he will be chosen as the proposer, but his payoff will decrease. If a player decreases his total bid, then his payoff is invariable, since other players will be chosen as the proposer. Thus, the strategy is the best response at stage 1. Hence, the above strategies constitute a SPE.

The proof of any subgame perfect equilibrium yields the Solidarity value needs five claims.

Claim a. In any SPE, at stage 3, each player $j \in N\backslash\{\alpha\}$ accepts any offer if $x_j^\alpha > \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$. The offer is rejected if $x_j^\alpha < \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$ for at least some $j \in N\backslash\{\alpha\}$.

If the proposal of player α is rejected, the excepted payoff of a player $j \neq \alpha$ is $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$ by the induction. Subsequently, the optimal strategy of the player j is that he accepts any offer higher than $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$ and rejects any offer lower than $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$.

Claim b. If there is a player $i \in N$, such that $v(N) > v(N\backslash\{i\})$, then the SPE strategies starting from stage 2 are as follows. At stage 2, the proposer α offers $x_j^\alpha = \frac{1}{n}\sum_{k\in N} Sol_j(N\backslash\{k\},v)$ to all $j \in N\backslash\{\alpha\}$, and at stage 3, $j \in N\backslash\{\alpha\}$ accepts any offer $x_j^\alpha \geq \frac{1}{n}\sum_{k\in N} Sol_j(N\backslash\{k\},v)$ and rejects the offer otherwise. If $v(N) = v(N\backslash\{k\})$ for all $k \in N$, there exist the SPE strategies besides the previous SPE strategies. At stage 2, the proposer α offers $x_j^\alpha \leq \frac{1}{n}\sum_{k\in N} Sol_j(N\backslash\{k\},v)$ to all $j \in N\backslash\{\alpha\}$, and at stage 3, $j \in N\backslash\{\alpha\}$ rejects any offer $x_j^\alpha \leq \frac{1}{n}\sum_{k\in N} Sol_j(N\backslash\{k\},v)$ and accepts the offer otherwise.

We prove that the previous strategies constitute a SPE. Suppose that there is a player $i \in N$, such that $v(N) > v(N\backslash\{i\})$. In the case of rejection, player α will expectantly receive $\frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\},v)$. Let $0 < \varepsilon < v(N) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v) - \frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\},v)$. The ε is definitely exists from Equation (14) and $v(N) > v(N\backslash\{i\})$. The player α can improve his payoff by offering any player $j \in N\backslash\{\alpha\}$ with $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v) + \frac{\varepsilon}{n-1}$. The offer will be accepted by claim a. Thus, in any SPE the offer of player α must be accepted. This implies that $x_j^\alpha \geq \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$ for all $j \in N\backslash\{\alpha\}$. Player α still can improve his payoff by offering $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v) + \frac{\varepsilon}{n-1}$ to player $j \in N\backslash\{\alpha\}$ with a smaller ε. These offers are accepted and α's payoff increases. Hence, $x_j^\alpha = \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$ for all $i \neq \alpha$ at any SPE and, at stage 3, every agent $i \neq \alpha$ accepts any offer if $x_j^\alpha \geq \frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\},v)$.

If $v(N) = v(N\backslash\{k\})$ for all $k \in N$, the proposer α offers at least $\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)$ to every player $j \neq \alpha$, so that the offer can be accepted by the same argument in the previous case. The proposer expectantly receive $\frac{1}{n}\sum_{k\in N\backslash\{\alpha\}} Sol_\alpha(N\backslash\{k\}, v)$ in the case of rejection, which is identical to the payoff in the case of acceptation. Therefore, any offer that leads to a rejection is also a SPE.

Claim c. In any SPE, $B^i = B^j = 0$ for all $i, j \in N$.

Claim d. In any SPE, the payoff of every player is invariable whoever is chosen as the proposer.

The proofs of the above two claims are similar to the proof of Theorem 1 in Pérez-Castrillo and Wettstein [33], and we omit them.

Claim e. In any SPE, the final payoff of every player coincides with the Solidarity value.

If a player i is the proposer, then the his payoff is $x_i^i = v(N) - \sum_{j\neq i}\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) - \sum_{j\neq i} b_j^i$. If a player $j \neq i$ is the proposer, the payoff of player i is $x_i^j = \frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v) + b_i^j$. Thus, we have

$$\begin{aligned}
\sum_{j\in N} x_i^j =& [v(N) - \sum_{j\neq i}\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) - \sum_{j\neq i} b_j^i] + \sum_{j\neq i}[\frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v) + b_i^j] \\
=& [v(N) - \sum_{j\neq i}\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v)] + \sum_{j\neq i}[\frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v)] - B^i \\
=& v(N) - \sum_{j\in N}\frac{1}{n}\sum_{k\in N\backslash\{j\}} Sol_j(N\backslash\{k\}, v) + \sum_{j\in N}[\frac{1}{n}\sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v)] \\
=& v(N) - \frac{1}{n}\sum_{k\in N} v(N\backslash\{k\}) + \sum_{k\in N\backslash\{i\}} Sol_i(N\backslash\{k\}, v) \\
=& nSol_i(N, v)
\end{aligned} \tag{15}$$

where the last equality holds by the recursive formula of the Solidarity value [42]. By claim (d), we have $x_i^j = x_i^k$ for all j, k. Thus $x_i^j = Sol_i(N, v)$ for all $i \in N$. $\square$

Remark 1. *Note that Xu et al. [31] also gave a bidding mechanism for the Solidarity value. When comparing our bidding mechanism with the work of Xu et al. [31], there is no difference between the two bidding mechanisms at Stage 1 and Stage 2, but an exceptional stage (Stage 0) is considered at the beginning of the mechanism in [31], where $a_i = \frac{1}{s}v(S\backslash\{i\})$ is distributed for player i that is in the active player set S. They call a_i as the compensation for player i from an ethical point of view. The main difference is at Stage 3. When the offer by proposer is rejected, a player β is randomly chosen from N with probability $\frac{1}{n}$ leaving the grand coalition in our bidding mechanism, while the rejected proposer leaves the grand coalition in the mechanism of Xu et al. [31].*

Remark 2. *When comparing our bidding mechanism with Pérez-Castrillo and Wettstein's mechanism [33] for the Shapley value, there is only difference at Stage 3. In Pérez-Castrillo and Wettstein's mechanism, if the offer is rejected, then the proposer α will leave from the grand coalition N and take away his individual worth $v(\{\alpha\})$. It means that the proposer himself takes the full responsibility when his proposal is rejected by others. In our mechanism for the Solidarity value, if the offer is rejected, a player β is randomly chosen from all players with probability $\frac{1}{n}$ leaving the grand coalition N and gets nothing. It indicates the preference of egalitarianism from the property of solidarity, which is, all players (not only the proposer or any rejector) need to take the responsibility for the breakdown of the coalition when the proposal is rejected in our mechanism.*

5.2. The Coalitional Bidding Mechanism for the Shapley-Solidarity Value

Vidal-Puga and Begantiños [34] extended the bidding mechanism for the Shapley value introduced in Pérez-Castrillo and Wettestein [33] to games with coalition structure. They used a two round coalitional bidding mechanism, where, at round 1, players in the same union play the bidding mechanism for obtaining the resources of the union and, at round 2, the players who have obtained the resources in round 1 play the bidding mechanism with the obtained resources.

In the former section, we have already proposed a new non-cooperative implementation of the Solidarity value. Next, we will use the coalitional bidding mechanism that was introduced in Vidal-Puga and Begantiños to implement the Shapley-Solidarity value. Considering the relationship between the Owen value and the Shapley-Solidarity value, we just need to change the first round of the coalitional bidding mechanism in the original implementation for the Owen value.

We will describe our bidding mechanism recursively. If there is only one player i, the player obtains $v(i)$. Suppose that the mechanism is played at most $n-1$ players. Then, for $N=\{1,\cdots,n\}$ and $C=\{C_1,\cdots,C_m\}$, the procedure of the bidding mechanism is as follows.

Round 1. At this round, the players in any union $C_h \in C$ play the bidding mechanism for obtaining the resources of C_h. If there is only one player i in C_h, this player has his resources. Assume the mechanism played by at most $|C_h|-1$ players. For $|C_h|$ the process is as follows.

- **Stage 1.** Every player $i \in C_h$ makes bids $b_j^i \in \mathbb{R}$ for every $j \in C_h \backslash \{i\}$. The net bid $B^i = \sum_{j\in C_h\backslash\{i\}} b_j^i - \sum_{j\in C_h\backslash\{i\}} b_i^j$. Let $\alpha_h = argmax_i\{B^i\}$. If the maximizer is not unique, randomly choosing any player from them.
- **Stage 2.** The proposer α_h makes an offer $x_j^{\alpha_h}$ to every player $i \in C_h\backslash\{\alpha_h\}$.
- **Stage 3.** The players in $C_h\backslash\{\alpha_h\}$ sequentially decides whether or not to accept the offer. If all players accept the offer, then the offer is accepted, otherwise the offer is rejected.

Every union in C plays the bidding mechanism sequentially in the order $C_1,\cdots,C_m$ until we find C_{l_0} and α_{l_0} such that the offer of α_{l_0} is rejected or for any $C_l \in C$ the offer of α_l is accepted.

If the offer of α_{l_0} is rejected by some player in C_{l_0}, then randomly choose one player from coalition C_{l_0}. Suppose that the player β_{l_0} is chosen then he leaves the game with nothing. Consider the bid in the first stage, all he acquires from the game is $b_{\beta_{l_0}}^{\alpha_{l_0}}$ if he is not the proposer otherwise he will lose $\sum_{j\in C_{l_0}\backslash\{\alpha_{l_0}\}} b_j^{\alpha_{l_0}}$. The left players play the subgame (N',v',C') with same rules, where $N'=N\backslash\{\beta_{l_0}\}$, $v'=v_{-\beta_{l_0}}$ and $C'=C_{-\beta_{l_0}}$.

If for any $C_l \in C$ the offer of α_l is accepted, then the player α_l becomes the "representative" of the coalition C_l. The final payoff of the players other than the proposer in the coalition C_l will get the bid in stage 1 and the offer in stage 3 that is given from the proposer. Accordingly, the payoff of the proposer in this round is $p_{\alpha_l}^1 = -\sum_{i\in C_l\backslash\{\alpha_l\}}(b_i^{\alpha_l}+x_i^{\alpha_l})$. He will go to round 2 with the resources of C_l and the other players leave the game.

When the first round is finished, any coalition $C_l \in C$ have one representative r_l. We denote C_l^r the set of players whose resources are obtained by representative r_l. The set of the players $C\backslash C_l^r$ is the players that have been removed in C_l.

Round 2. All of representatives $N^r=\{r_1,\cdots,r_m\}$ play the bidding mechanism introduced by Pérez-Castrillo and Wettstein [33] according to the game (N^r,v^r), which is defined by for any $S\subset N^r$, $v^r(S)=v(\cup_{r_l\in S}C_l^r)$. For any $r_l\in N^r$, the payoff obtained by r_l at round 2 is denoted by $p_{r_l}^2$.

The bidding mechanism for the Shapley value in Pérez-Castrillo and Wettstein needs that the underlying game is zero monotonic game i.e., the game (N^r,v^r) should be a zero monotonic game. Thus, the game v should be a superadditive game. In our bidding mechanism for the Solidarity value, we restrict the underlying game to monotonic games. It is easy to check that, if a superadditive game is non-negative game, then the game is a monotonic game. Therefore, we will restrict the underlying game for implementing the Shapley–Solidarity value to the superadditive game where every coalition has non-negative worth.

Theorem 4. *Given any game (N,v,C), if (N,v,C) is superadditive and non-negative, then above coalitional bidding mechanism implements the Shapley–Solidarity value.*

Proof. The result holds when there is only one player. Suppose the result holds for at most $n-1$ players. Let $N=\{1,2,...,n\}$, and we consider the following strategies.

Round 1. For any coalition $C_h \in C$, we describe their strategies as follows.

- **Stage 1.** Every player $i \in C_h$ makes bids $b^i_j = SS_j(N, v, C) - \frac{1}{|C_h|}\sum_{k \in C_h} SS_j(N \backslash \{k\}, v, C_{N \backslash \{k\}})$ for every $j \in C_h \backslash \{i\}$.
- **Stage 2.** The proposer α_h offers $x^{\alpha_h}_j = \frac{1}{|C_h|}\sum_{k \in C_h} SS_j(N \backslash \{k\}, v, C_{N \backslash \{k\}})$ to every player $j \in C_h \backslash \{\alpha_h\}$.
- **Stage 3.** Any player $j \in C_h \backslash \{\alpha_h\}$ accepts the offer of α_h if and only if $x^{\alpha_h}_j \geq \frac{1}{|C_h|}\sum_{k \in C_h} SS_j(N \backslash \{k\}, v, C_{N \backslash \{k\}})$.

Round 2. The representatives play the bidding mechanism in terms of the strategies described in Pérez-Castrillo and Wettstein [33].

Following the similar proof procedure in Theorem 2 of Vidual-Puga and Bergantiños [34], we can prove the theorem and we will omit it. □

Remark 3. *Our non-cooperative implementation for the Shapley–Solidarity value is quit similar to the implementation for the Owen value. We just change the way to punish the players when the bargaining is broken in the first round.*

6. Conclusions

Coalition structure is adopted in order to describe the situation where the players form groups for bargaining payoffs in cooperative games. The Shapley-Solidarity value provides a way of dividing the total payoffs among the players in a game with coalition structure. It employs the Shapley value among the unions and the Solidarity value among the members inside each union, by considering that the players within a union were more willing to show their solidarity and each union was more inclined to protect its revenue. In this paper, we study the characterizations of the Shapley–Solidarity value from both cooperative and non-cooperative aspects.

Firstly, we present two axiomatic characterizations of the Shapley-Solidarity value. We characterized the Shapley–Solidarity value is the only coalitional value that satisfies efficiency, coalitional symmetry, coalitional strong marginality, intracoalitional equal averaged gains (intracoalitional quasi-balanced contributions). Additioally, the Shapley–Solidarity value is the only coalitional value that satisfies efficiency, additivity, coalitional symmetry, null union, intracoalitional equal averaged gains (intracoalitional quasi-balanced contributions). In the axiom system, the axiom of intracoalitional equal averaged gains can be replaced by the axiom of intracoalitional quasi-balanced contributions for these two properties are equivalent for a coalitional value.

Secondly, we combine the A-potential function for the Solidarity value and the potential function for the Owen value, in order to obtain the potential function for the Shapley–Solidarity value. We prove that there exists a unique adjusted potential function for games with coalition structure. Moreover, for each player of a game with with coalition structure, the player's marginal contribution of the unique adjusted potential function coincides with its Shapley–Solidarity value.

Finally, we propose a bidding mechanism in order to implement the Solidarity value. At beginning, we introduce a new bidding mechanism for the Solidarity value, inspiring from Pérez-Castrillo and Wettestein's bidding mechanism for the Shapley value. In our bidding mechanism for the Solidarity value, if the offer is rejected, then a player is randomly chosen from all players with probability $\frac{1}{n}$ leaving the grand coalition and gets nothing. It means that all players (not only the proposer or any rejector) need to take the responsibility for the breakdown of the coalition when the proposal is rejected, from the property of solidarity. This is the main difference with the Pérez-Castrillo and Wettstein's implementation of the Shapley value. Subsequengly, by changing the first round of the two round mechanism in the original implementation for the Owen value, we extend the coalitional bidding mechanism that was introduced by Vidal-Puga and Begantiños to implement the Shapley–Solidarity value.

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