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A New Algorithm for the Common Solutions of a Generalized Variational Inequality System and a Nonlinear Operator Equation in Banach Spaces

Yuanheng Wang ^{*}, Cancan Li [†] and Lirong Lu [†]

College of Mathematics and Computer Science, with College of Xingzhi, Zhejiang Normal University, 268 Yingbin Road, Jinhua 321004, Zhejiang, China; ccli521133@163.com (C.L.); yhwang@zjnu.edu.cn (L.L.)

* Correspondence: yhwang@zjnu.cn; Tel.: +86-579-8229-8258

† These authors contributed equally to this work.

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Abstract: We study a new algorithm for the common solutions of a generalized variational inequality system and the fixed points of an asymptotically non-expansive mapping in Banach spaces. Under some specific assumptions imposed on the control parameters, some strong convergence theorems for the sequence generated by our new viscosity iterative scheme to approximate their common solutions are proved. As an application of our main results, we solve the standard constrained convex optimization problem. The results here generalize and improve some other authors' recently corresponding results.

Keywords: asymptotically non-expansive mapping; general variational inequality system; strong convergence; Banach space

MSC: 47H10; 47H09; 47H05; 47J25

1. Introduction

In the paper, we assume that X is a real Banach space and X^* is the dual space of X . The normalized duality mapping $J : X \rightarrow 2^{X^*}$ is defined by

$$J(v) = \{f \in X^* : \langle v, f \rangle = \|v\| \|f\|, \|v\| = \|f\|\}, \forall v \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^* . As we all know, the mapping J is well defined and J is identity mapping if and only if X is a Hilbert space. In general, J is multiple-valued and nonlinear, $j \in J$.

A Banach space X is called strictly convex if $\frac{\|x+y\|}{2} < 1$ for $\|x\| = \|y\| = 1$ and $x \neq y$. The modulus of convexity of X is defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{1}{2} \|v+u\| : \|v\| = \|u\| \leq 1, \|v-u\| = \epsilon \right\},$$

for all $0 < \epsilon \leq 2$. X is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \leq 2$.

Let $\rho_X : [0, +\infty) \rightarrow [0, +\infty)$ be the modulus of smoothness of X defined by

$$\rho_X(\zeta) = \sup \left\{ \frac{1}{2} (\|v+u\| + \|v-u\|) - 1 : \|v\| = 1, \|u\| = \zeta \right\}.$$

A Banach space X is called uniformly smooth if $\frac{\rho_X(\zeta)}{\zeta} \rightarrow 0$ as $\zeta \rightarrow 0$. Moreover, X is uniformly smooth if the norm of X is uniformly Fréchet differentiable.

Suppose $\emptyset \neq C \subset X, T : C \rightarrow C$ is an operator and the set of fixed points of T is denoted by $\text{Fix}(T)$, that is $\text{Fix}(T) = \{v \in C : v = Tv\}$. T is said to be asymptotically non-expansive, if there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|T^n v - T^n u\| \leq k_n \|v - u\|, \forall v, u \in C.$$

If $k_n \equiv 1$, then we call that T is non-expansive. T is said to be uniformly asymptotically regular iff $\lim_{n \rightarrow \infty} \|T^{n+1}v - T^n v\| = 0, \forall v \in C$. T is said to be a contraction when there exists a number $\eta \in (0, 1)$ such that $\|Tv - Tu\| \leq \eta \|v - u\|, \forall v, u \in C$.

A mapping $\widehat{B} : C \rightarrow X$ is called inverse strongly accretive if there exists $e \in \mathbb{R}^+$ such that

$$\langle \widehat{B}v - \widehat{B}u, j(v - u) \rangle \geq e \|\widehat{B}v - \widehat{B}u\|^2, \forall v, u \in C.$$

A bounded linear operator $\widehat{B}$ on X is called strongly positive, if there exists a fixed number $\bar{\delta} > 0$ such that

$$\langle \widehat{B}v, j(v) \rangle \geq \bar{\delta} \|v\|^2, \|\widehat{B}\| = \sup_{\|v\| \leq 1} | \langle (e_1 I - e_2 \widehat{B})v, j(v) \rangle |, e_1 \in [0, 1], e_2 \in [-1, 1], \forall v \in X.$$

Assume $E \neq \emptyset, E \subset C \subseteq X$ is a convex and closed set. An operator $Q : C \rightarrow E$ is called sunny when Q has the following relation:

$$Q(v + \alpha(v - Q(v))) = Q(v), \forall v \in C, \alpha \geq 0,$$

for $v + \alpha(v - Q(v)) \in C$. A mapping $Q : C \rightarrow E$ is called retraction if $Qv = v, \forall v \in E$. Moreover, Q is called sunny non-expansive retraction from C onto E iff Q is a sunny, retraction, and non-expansive operator from C onto E .

Let C be a nonempty closed convex subset of X and $A, B : C \rightarrow X$ be two nonlinear mappings, respectively. Recall the classical variational inequality problem is to find a point $p \in C$ such that

$$\langle Ap, j(v - p) \rangle \geq 0, \forall v \in C. \quad (1)$$

The set of the solutions of the variational inequality (1) is denoted by $VI(C, A)$. Lots of problems in physics, optimization, differential equation (inclusion), finance and minimax problem reduce to find an element of (1) and relevant numerical analysis methods can be considered to solve the problems, see [1–16].

In 2011, Katchang et al. [17] introduced the following system of general variational inequalities. They considered the problem of finding $(t, s) \in C \times C$ such that:

$$\begin{cases} \langle \lambda As + t - s, j(v - t) \rangle \geq 0, \forall v \in C, \\ \langle \mu Bt + s - t, j(v - s) \rangle \geq 0, \forall v \in C, \end{cases} \quad (2)$$

and proved a strong convergence theorem for the following sequence of problem (2):

$$\begin{cases} u_n = Q_C(I - \mu B)v_n, \\ v_{n+1} = \alpha_n f(v_n) + \beta_n v_n + \gamma_n [\beta S v_n + (1 - \beta) Q_C(I - \lambda A)u_n], \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$, λ and μ are two positive real numbers, $0 < \beta < 1$, Q_C is the sunny non-expansive retraction from X to C , f is a contractive mapping on C and $S : C \rightarrow C$ is a non-expansive mapping.

In 2020, Wang et al. [18] investigated a new general variational inequality system:

$$\begin{cases} \langle (I - \lambda A)(bt + (1 - b)s) - t, j(v - t) \rangle \leq 0, \forall v \in C, \\ \langle (I - \mu B)t - s, j(v - s) \rangle \leq 0, \forall v \in C, \end{cases} \quad (3)$$

and proposed the following iterative method:

$$\begin{cases} w_n = Q_C(I - \mu B)v_n, \\ l_n = Q_C(I - \lambda A)(bv_n + (1 - b)l_n), \\ u_n = \delta_n v_n + (1 - \delta_n)l_n, \\ v_{n+1} = \alpha_n f(v_n) + \beta_n v_n + \gamma_n T^n u_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$, λ and μ are two positive real numbers, $0 \leq b \leq 1$, Q_C is the sunny non-expansive retraction from X to C , f is a contractive mapping on C and $T : C \rightarrow C$ is an asymptotically non-expansive mapping. Then they proved that v_n converges to the common element of the fixed points of an asymptotically non-expansive mapping and the solutions of the general variational inequality system (3).

Inspired by the above mentioned results, we study the following new generalized variational inequality system in uniformly convex and uniformly smooth Banach spaces:

$$\begin{cases} \langle t - (I - \lambda_1 B_1)(bt + (1 - b)s), j(v - t) \rangle \geq 0, \forall v \in C, \\ \langle s - (I - \lambda_2 B_2)(bt + (1 - b)w), j(v - s) \rangle \geq 0, \forall v \in C, \\ \langle w - (I - \lambda_3 B_3)t, j(v - w) \rangle \geq 0, \forall v \in C. \end{cases} \quad (4)$$

This is the main problem which we seek to solve in this work and construct the suitable abstract iterative method. The new generalized variational inequality system problem (4) contains (1)–(3) as special cases. We construct a new iterative algorithm to approximate the common solutions of the generalized system of variational inequality (4) and an asymptotically non-expansive mapping. Some strong convergence theorems that depend on some suitable conditions are proved. Eventually, we show an application for solving the standard constrained problem of convex optimization to illustrate the efficiency of our main theorem. Some other outcomes proposed by other authors are also improved, see [9,10,17–22].

2. Preliminaries

This section contains the useful lemmas that are indispensable for proving our convergence theorem in the following section. Generally speaking, we suppose that space X is uniformly smooth and uniformly convex in this section.

Lemma 1 ([18]). Suppose that $X \supset C \neq \emptyset$ and C is a closed convex set. Let J be the normalized duality mapping $J : X \rightarrow 2^{X^*}$ and $Q : X \rightarrow C$ be a retraction. Then the followings are equivalent:

- (I) Q is sunny and non-expansive;
- (II) $\langle v - Qv, J(u - Qv) \rangle \leq 0, \forall v \in X, u \in C$;
- (III) $\|Qv - Qu\|^2 \leq \langle v - u, J(Qv - Qu) \rangle, \forall v, u \in X$.

Lemma 2 ([23]). Assume X is uniformly convex and $V_R(0)$ is a closed ball of X . Then there exists a continuous strictly increasing convex function $m_1 : [0, \infty) \rightarrow [0, \infty)$ with $m_1(0) = 0$ such that

$$\|a_1 v_1 + a_2 v_2 + \cdots + a_n v_n\|^2 = \sum_{i=1}^n a_i \|v_i\|^2 - a_x a_y m_1(\|v_x - v_y\|),$$

for any $x, y \in \{1, 2, \dots, n\}$ and for $v_i \in V_R(0) := \{v \in X : \|v\| \leq R\}$, $i = 1, 2, \dots, n$ with $\sum_{i=1}^n a_i = 1$.

Lemma 3 ([24]). Suppose $\{v_n\}$ and $\{z_n\}$ are two bounded sequences of X . Let $\{\mu_n\} \subset [0, 1]$ be a number sequence with $0 < \liminf_{n \rightarrow \infty} \mu_n \leq \limsup_{n \rightarrow \infty} \mu_n < 1$. Assume that $v_{n+1} = (1 - \mu_n)v_n + \mu_n z_n$, $n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|v_{n+1} - v_n\|) \leq 0$. Then $\lim_{n \rightarrow \infty} \|z_n - v_n\| = 0$.

Lemma 4 ([25]). For $\forall v, u \in X$, $j(v + u) \in J(v + u)$, there holds the relation:

$$\|v + u\|^2 \leq \|v\|^2 + 2\langle u, j(v + u) \rangle.$$

Lemma 5 ([26]). Let $\{c_n\}$ be a nonnegative number sequence satisfying:

$$c_{n+1} \leq (1 - \sigma_n)c_n + \sigma_n \theta_n, \quad \forall n \geq 0,$$

where

$$\{\sigma_n\} \subset [0, 1], \quad \sum_{n=0}^{\infty} \sigma_n = +\infty; \quad \limsup_{n \rightarrow \infty} \theta_n \leq 0 \text{ or } \sum_{n=1}^{\infty} |\sigma_n \theta_n| < +\infty.$$

Then $\lim_{n \rightarrow \infty} c_n = 0$.

Lemma 6 ([27]). Suppose that $\bar{V}$ is a strongly positive linear bounded operator on X with coefficient $\bar{\delta} > 0$ and $0 < \rho \leq \|\bar{V}\|^{-1}$. Then, $\|I - \rho \bar{V}\| \leq 1 - \rho \bar{\delta}$.

Lemma 7 ([18]). Assume C is a nonempty convex and closed subset of X . Assume mapping $\hat{B} : C \rightarrow X$ is a $\hat{b}$ -inverse strongly accretive. Then,

$$\|(I - \lambda \hat{B})v - (I - \lambda \hat{B})u\|^2 \leq \|v - u\|^2 - \lambda(2\hat{b} - \lambda)\|\hat{B}v - \hat{B}u\|^2, \quad \lambda > 0.$$

If $0 < \lambda < 2\hat{b}$, then $I - \lambda \hat{B}$ is non-expansive.

Lemma 8 ([28]). Let $C \neq \emptyset$ be a closed convex subset of X . Suppose $Q_C : X \rightarrow C$ is a sunny non-expansive mapping and $\hat{B}$ is an accretive operator from C into X . Then,

$$VI(C, \hat{B}) = \text{Fix}(Q_C(I - \lambda \hat{B})), \quad \lambda > 0,$$

where $VI(C, \hat{B}) = \{t \in C : \langle \hat{B}t, j(v - t) \rangle \geq 0, \forall v \in C\}$.

Lemma 9 ([18]). There exists a convex continuous strictly increasing function $m_2 : [0, 2r] \rightarrow \mathbb{R}$, $m_2(0) = 0$ such that

$$m_2(\|v - u\|) \leq \|v\|^2 - 2\langle v, j(u) \rangle + \|u\|^2, \quad \forall r > 0, v, u \in B_r.$$

Lemma 10 ([9]). Assume $\emptyset \neq C \subset X$ is a closed convex set. Suppose $T : C \rightarrow C$ is an asymptotically non-expansive mapping and $\text{Fix}(T) \neq \emptyset$. If X has a weakly sequentially continuous duality mapping J , then the mapping $I - T$ is demi-closed at zero, where I is the identity mapping, i.e., if $v_n \rightarrow v$ and $\|v_n - Tv_n\| \rightarrow 0$, then $v = Tv$.

Lemma 11. Assume $X \supset C \neq \emptyset$ is a closed convex set. Let $B_1, B_2, B_3 : C \rightarrow X$ be three mappings, respectively. When $\lambda_1, \lambda_2, \lambda_3 > 0, b \in [0, 1]$, then the following statements are the same:

- (i) $(t, s, w) \in C \times C \times C$ is a solution of the problem (4);
- (ii) t is a fixed point of the mapping L , i.e., $t \in \text{Fix}(L)$, where $L : C \rightarrow C$ is defined by

$$L(v) = Q_C(I - \lambda_1 B_1)(bv + (1 - b)Q_C(I - \lambda_2 B_2)(bv + (1 - b)Q_C(I - \lambda_3 B_3)v)), \quad \forall v \in C, \quad (5)$$

$$\text{and } s = Q_C(I - \lambda_2 B_2)(bt + (1 - b)w), \quad w = Q_C(I - \lambda_3 B_3)t.$$

Proof. (i) $\Rightarrow$ (ii). Assume that $(t, s, w) \in C \times C \times C$ is a solution of (4). For all $v \in C$, we have

$$\begin{aligned} \langle t - (I - \lambda_1 B_1)(bt + (1 - b)s), j(v - t) \rangle &\geq 0, \\ \langle s - (I - \lambda_2 B_2)(bt + (1 - b)w), j(v - s) \rangle &\geq 0, \\ \langle w - (I - \lambda_3 B_3)t, j(v - w) \rangle &\geq 0. \end{aligned}$$

From property of Q_C , we get

$$\begin{aligned} t &= Q_C(I - \lambda_1 B_1)(bt + (1 - b)s), \\ s &= Q_C(I - \lambda_2 B_2)(bt + (1 - b)w), \\ w &= Q_C(I - \lambda_3 B_3)t. \end{aligned}$$

It implies that

$$t = Q_C(I - \lambda_1 B_1)(bt + (1 - b)Q_C(I - \lambda_2 B_2)(bt + (1 - b)Q_C(I - \lambda_3 B_3)t)) = L(t).$$

Then $t \in \text{Fix}(L)$, where $s = Q_C(I - \lambda_2 B_2)(bt + (1 - b)w)$ and $w = Q_C(I - \lambda_3 B_3)t$.

(ii) $\Rightarrow$ (i). Let $t \in \text{Fix}(L)$, $s = Q_C(I - \lambda_2 B_2)(bt + (1 - b)w)$ and $w = Q_C(I - \lambda_3 B_3)t$. Since $t \in \text{Fix}(L)$, we have

$$\begin{aligned} t &= Q_C(I - \lambda_1 B_1)(bt + (1 - b)Q_C(I - \lambda_2 B_2)(bt + (1 - b)Q_C(I - \lambda_3 B_3)t)) \\ &= Q_C(I - \lambda_1 B_1)(bt + (1 - b)s). \end{aligned}$$

In combination with $s = Q_C(I - \lambda_2 B_2)(bt + (1 - b)w)$, $w = Q_C(I - \lambda_3 B_3)t$ and the property of Q_C , we get

$$\begin{aligned} \langle t - (I - \lambda_1 B_1)(bt + (1 - b)s), j(v - t) \rangle &\geq 0, \quad \forall v \in C, \\ \langle s - (I - \lambda_2 B_2)(bt + (1 - b)w), j(v - s) \rangle &\geq 0, \quad \forall v \in C, \\ \langle w - (I - \lambda_3 B_3)t, j(v - w) \rangle &\geq 0, \quad \forall v \in C. \end{aligned}$$

Then $(t, s, w) \in C \times C \times C$ is a solution of the problem (4). $\square$

3. Results

Theorem 1. Suppose X is a uniformly smooth and uniformly convex Banach space and $C \neq \emptyset$ is a closed convex subset of X . Assume $Q_C : X \rightarrow C$ is a sunny non-expansive retraction. Let $\widehat{B}, B_1, B_2, B_3 : C \rightarrow X$ be $\widehat{b}, b_1, b_2, b_3$ -inverse strongly accretive, respectively. L is the mapping defined by (5) in Lemma 11 and $b \in [0, 1)$. Assume $\overline{V} : X \rightarrow X$ is a bounded linear strongly positive operator on X with coefficient $\overline{\delta} > 0$ and $0 < \overline{\delta} < \|\overline{V}\|^{-1}$. Let g be an $\eta \in (0, 1)$ -contraction mapping on C and $T : C \rightarrow C$ be an asymptotically non-expansive mapping with $\Psi = \text{Fix}(T) \cap \text{Fix}(L) \cap VI(C, \widehat{B}) \neq \emptyset$. Assume $\{v_n\}$ is defined as follows:

$$\begin{cases} v_1 \in C, \\ u_n = \beta_n \delta g(v_n) + (I - \beta_n \overline{V})Lv_n, \\ v_{n+1} = \mu_n v_n + \nu_n T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) + \omega_n Q_C(I - \lambda \widehat{B})u_n, \end{cases} \quad n \geq 0, \quad (6)$$

where $\lambda \in (0, 2\widehat{b})$, $\{\beta_n\} \subset (0, 1)$, $\mu_n + \nu_n + \omega_n = 1$, and the following coefficient conditions are satisfying:

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
- (ii) $0 < h_1 \leq \mu_n, \nu_n, \omega_n \leq h_2 < 1$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$, $\sum_{n=1}^{\infty} |\nu_{n+1} - \nu_n| < \infty$, $\sum_{n=1}^{\infty} |\omega_{n+1} - \omega_n| < \infty$;
- (iii) $0 \leq \xi_n \leq \xi_{n+1} \leq 1$, $0 < \delta < \frac{h_1 \bar{\delta}}{2\eta}$, $k_n^2 - 1 = \varepsilon \beta_n$, $0 < \varepsilon < h_1 \bar{\delta} - 2h_1 \delta \eta$;
- (iv) $\nu_n k_n^2 (1 - \xi_n) < 1 - \tau$, $h_2 < \tau < 1$.

Then the sequence $\{v_n\}$ converges strongly to $t = Q_{\Psi}(I - \bar{V} + \delta g)t \in \Psi$ and (t, s, w) is a solution of (4), where $s = Q_C(I - \lambda_2 B_2)(bt + (1 - b)w)$ and $w = Q_C(I - \lambda_3 B_3)t$, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\bar{b})$ with $\bar{b} = \min\{b_1, b_2, b_3\}$.

Proof. Our proof process can be divided into 6 steps.

Step 1: Let $\hat{B}, B_1, B_2, B_3$ be $\hat{b}, b_1, b_2, b_3$ -inverse strongly accretive mapping, respectively, and $\lambda \in (0, 2\hat{b}), \lambda_1, \lambda_2, \lambda_3 \in (0, 2\bar{b})$ with $\bar{b} = \min\{b_1, b_2, b_3\}$. From Lemma 7, we know that $(I - \lambda \hat{B}), (I - \lambda_1 B_1), (I - \lambda_2 B_2)$ and $(I - \lambda_3 B_3)$ are non-expansive mappings. Moreover, we attain that $Q_C(I - \lambda \hat{B}), Q_C(I - \lambda_1 B_1), Q_C(I - \lambda_2 B_2)$ and $Q_C(I - \lambda_3 B_3)$ are non-expansive mapping.

So, we have

$$\begin{aligned}
 \|Lv - Lu\| &= \|Q_C(I - \lambda_1 B_1)(bv + (1 - b)Q_C(I - \lambda_2 B_2)(bv + (1 - b)Q_C(I - \lambda_3 B_3)v)) \\
 &\quad - Q_C(I - \lambda_1 B_1)(bu + (1 - b)Q_C(I - \lambda_2 B_2)(bu + (1 - b)Q_C(I - \lambda_3 B_3)u)))\| \\
 &\leq \|(bv + (1 - b)Q_C(I - \lambda_2 B_2)(bv + (1 - b)Q_C(I - \lambda_3 B_3)v)) \\
 &\quad - (bu + (1 - b)Q_C(I - \lambda_2 B_2)(bu + (1 - b)Q_C(I - \lambda_3 B_3)u)))\| \\
 &\leq b\|v - u\| + (1 - b)\|Q_C(I - \lambda_2 B_2)(bv + (1 - b)Q_C(I - \lambda_3 B_3)v) \\
 &\quad - Q_C(I - \lambda_2 B_2)(bu + (1 - b)Q_C(I - \lambda_3 B_3)u))\| \\
 &\leq b\|v - u\| + (1 - b)\|(bv + (1 - b)Q_C(I - \lambda_3 B_3)v) - (bu + (1 - b)Q_C(I - \lambda_3 B_3)u)\| \\
 &\leq b\|v - u\| + (1 - b)(b\|v - u\| + (1 - b)\|Q_C(I - \lambda_3 B_3)v - Q_C(I - \lambda_3 B_3)u\|) \\
 &\leq b\|v - u\| + (1 - b)(b\|v - u\| + (1 - b)\|v - u\|) \\
 &= \|v - u\|.
 \end{aligned}$$

Then, L is non-expansive.

Step 2: For any given $q \in \Psi$, from (6), we have

$$\begin{aligned}
 \|T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) - q\| &\leq k_n \|\xi_n v_n + (1 - \xi_n)v_{n+1} - q\| \\
 &\leq k_n \xi_n \|v_n - q\| + k_n(1 - \xi_n) \|v_{n+1} - q\|,
 \end{aligned}$$

and

$$\begin{aligned}
 \|u_n - q\| &= \|\beta_n \delta g(v_n) + (I - \beta_n \bar{V})Lv_n - q\| \\
 &\leq \beta_n \|\delta g(v_n) - \bar{V}q\| + \|(I - \beta_n \bar{V})(Lv_n - q)\| \\
 &\leq \beta_n \|\delta g(v_n) - \delta g(q)\| + \beta_n \|\delta g(q) - \bar{V}q\| + (1 - \beta_n \bar{\delta}) \|v_n - q\| \\
 &\leq \beta_n \delta \eta \|v_n - q\| + \beta_n \|\delta g(q) - \bar{V}q\| + (1 - \beta_n \bar{\delta}) \|v_n - q\| \\
 &\leq [1 - \beta_n (\bar{\delta} - \delta \eta)] \|v_n - q\| + \beta_n \|\delta g(q) - \bar{V}q\|.
 \end{aligned}$$

Then we compute

$$\begin{aligned}
 \|v_{n+1} - q\| &= \|\mu_n v_n + \nu_n T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) + \omega_n Q_C(I - \lambda \hat{B})u_n - q\| \\
 &\leq \|\mu_n(v_n - q) + \nu_n(T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) - q) + \omega_n(Q_C(I - \lambda \hat{B})u_n - q)\| \\
 &\leq \mu_n \|v_n - q\| + \nu_n \|T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) - q\| + \omega_n \|u_n - q\| \\
 &\leq \mu_n \|v_n - q\| + \nu_n k_n \xi_n \|v_n - q\| + \nu_n k_n (1 - \xi_n) \|v_{n+1} - q\| \\
 &\quad + \omega_n [1 - \beta_n(\bar{\delta} - \delta\eta)] \|v_n - q\| + \omega_n \beta_n \|\delta g(q) - \bar{V}q\| \\
 &\leq (\mu_n + \nu_n k_n \xi_n + \omega_n [1 - \beta_n(\bar{\delta} - \delta\eta)]) \|v_n - q\| + \nu_n k_n (1 - \xi_n) \|v_{n+1} - q\| \\
 &\quad + \omega_n \beta_n \|\delta g(q) - \bar{V}q\|.
 \end{aligned}$$

It follows that

$$(1 - \nu_n k_n (1 - \xi_n)) \|v_{n+1} - q\| \leq (\mu_n + \nu_n k_n \xi_n + \omega_n [1 - \beta_n(\bar{\delta} - \delta\eta)]) \|v_n - q\| + \omega_n \beta_n \|\delta g(q) - \bar{V}q\|.$$

From conditions (ii), (iii) and (iv) in Theorem 1, we get

$$\begin{aligned}
 \|v_{n+1} - q\| &\leq \frac{\mu_n + \nu_n k_n \xi_n + \omega_n [1 - \beta_n(\bar{\delta} - \delta\eta)]}{1 - \nu_n k_n (1 - \xi_n)} \|v_n - q\| + \frac{\omega_n \beta_n}{1 - \nu_n k_n (1 - \xi_n)} \|\delta g(q) - \bar{V}q\| \\
 &\leq (1 - \frac{\omega_n \beta_n (\bar{\delta} - \delta\eta) - \nu_n (k_n - 1)}{1 - \nu_n k_n (1 - \xi_n)}) \|v_n - q\| + \frac{\omega_n \beta_n}{1 - \nu_n k_n (1 - \xi_n)} \|\delta g(q) - \bar{V}q\| \\
 &\leq (1 - \frac{h_1 \beta_n (\bar{\delta} - \delta\eta) - \varepsilon \beta_n}{1 - \nu_n k_n (1 - \xi_n)}) \|v_n - q\| + \frac{\beta_n}{1 - \nu_n k_n (1 - \xi_n)} \|\delta g(q) - \bar{V}q\| \\
 &\leq (1 - \frac{\beta_n (h_1 \bar{\delta} - h_1 \delta\eta - \varepsilon)}{1 - \nu_n k_n (1 - \xi_n)}) \|v_n - q\| + \frac{\beta_n (h_1 \bar{\delta} - h_1 \delta\eta - \varepsilon)}{1 - \nu_n k_n (1 - \xi_n)} \frac{\|\delta g(q) - \bar{V}q\|}{h_1 \bar{\delta} - h_1 \delta\eta - \varepsilon} \\
 &\leq \max\{\|v_n - q\|, \frac{\|\delta g(q) - \bar{V}q\|}{h_1 \bar{\delta} - h_1 \delta\eta - \varepsilon}\}.
 \end{aligned}$$

By induction, we readily obtain

$$\|v_{n+1} - q\| \leq \max\{\|v_1 - q\|, \frac{\|\delta g(q) - \bar{V}q\|}{h_1 \bar{\delta} - h_1 \delta\eta - \varepsilon}\}.$$

This implies that $\{v_n\}$ is bounded, and so are $\{u_n\}$, $\{g(v_n)\}$, $\{Lv_n\}$, $\{T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\}$.

Step 3: By (6) and Lemma 6, we gain that

$$\begin{aligned}
 \|u_{n+1} - u_n\| &= \|\beta_{n+1} \delta g(v_{n+1}) + (I - \beta_{n+1} \bar{V})Lv_{n+1} - (\beta_n \delta g(v_n) + (I - \beta_n \bar{V})Lv_n)\| \\
 &= \|\beta_{n+1} \delta g(v_{n+1}) - \beta_{n+1} \delta g(v_n) + \beta_{n+1} \delta g(v_n) - \beta_n \delta g(v_n) \\
 &\quad + (I - \beta_{n+1} \bar{V})Lv_{n+1} - (I - \beta_{n+1} \bar{V})Lv_n + (I - \beta_{n+1} \bar{V})Lv_n - (I - \beta_n \bar{V})Lv_n\| \\
 &= \|\beta_{n+1} [\delta g(v_{n+1}) - \delta g(v_n)] + (\beta_{n+1} - \beta_n) \delta g(v_n) + (I - \beta_{n+1} \bar{V})(Lv_{n+1} - Lv_n) \\
 &\quad - (\beta_{n+1} - \beta_n) \bar{V}Lv_n\| \\
 &\leq \beta_{n+1} \delta \|g(v_{n+1}) - g(v_n)\| + |\beta_{n+1} - \beta_n| \|\delta g(v_n) - \bar{V}Lv_n\| + \|I - \beta_{n+1} \bar{V}\| \|Lv_{n+1} - Lv_n\| \\
 &\leq \beta_{n+1} \delta \|v_{n+1} - v_n\| + |\beta_{n+1} - \beta_n| \|\delta g(v_n) - \bar{V}Lv_n\| + (1 - \beta_{n+1} \bar{\delta}) \|v_{n+1} - v_n\| \\
 &= (1 - \beta_{n+1} (\bar{\delta} - \delta\eta)) \|v_{n+1} - v_n\| + |\beta_{n+1} - \beta_n| \|\delta g(v_n) - \bar{V}Lv_n\| \\
 &\leq \|v_{n+1} - v_n\| + |\beta_{n+1} - \beta_n| \|\delta g(v_n) - \bar{V}Lv_n\|.
 \end{aligned} \tag{7}$$

By (6) and (7), we can acquire

$$\begin{aligned}
\|v_{n+2} - v_{n+1}\| &= \|\mu_{n+1}v_{n+1} + \nu_{n+1}T^{n+1}(\xi_{n+1}v_{n+1} + (1 - \xi_{n+1})v_{n+2}) + \omega_{n+1}Q_C(I - \lambda\hat{B})u_{n+1} \\
&\quad - [\mu_nv_n + \nu_nT^n(\xi_nv_n + (1 - \xi_n)v_{n+1}) + \omega_nQ_C(I - \lambda\hat{B})u_n]\| \\
&= \|\mu_{n+1}(v_{n+1} - v_n) + (\mu_{n+1} - \mu_n)v_n + \nu_{n+1}(T^{n+1}(\xi_{n+1}v_{n+1} + (1 - \xi_{n+1})v_{n+2}) \\
&\quad - T^{n+1}(\xi_nv_n + (1 - \xi_n)v_{n+1})) + (\nu_{n+1} - \nu_n)T^{n+1}(\xi_nv_n + (1 - \xi_n)v_{n+1}) \\
&\quad + \nu_n[T^{n+1}(\xi_nv_n + (1 - \xi_n)v_{n+1}) - T^n(\xi_nv_n + (1 - \xi_n)v_{n+1})] \\
&\quad + \omega_{n+1}[Q_C(I - \lambda\hat{B})u_{n+1} - Q_C(I - \lambda\hat{B})u_n] + (\omega_{n+1} - \omega_n)Q_C(I - \lambda\hat{B})u_n\| \\
&\leq \mu_{n+1}\|v_{n+1} - v_n\| + |\mu_{n+1} - \mu_n|\|v_n\| + \nu_{n+1}k_{n+1}\|(\xi_{n+1}v_{n+1} + (1 - \xi_{n+1})v_{n+2}) \\
&\quad - (\xi_nv_n + (1 - \xi_n)v_{n+1}))\| + |\nu_{n+1} - \nu_n|\|T^{n+1}v\| + \nu_n \sup_{v \in C} \|T^{n+1}v - T^n v\| \\
&\quad + \omega_{n+1}\|Q_C(I - \lambda\hat{B})u_{n+1} - Q_C(I - \lambda\hat{B})u_n\| + |\omega_{n+1} - \omega_n|\|Q_C(I - \lambda\hat{B})u_n\| \\
&\leq \mu_{n+1}\|v_{n+1} - v_n\| + |\mu_{n+1} - \mu_n|\|v_n\| + \nu_{n+1}k_{n+1}\xi_n\|v_{n+1} - v_n\| \\
&\quad + \nu_{n+1}k_{n+1}(1 - \xi_{n+1})\|v_{n+2} - v_{n+1}\| + |\nu_{n+1} - \nu_n|\|T^{n+1}v\| \\
&\quad + \nu_n \sup_{v \in C} \|T^{n+1}v - T^n v\| + \omega_{n+1}\|u_{n+1} - u_n\| + |\omega_{n+1} - \omega_n|\|Q_C(I - \lambda\hat{B})u_n\| \\
&\leq \mu_{n+1}\|v_{n+1} - v_n\| + |\mu_{n+1} - \mu_n|\|v_n\| + \nu_{n+1}k_{n+1}\xi_n\|v_{n+1} - v_n\| \\
&\quad + \nu_{n+1}k_{n+1}(1 - \xi_{n+1})\|v_{n+2} - v_{n+1}\| + |\nu_{n+1} - \nu_n|\|T^{n+1}v\| \\
&\quad + \nu_n \sup_{v \in C} \|T^{n+1}v - T^n v\| + \omega_{n+1}(\|v_{n+1} - v_n\| + |\beta_{n+1} - \beta_n|\|\delta g(v_n) - \bar{V}Lv_n\|) \\
&\quad + |\omega_{n+1} - \omega_n|\|Q_C(I - \lambda\hat{B})u_n\| \\
&\leq (\mu_{n+1} + \nu_{n+1}k_{n+1}\xi_n + \omega_{n+1})\|v_{n+1} - v_n\| + \nu_{n+1}k_{n+1}(1 - \xi_{n+1})\|v_{n+2} - v_{n+1}\| \\
&\quad + |\mu_{n+1} - \mu_n|\|v_n\| + |\nu_{n+1} - \nu_n|\|T^{n+1}v\| + \nu_n \sup_{v \in C} \|T^{n+1}v - T^n v\| \\
&\quad + \omega_{n+1}|\beta_{n+1} - \beta_n|\|\delta g(v_n) - \bar{V}Lv_n\| + |\omega_{n+1} - \omega_n|\|Q_C(I - \lambda\hat{B})u_n\|.
\end{aligned}$$

Which implies that

$$\begin{aligned}
&[1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})]\|v_{n+2} - v_{n+1}\| \\
&\leq (\mu_{n+1} + \nu_{n+1}k_{n+1}\xi_n + \omega_{n+1})\|v_{n+1} - v_n\| + |\mu_{n+1} - \mu_n|\|v_n\| + |\nu_{n+1} - \nu_n|\|T^{n+1}v\| \\
&\quad + \nu_n \sup_{v \in C} \|T^{n+1}v - T^n v\| + \omega_{n+1}|\beta_{n+1} - \beta_n|\|\delta g(v_n) - \bar{V}Lv_n\| + |\omega_{n+1} - \omega_n|\|Q_C(I - \lambda\hat{B})u_n\|.
\end{aligned}$$

By conditions (ii)–(iv) in Theorem 1, we have

$$\begin{aligned}
 & \|v_{n+2} - v_{n+1}\| \\
 & \leq \frac{\mu_{n+1} + \nu_{n+1}k_{n+1}\xi_n + \omega_{n+1}}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|v_{n+1} - v_n\| + \frac{|\mu_{n+1} - \mu_n|}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|v_n\| \\
 & \quad + \frac{|\nu_{n+1} - \nu_n|}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|T^{n+1}v\| + \frac{\nu_n}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \sup_{v \in C} \|T^{n+1}v - T^n v\| \\
 & \quad + \frac{\omega_{n+1}|\beta_{n+1} - \beta_n|}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|\delta g(v_n) - \bar{V}Lv_n\| + \frac{|\omega_{n+1} - \omega_n|}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|Q_C(I - \lambda\hat{B})u_n\| \\
 & \leq [1 + \frac{\nu_{n+1}(k_{n+1} - 1) - \nu_{n+1}k_{n+1}(\xi_{n+1} - \xi_n)}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})}] \|v_{n+1} - v_n\| + \frac{|\mu_{n+1} - \mu_n|}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|v_n\| \\
 & \quad + \frac{|\nu_{n+1} - \nu_n|}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|T^{n+1}v\| + \frac{\nu_n}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \sup_{v \in C} \|T^{n+1}v - T^n v\| \quad (8) \\
 & \quad + \frac{\omega_{n+1}|\beta_{n+1} - \beta_n|}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|\delta g(v_n) - \bar{V}Lv_n\| + \frac{|\omega_{n+1} - \omega_n|}{1 - \nu_{n+1}k_{n+1}(1 - \xi_{n+1})} \|Q_C(I - \lambda\hat{B})u_n\| \\
 & \leq [1 + \frac{\nu_{n+1}(k_{n+1} - 1)}{\tau}] \|v_{n+1} - v_n\| + \frac{\omega_{n+1}|\beta_{n+1} - \beta_n|}{\tau} \|\delta g(v_n) - \bar{V}Lv_n\| \\
 & \quad + \frac{|\mu_{n+1} - \mu_n|}{\tau} \|v_n\| + \frac{|\nu_{n+1} - \nu_n|}{\tau} \|T^{n+1}v\| + \frac{\nu_n}{\tau} \sup_{v \in C} \|T^{n+1}v - T^n v\| \\
 & \quad + \frac{|\omega_{n+1} - \omega_n|}{\tau} \|Q_C(I - \lambda\hat{B})u_n\| \\
 & \leq (1 + \varepsilon\beta_{n+1}) \|v_{n+1} - v_n\| + M_{n_1},
 \end{aligned}$$

where

$$\begin{aligned}
 M_{n_1} = & \frac{|\mu_{n+1} - \mu_n|}{\tau} \|v_n\| + \frac{|\nu_{n+1} - \nu_n|}{\tau} \|T^{n+1}v\| + \frac{\nu_n}{\tau} \sup_{v \in C} \|T^{n+1}v - T^n v\| \\
 & + \frac{\omega_{n+1}|\beta_{n+1} - \beta_n|}{\tau} \|\delta g(v_n) - \bar{V}Lv_n\| + \frac{|\omega_{n+1} - \omega_n|}{\tau} \|Q_C(I - \lambda\hat{B})u_n\|.
 \end{aligned}$$

We set

$$v_{n+1} = (1 - \mu_n)z_n + \mu_nv_n,$$

where

$$z_n = \frac{v_{n+1} - \mu_nv_n}{1 - \mu_n}, \quad \forall n \geq 0.$$

Since

$$v_{n+1} - \mu_nv_n = \nu_n T^n(\xi_nv_n + (1 - \xi_n)v_{n+1}) + \omega_n Q_C(I - \lambda\hat{B})u_n,$$

we attain

$$\begin{aligned}
 & z_{n+1} - z_n \\
 &= \frac{v_{n+2} - \mu_n v_{n+1}}{1 - \mu_{n+1}} - \frac{v_{n+1} - \mu_n v_n}{1 - \mu_n} \\
 &= \frac{v_{n+1} T^{n+1}(\xi_{n+1} v_{n+1} + (1 - \xi_{n+1}) v_{n+2}) + \omega_{n+1} Q_C(I - \lambda \hat{B}) u_{n+1}}{1 - \mu_{n+1}} \\
 &\quad - \frac{v_n T^n(\xi_n v_n + (1 - \xi_n) v_{n+1}) + \omega_n Q_C(I - \lambda \hat{B}) u_n}{1 - \mu_n} \\
 &= \frac{v_{n+1}}{1 - \mu_{n+1}} T^{n+1}(\xi_{n+1} v_{n+1} + (1 - \xi_{n+1}) v_{n+2}) - \frac{v_n}{1 - \mu_n} T^n(\xi_n v_n + (1 - \xi_n) v_{n+1}) \\
 &\quad + \frac{\omega_{n+1}}{1 - \mu_{n+1}} Q_C(I - \lambda \hat{B}) u_{n+1} - \frac{\omega_n}{1 - \mu_n} Q_C(I - \lambda \hat{B}) u_n \\
 &= \frac{v_{n+1}}{1 - \mu_{n+1}} T^{n+1}(\xi_{n+1} v_{n+1} + (1 - \xi_{n+1}) v_{n+2}) - \frac{v_{n+1}}{1 - \mu_{n+1}} T^{n+1}(\xi_n v_n + (1 - \xi_n) v_{n+1}) \\
 &\quad + \frac{v_{n+1}}{1 - \mu_{n+1}} T^{n+1}(\xi_n v_n + (1 - \xi_n) v_{n+1}) - \frac{v_n}{1 - \mu_n} T^{n+1}(\xi_n v_n + (1 - \xi_n) v_{n+1}) \\
 &\quad + \frac{v_n}{1 - \mu_n} T^{n+1}(\xi_n v_n + (1 - \xi_n) v_{n+1}) - \frac{v_n}{1 - \mu_n} T^n(\xi_n v_n + (1 - \xi_n) v_{n+1}) \\
 &\quad + \frac{\omega_{n+1}}{1 - \mu_{n+1}} Q_C(I - \lambda \hat{B}) u_{n+1} - \frac{\omega_{n+1}}{1 - \mu_{n+1}} Q_C(I - \lambda \hat{B}) u_n + \frac{\omega_{n+1}}{1 - \mu_{n+1}} Q_C(I - \lambda \hat{B}) u_n \\
 &\quad - \frac{\omega_n}{1 - \mu_n} Q_C(I - \lambda \hat{B}) u_n \\
 &= \frac{v_{n+1}}{1 - \mu_{n+1}} (T^{n+1}(\xi_{n+1} v_{n+1} + (1 - \xi_{n+1}) v_{n+2}) - T^{n+1}(\xi_n v_n + (1 - \xi_n) v_{n+1})) \\
 &\quad + \left(\frac{v_{n+1}}{1 - \mu_{n+1}} - \frac{v_n}{1 - \mu_n} \right) T^{n+1}(\xi_n v_n + (1 - \xi_n) v_{n+1}) + \frac{v_n}{1 - \mu_n} (T^{n+1}(\xi_n v_n + (1 - \xi_n) v_{n+1}) \\
 &\quad - T^n(\xi_n v_n + (1 - \xi_n) v_{n+1})) + \frac{\omega_{n+1}}{1 - \mu_{n+1}} (Q_C(I - \lambda \hat{B}) u_{n+1} - Q_C(I - \lambda \hat{B}) u_n) \\
 &\quad + \left(\frac{\omega_{n+1}}{1 - \mu_{n+1}} - \frac{\omega_n}{1 - \mu_n} \right) Q_C(I - \lambda \hat{B}) u_n.
 \end{aligned}$$

Thus, we acquire

$$\begin{aligned}
 & \|z_{n+1} - z_n\| \\
 &\leq \frac{v_{n+1}}{1 - \mu_{n+1}} \|T^{n+1}(\xi_{n+1} v_{n+1} + (1 - \xi_{n+1}) v_{n+2}) - T^{n+1}(\xi_n v_n + (1 - \xi_n) v_{n+1})\| \\
 &\quad + \left| \frac{v_{n+1}}{1 - \mu_{n+1}} - \frac{v_n}{1 - \mu_n} \right| \|T^{n+1} v\| + \frac{v_n}{1 - \mu_n} \|T^{n+1} v - T^n v\| + \frac{\omega_{n+1}}{1 - \mu_{n+1}} \|Q_C(I - \lambda \hat{B}) u_{n+1} \\
 &\quad - Q_C(I - \lambda \hat{B}) u_n\| + \left| \frac{\omega_{n+1}}{1 - \mu_{n+1}} - \frac{\omega_n}{1 - \mu_n} \right| \|Q_C(I - \lambda \hat{B}) u_n\|
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\nu_{n+1}}{1-\mu_{n+1}} k_{n+1} \|\xi_{n+1} v_{n+1} + (1-\xi_{n+1}) v_{n+2} - [\xi_n v_n + (1-\xi_n) v_{n+1}]\| + \left| \frac{\nu_{n+1}}{1-\mu_{n+1}} \right. \\
&\quad \left. - \frac{\nu_n}{1-\mu_n} \right| \|T^{n+1} v\| + \frac{\nu_n}{1-\mu_n} \|T^{n+1} v - T^n v\| + \frac{\omega_{n+1}}{1-\mu_{n+1}} \|u_{n+1} - u_n\| \\
&\quad + \left| \frac{\omega_{n+1}}{1-\mu_{n+1}} - \frac{\omega_n}{1-\mu_n} \right| \|Q_C(I - \lambda \hat{B}) u_n\| \\
&\leq \frac{\nu_{n+1}}{1-\mu_{n+1}} k_{n+1} \|\xi_n (v_{n+1} - v_n) + (1-\xi_{n+1})(v_{n+2} - v_{n+1})\| + \left| \frac{\nu_{n+1}}{1-\mu_{n+1}} - \frac{\nu_n}{1-\mu_{n+1}} \right. \\
&\quad \left. + \frac{\nu_n}{1-\mu_{n+1}} - \frac{\nu_n}{1-\mu_n} \right| \|T^{n+1} v\| + \frac{\nu_n}{1-\mu_n} \sup_{v \in C} \|T^{n+1} v - T^n v\| + \frac{\omega_{n+1}}{1-\mu_{n+1}} \|u_{n+1} - u_n\| \\
&\quad + \left| \frac{\omega_{n+1}}{1-\mu_{n+1}} - \frac{\omega_n}{1-\mu_{n+1}} + \frac{\omega_n}{1-\mu_{n+1}} - \frac{\omega_n}{1-\mu_n} \right| \|Q_C(I - \lambda \hat{B}) u_n\| \\
&\leq \frac{\nu_{n+1} k_{n+1} \xi_n}{1-\mu_{n+1}} \|v_{n+1} - v_n\| + \frac{\nu_{n+1} k_{n+1} (1-\xi_{n+1})}{1-\mu_{n+1}} \|v_{n+2} - v_{n+1}\| + \left| \frac{\nu_{n+1} - \nu_n}{1-\mu_{n+1}} \right| \|T^{n+1} v\| \\
&\quad + \nu_n \frac{|\mu_{n+1} - \mu_n|}{(1-\mu_{n+1})(1-\mu_n)} \|T^{n+1} v\| + \frac{\nu_n}{1-\mu_n} \sup_{v \in C} \|T^{n+1} v - T^n v\| + \frac{\omega_{n+1}}{1-\mu_{n+1}} \|u_{n+1} - u_n\| \\
&\quad + \frac{|\omega_{n+1} - \omega_n|}{1-\mu_{n+1}} \|Q_C(I - \lambda \hat{B}) u_n\| + \omega_n \frac{|\mu_{n+1} - \mu_n|}{(1-\mu_{n+1})(1-\mu_n)} \|Q_C(I - \lambda \hat{B}) u_n\| \\
&\leq \frac{\nu_{n+1} k_{n+1} \xi_n}{1-\mu_{n+1}} \|v_{n+1} - v_n\| + \frac{\nu_{n+1} k_{n+1} (1-\xi_{n+1})}{1-\mu_{n+1}} \|v_{n+2} - v_{n+1}\| + \left| \frac{\nu_{n+1} - \nu_n}{1-\mu_{n+1}} \right| \|T^{n+1} v\| \\
&\quad + \frac{|\mu_{n+1} - \mu_n|}{(1-\mu_{n+1})(1-\mu_n)} \|T^{n+1} v\| + \frac{\nu_n}{1-\mu_n} \sup_{v \in C} \|T^{n+1} v - T^n v\| + \frac{\omega_{n+1}}{1-\mu_{n+1}} \|u_{n+1} - u_n\| \\
&\quad + \frac{|\omega_{n+1} - \omega_n|}{1-\mu_{n+1}} \|Q_C(I - \lambda \hat{B}) u_n\| + \frac{|\mu_{n+1} - \mu_n|}{(1-\mu_{n+1})(1-\mu_n)} \|Q_C(I - \lambda \hat{B}) u_n\|.
\end{aligned}$$

From (7), (8) and condition (ii), (iii) in Theorem 1, we can compute

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\leq \frac{\nu_{n+1} k_{n+1} \xi_n}{1-\mu_{n+1}} \|v_{n+1} - v_n\| + \frac{\nu_{n+1} k_{n+1} (1-\xi_{n+1})}{1-\mu_{n+1}} [(1 + \varepsilon \beta_{n+1}) \|v_{n+1} - v_n\| + M_{n_1}] \\
&\quad + \left| \frac{\nu_{n+1} - \nu_n}{1-\mu_{n+1}} \right| \|T^{n+1} v\| + \frac{|\mu_{n+1} - \mu_n|}{(1-\mu_{n+1})(1-\mu_n)} \|T^{n+1} v\| + \frac{\nu_n}{1-\mu_n} \sup_{v \in C} \|T^{n+1} v - T^n v\| \\
&\quad + \frac{\omega_{n+1}}{1-\mu_{n+1}} (\|v_{n+1} - v_n\| + |\beta_{n+1} - \beta_n| \|\delta g(v_n) - \bar{V} L v_n\|) + \frac{|\omega_{n+1} - \omega_n|}{1-\mu_{n+1}} \|Q_C(I - \lambda \hat{B}) u_n\| \\
&\quad + \frac{|\mu_{n+1} - \mu_n|}{(1-\mu_{n+1})(1-\mu_n)} \|Q_C(I - \lambda \hat{B}) u_n\| \\
&\leq \left[\frac{\nu_{n+1} k_{n+1} \xi_n}{1-\mu_{n+1}} + \frac{\nu_{n+1} k_{n+1} (1-\xi_{n+1})}{1-\mu_{n+1}} (1 + \varepsilon \beta_{n+1}) + \frac{\omega_{n+1}}{1-\mu_{n+1}} \right] \|v_{n+1} - v_n\| \\
&\quad + \frac{\nu_{n+1} k_{n+1} (1-\xi_{n+1})}{1-\mu_{n+1}} M_{n_1} + \frac{|\nu_{n+1} - \nu_n|}{1-h_2} \|T^{n+1} v\| + \frac{|\mu_{n+1} - \mu_n|}{(1-h_2)^2} \|T^{n+1} v\| \\
&\quad + \frac{\nu_n}{1-h_2} \sup_{v \in C} \|T^{n+1} v - T^n v\| + \frac{\omega_{n+1} |\beta_{n+1} - \beta_n|}{1-h_2} \|\delta g(v_n) - \bar{V} L v_n\| \\
&\quad + \frac{|\omega_{n+1} - \omega_n|}{1-h_2} \|Q_C(I - \lambda \hat{B}) u_n\| + \frac{|\mu_{n+1} - \mu_n|}{(1-h_2)^2} \|Q_C(I - \lambda \hat{B}) u_n\| \\
&\leq \left[\frac{\nu_{n+1} k_{n+1} \xi_n + \nu_{n+1} k_{n+1} (1-\xi_{n+1}) + \omega_{n+1}}{1-\mu_{n+1}} + \frac{\nu_{n+1} k_{n+1} (1-\xi_{n+1}) \varepsilon \beta_{n+1}}{1-\mu_{n+1}} \right] \|v_{n+1} - v_n\| \\
&\quad + \frac{\nu_{n+1} k_{n+1} (1-\xi_{n+1})}{1-\mu_{n+1}} M_{n_1} + M_{n_2} \\
&\leq \left[1 + \frac{\nu_{n+1} (k_{n+1} (\xi_n + 1 - \xi_{n+1}) - 1)}{1-\mu_{n+1}} + \frac{\nu_{n+1} k_{n+1} (1-\xi_{n+1}) \varepsilon \beta_{n+1}}{1-\mu_{n+1}} \right] \|v_{n+1} - v_n\|
\end{aligned}$$

$$\begin{aligned}
 & + \frac{\nu_{n+1}k_{n+1}(1-\xi_{n+1})}{1-\mu_{n+1}}M_{n_1} + M_{n_2} \\
 \leq & \left[1 + \frac{\nu_{n+1}(k_{n+1}-1)}{1-\mu_{n+1}} + \frac{\nu_{n+1}k_{n+1}(1-\xi_{n+1})\varepsilon\beta_{n+1}}{1-\mu_{n+1}}\right]\|v_{n+1}-v_n\| \\
 & + \frac{\nu_{n+1}k_{n+1}(1-\xi_{n+1})}{1-\mu_{n+1}}M_{n_1} + M_{n_2} \\
 \leq & \left[1 + \frac{\nu_{n+1}\varepsilon\beta_{n+1}}{1-\mu_{n+1}} + \frac{\nu_{n+1}k_{n+1}\varepsilon\beta_{n+1}}{1-\mu_{n+1}}\right]\|v_{n+1}-v_n\| + \frac{\nu_{n+1}k_{n+1}(1-\xi_{n+1})}{1-\mu_{n+1}}M_{n_1} + M_{n_2} \\
 \leq & \left[1 + \frac{(k_{n+1}+1)\nu_{n+1}\varepsilon\beta_{n+1}}{1-h_2}\right]\|v_{n+1}-v_n\| + \frac{\nu_{n+1}k_{n+1}(1-\xi_{n+1})}{1-h_2}M_{n_1} + M_{n_2},
 \end{aligned}$$

where

$$\begin{aligned}
 M_{n_2} = & \frac{|\nu_{n+1}-\nu_n|}{1-h_2}\|T^{n+1}v\| + \frac{|\mu_{n+1}-\mu_n|}{(1-h_2)^2}\|T^{n+1}v\| \\
 & + \frac{\nu_n}{1-h_2}\sup_{x\in C}\|T^{n+1}v-T^n v\| + \frac{\omega_{n+1}|\beta_{n+1}-\beta_n|}{1-h_2}\|\delta g(v_n)-\bar{V}Lv_n\| \\
 & + \frac{|\omega_{n+1}-\omega_n|}{1-h_2}\|Q_C(I-\lambda\hat{B})u_n\| + \frac{|\mu_{n+1}-\mu_n|}{(1-h_2)^2}\|Q_C(I-\lambda\hat{B})u_n\|.
 \end{aligned}$$

So, by conditions (i) and (ii) in Theorem 1, we gain

$$\limsup_{n\rightarrow\infty}(\|z_{n+1}-z_n\|-\|v_{n+1}-v_n\|)\leq 0.$$

Adopting Lemma 3, we gain

$$\lim_{n\rightarrow\infty}\|z_n-v_n\|=0.$$

Taking notice of $v_{n+1}-v_n=(1-\mu_n)(z_n-v_n)$. Then

$$\lim_{n\rightarrow\infty}\|v_{n+1}-v_n\|=0. \quad (9)$$

From (7), (9) and condition (i) in Theorem 1, we receive

$$\lim_{n\rightarrow\infty}\|u_{n+1}-u_n\|=0.$$

Step 4: Again by the definition of u_n and Lemma 4, we get

$$\begin{aligned}
 \|u_n-q\|^2 & = \|\beta_n\delta g(v_n)+(I-\beta_n\bar{V})Lv_n-q\|^2 \\
 & = \|(Lv_n-q)+(\beta_n\delta g(v_n)-\beta_n\bar{V}Lv_n)\|^2 \\
 & \leq \|Lv_n-q\|^2+2\beta_n\langle\delta g(v_n)-\bar{V}Lv_n,j(u_n-q)\rangle \\
 & \leq \|v_n-q\|^2+2\beta_n\|\delta g(v_n)-\bar{V}Lv_n\|\|u_n-q\|.
 \end{aligned} \quad (10)$$

By the non-expansiveness of Q_C , Lemma 7 and (10), we have

$$\begin{aligned}
 \|v_{n+1}-q\|^2 & = \|\mu_nv_n+\nu_nT^n(\xi_nv_n+(1-\xi_n)v_{n+1})+\omega_nQ_C(I-\lambda\hat{B})u_n-q\|^2 \\
 & \leq \mu_n\|v_n-q\|^2+\nu_n\|T^n(\xi_nv_n+(1-\xi_n)v_{n+1})-q\|^2+\omega_n\|Q_C(I-\lambda\hat{B})u_n-q\|^2 \\
 & \leq \mu_n\|v_n-q\|^2+\nu_nk_n^2\|(\xi_nv_n+(1-\xi_n)v_{n+1})-q\|^2+\omega_n\|(I-\lambda\hat{B})u_n-(I-\lambda\hat{B})q\|^2 \\
 & \leq \mu_n\|v_n-q\|^2+\nu_nk_n^2\xi_n\|v_n-q\|^2+\nu_nk_n^2(1-\xi_n)\|v_{n+1}-q\|^2+\omega_n\|u_n-q\|^2 \\
 & \quad -\omega_n\lambda(2\hat{b}-\lambda)\|\hat{B}u_n-\hat{B}q\|^2
 \end{aligned}$$

$$\begin{aligned}
&\leq \mu_n \|v_n - q\|^2 + \nu_n k_n^2 \xi_n \|v_n - q\|^2 + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - q\|^2 + \omega_n \|v_n - q\|^2 \\
&\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| - \omega_n \lambda (2\hat{b} - \lambda) \|\hat{B}u_n - \hat{B}q\|^2 \\
&\leq (\mu_n + \nu_n k_n^2 \xi_n + \omega_n) \|v_n - q\|^2 + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - q\|^2 + 2\omega_n \beta_n \|\delta g(v_n) \\
&\quad - \bar{V}Lv_n\| \|u_n - q\| - \omega_n \lambda (2\hat{b} - \lambda) \|\hat{B}u_n - \hat{B}q\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\omega_n \lambda (2\hat{b} - \lambda) \|\hat{B}u_n - \hat{B}q\|^2 \leq (\mu_n + \nu_n k_n^2 \xi_n + \omega_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 \\
&\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\
&\leq (1 - \nu_n + \nu_n k_n^2 \xi_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\
&\leq (1 - \nu_n k_n^2 + \nu_n k_n^2 \xi_n + \nu_n k_n^2 - \nu_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 \\
&\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\
&\leq (1 - \nu_n k_n^2 (1 - \xi_n)) [\|v_n - q\|^2 - \|v_{n+1} - q\|^2] + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\
&\quad + \nu_n (k_n^2 - 1) \|v_n - q\|^2 \\
&\leq (\|v_n - q\| + \|v_{n+1} - q\|) \|v_n - v_{n+1}\| + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| + \nu_n \varepsilon \beta_n \|v_n - q\|^2.
\end{aligned}$$

By conditions (i), (ii) in Theorem 1 and (9), we can acquire

$$\lim_{n \rightarrow \infty} \|\hat{B}u_n - \hat{B}q\| = 0. \quad (11)$$

Applying Lemma 2 and (10), we obtain

$$\begin{aligned}
\|v_{n+1} - q\|^2 &= \|\mu_n v_n + \nu_n T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) + \omega_n Q_C(I - \lambda \hat{B})u_n - q\|^2 \\
&\leq \mu_n \|v_n - q\|^2 + \nu_n \|T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) - q\|^2 + \omega_n \|Q_C(I - \lambda \hat{B})u_n - q\|^2 \\
&\quad - \mu_n \nu_n m_1(\|v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\|) - \mu_n \omega_n m_1(\|v_n - Q_C(I - \lambda \hat{B})u_n\|) \\
&\leq \mu_n \|v_n - q\|^2 + \nu_n k_n^2 \|\xi_n v_n + (1 - \xi_n)v_{n+1} - q\|^2 + \omega_n \|u_n - q\|^2 \\
&\quad - \mu_n \nu_n m_1(\|v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\|) - \mu_n \omega_n m_1(\|v_n - Q_C(I - \lambda \hat{B})u_n\|) \\
&\leq \mu_n \|v_n - q\|^2 + \nu_n k_n^2 \xi_n \|v_n - q\|^2 + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - q\|^2 + \omega_n \|u_n - q\|^2 \\
&\quad - \mu_n \nu_n m_1(\|v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\|) - \mu_n \omega_n m_1(\|v_n - Q_C(I - \lambda \hat{B})u_n\|) \\
&\leq (\mu_n + \nu_n k_n^2 \xi_n + \omega_n) \|v_n - q\|^2 + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - q\|^2 \\
&\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| - \mu_n \nu_n m_1(\|v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\|) \\
&\quad - \mu_n \omega_n m_1(\|v_n - Q_C(I - \lambda \hat{B})u_n\|).
\end{aligned}$$

Which implies that

$$\begin{aligned}
&\mu_n \nu_n m_1(\|v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\|) + \mu_n \omega_n m_1(\|v_n - Q_C(I - \lambda \hat{B})u_n\|) \\
&\leq (\mu_n + \nu_n k_n^2 \xi_n + \omega_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 \\
&\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\
&\leq (1 - \nu_n + \nu_n k_n^2 \xi_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 \\
&\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\
&\leq (1 - \nu_n k_n^2 + \nu_n k_n^2 \xi_n + \nu_n k_n^2 - \nu_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 \\
&\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\
&\leq [1 - \nu_n k_n^2 (1 - \xi_n)] (\|v_n - q\|^2 - \|v_{n+1} - q\|^2) + \nu_n (k_n^2 - 1) \|v_n - q\|^2 \\
&\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\
&\leq (\|v_n - q\| + \|v_{n+1} - q\|) \|v_n - v_{n+1}\| + \nu_n \varepsilon \beta_n \|v_n - q\|^2 + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\|.
\end{aligned}$$

From conditions (i), (ii) in Theorem 1 and (9), and the properties of the function m_1 , we acquire

$$\lim_{n \rightarrow \infty} \|v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| = 0, \quad (12)$$

and

$$\lim_{n \rightarrow \infty} \|v_n - Q_C(I - \lambda \hat{B})u_n\| = 0. \quad (13)$$

Applying Lemma 1, Lemma 9 and (10), we gain

$$\begin{aligned} \|Q_C(I - \lambda \hat{B})u_n - q\|^2 &\leq \langle (I - \lambda \hat{B})u_n - (I - \lambda \hat{B})q, j(Q_C(I - \lambda \hat{B})u_n - q) \rangle \\ &\leq \langle u_n - q, j(Q_C(I - \lambda \hat{B})u_n - q) \rangle - \lambda \langle \hat{B}u_n - \hat{B}q, j(Q_C(I - \lambda \hat{B})u_n - q) \rangle \\ &\leq \frac{1}{2} [\|u_n - q\|^2 + \|Q_C(I - \lambda \hat{B})u_n - q\|^2 - m_2(\|u_n - q - (Q_C(I - \lambda \hat{B})u_n - q)\|)] \\ &\quad - \lambda \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\| \\ &\leq \frac{1}{2} [\|u_n - q\|^2 + \|Q_C(I - \lambda \hat{B})u_n - q\|^2 - m_2(\|u_n - Q_C(I - \lambda \hat{B})u_n\|)] \\ &\quad - \lambda \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\| \\ &\leq \frac{1}{2} [\|v_n - q\|^2 + 2\beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| + \|Q_C(I - \lambda \hat{B})u_n - q\|^2 \\ &\quad - m_2(\|u_n - Q_C(I - \lambda \hat{B})u_n\|)] - \lambda \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|Q_C(I - \lambda \hat{B})u_n - q\|^2 &\leq \|v_n - q\|^2 + 2\beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\ &\quad - m_2(\|u_n - Q_C(I - \lambda \hat{B})u_n\|) - 2\lambda \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\|. \end{aligned}$$

Hence, we know that

$$\begin{aligned} \|v_{n+1} - q\|^2 &= \|\mu_n v_n + \nu_n T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) + \omega_n Q_C(I - \lambda \hat{B})u_n - q\|^2 \\ &\leq \mu_n \|v_n - q\|^2 + \nu_n \|T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) - q\|^2 + \omega_n \|Q_C(I - \lambda \hat{B})u_n - q\|^2 \\ &\leq \mu_n \|v_n - q\|^2 + \nu_n k_n^2 \|(\xi_n v_n + (1 - \xi_n)v_{n+1}) - q\|^2 + \omega_n \|Q_C(I - \lambda \hat{B})u_n - q\|^2 \\ &\leq \mu_n \|v_n - q\|^2 + \nu_n k_n^2 \xi_n \|v_n - q\|^2 + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - q\|^2 \\ &\quad + \omega_n [\|v_n - q\|^2 + 2\beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| \\ &\quad - m_2(\|u_n - Q_C(I - \lambda \hat{B})u_n\|) - 2\lambda \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\|] \\ &\leq (\mu_n + \nu_n k_n^2 \xi_n + \omega_n) \|v_n - q\|^2 + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - q\|^2 \\ &\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| - \omega_n m_2(\|u_n - Q_C(I - \lambda \hat{B})u_n\|) \\ &\quad - 2\lambda \omega_n \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\|. \end{aligned}$$

Which implies that

$$\begin{aligned} \omega_n m_2(\|u_n - Q_C(I - \lambda \hat{B})u_n\|) &\leq (\mu_n + \nu_n k_n^2 \xi_n + \omega_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 \\ &\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| - 2\lambda \omega_n \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\| \\ &\leq (1 - \nu_n + \nu_n k_n^2 \xi_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 \\ &\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| - 2\lambda \omega_n \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\| \\ &\leq (1 - \nu_n k_n^2 + \nu_n k_n^2 \xi_n + \nu_n k_n^2 - \nu_n) \|v_n - q\|^2 - [1 - \nu_n k_n^2 (1 - \xi_n)] \|v_{n+1} - q\|^2 \\ &\quad + 2\omega_n \beta_n \|\delta g(v_n) - \bar{V}Lv_n\| \|u_n - q\| - 2\lambda \omega_n \|\hat{B}u_n - \hat{B}q\| \|Q_C(I - \lambda \hat{B})u_n - q\| \\ &\leq [1 - \nu_n k_n^2 (1 - \xi_n)] (\|v_n - q\|^2 - \|v_{n+1} - q\|^2) + \nu_n (k_n^2 - 1) \|v_n - q\|^2 \end{aligned}$$

$$\begin{aligned}
& + 2\omega_n\beta_n\|\delta g(v_n) - \bar{V}Lv_n\|\|u_n - q\| - 2\lambda\omega_n\|\hat{B}u_n - \hat{B}q\|\|Q_C(I - \lambda\hat{B})u_n - q\| \\
& \leq [1 - \nu_n k_n^2(1 - \xi_n)](\|v_n - q\| + \|v_{n+1} - q\|)\|v_n - v_{n+1}\| + \nu_n(k_n^2 - 1)\|v_n - q\|^2 \\
& + 2\omega_n\beta_n\|\delta g(v_n) - \bar{V}Lv_n\|\|u_n - q\| - 2\lambda\omega_n\|\hat{B}u_n - \hat{B}q\|\|Q_C(I - \lambda\hat{B})u_n - q\| \\
& \leq (\|v_n - q\| + \|v_{n+1} - q\|)\|v_n - v_{n+1}\| + \nu_n\epsilon\beta_n\|v_n - q\|^2 \\
& + 2\omega_n\beta_n\|\delta g(v_n) - \bar{V}Lv_n\|\|u_n - q\| - 2\lambda\omega_n\|\hat{B}u_n - \hat{B}q\|\|Q_C(I - \lambda\hat{B})u_n - q\|.
\end{aligned}$$

Similarly, by conditions (i), (ii) in Theorem 1 and (9), (11) and the properties of the function m_2 , we attain

$$\lim_{n \rightarrow \infty} \|u_n - Q_C(I - \lambda\hat{B})u_n\| = 0. \quad (14)$$

Thinking

$$\|v_n - u_n\| \leq \|v_n - Q_C(I - \lambda\hat{B})u_n\| + \|Q_C(I - \lambda\hat{B})u_n - u_n\|,$$

and (13) and (14), we receive

$$\lim_{n \rightarrow \infty} \|v_n - u_n\| = 0. \quad (15)$$

Adopting condition (i) in Theorem 1, we can get

$$\begin{aligned}
\|u_n - Lv_n\| &= \|\beta_n\delta g(v_n) + (I - \beta_n\bar{V})Lv_n - Lv_n\| \\
&= \beta_n\|\delta g(v_n) - \bar{V}Lv_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

and also

$$\|v_n - Lv_n\| \leq \|v_n - u_n\| + \|u_n - Lv_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (16)$$

Moreover, from (12), (14) and (15), we have

$$\begin{aligned}
& \|Q_C(I - \lambda\hat{B})u_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| \\
&= \|Q_C(I - \lambda\hat{B})u_n - u_n\| + \|u_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| \\
&= \|Q_C(I - \lambda\hat{B})u_n - u_n\| + \|u_n - v_n\| + \|v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \quad (17)$$

Since

$$\begin{aligned}
& \|v_{n+1} - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| \\
&= \|\mu_n v_n + \nu_n T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) + \omega_n Q_C(I - \lambda\hat{B})u_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| \\
&= \|\mu_n[v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})] + \omega_n[Q_C(I - \lambda\hat{B})u_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})]\| \\
&\leq \mu_n\|v_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| + \omega_n\|Q_C(I - \lambda\hat{B})u_n - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\|.
\end{aligned}$$

By condition (ii) in Theorem 1 and (12), (17), we obtain

$$\|v_{n+1} - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (18)$$

We know that

$$\begin{aligned}
\|v_n - T^n v_n\| &= \|v_n - v_{n+1} + v_{n+1} - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) + T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) - T^n v_n\| \\
&\leq \|v_n - v_{n+1}\| + \|v_{n+1} - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| + \|T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) - T^n v_n\| \\
&\leq \|v_n - v_{n+1}\| + \|v_{n+1} - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| + k_n\|\xi_n v_n + (1 - \xi_n)v_{n+1} - v_n\|
\end{aligned}$$

$$\leq \|v_n - v_{n+1}\| + \|v_{n+1} - T^n(\xi_n v_n + (1 - \xi_n)v_{n+1})\| + k_n(1 - \xi_n)\|v_{n+1} - v_n\|.$$

By (9) and (18), we acquire

$$\|v_n - T^n v_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (19)$$

Because T is an asymptotically non-expansive mapping, we attain

$$\begin{aligned} \|v_n - T v_n\| &= \|v_n - v_{n+1} + v_{n+1} - T^{n+1} v_{n+1} + T^{n+1} v_{n+1} - T^{n+1} v_n + T^{n+1} v_n - T v_n\| \\ &\leq \|v_n - v_{n+1}\| + \|v_{n+1} - T^{n+1} v_{n+1}\| + \|T^{n+1} v_{n+1} - T^{n+1} v_n\| + \|T^{n+1} v_n - T v_n\| \\ &\leq \|v_n - v_{n+1}\| + \|v_{n+1} - T^{n+1} v_{n+1}\| + k_{n+1}\|v_{n+1} - v_n\| + k_1\|T^n v_n - v_n\| \\ &\leq (1 + k_{n+1})\|v_n - v_{n+1}\| + \|v_{n+1} - T^{n+1} v_{n+1}\| + k_1\|T^n v_n - v_n\|. \end{aligned}$$

By (9) and (19), we acquire

$$\|v_n - T v_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (20)$$

From (15), (16) and (20), we have

$$\begin{aligned} \|u_n - T u_n\| &\leq \|u_n - v_n\| + \|v_n - T v_n\| + \|T v_n - T u_n\| \\ &\leq \|u_n - v_n\| + \|v_n - T v_n\| + k_1\|v_n - u_n\| \rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned} \quad (21)$$

and

$$\begin{aligned} \|u_n - L u_n\| &\leq \|u_n - v_n\| + \|v_n - L v_n\| + \|L v_n - L u_n\| \\ &\leq \|u_n - v_n\| + \|v_n - L v_n\| + \|v_n - u_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (22)$$

Step 5: We can take a subsequence u_{n_i} of $\{u_n\}$ and it is satisfied with

$$\lim_{i \rightarrow \infty} \langle \delta g(t) - \bar{V}t, j(u_{n_i} - t) \rangle = \limsup_{n \rightarrow \infty} \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle. \quad (23)$$

Since X is a uniformly smooth Banach space and $\{v_n\}$ is bounded, there exists a subsequence $v_{n_i} \rightharpoonup \bar{v} \in C$ as $i \rightarrow \infty$. By (15) and (23), we know

$$u_{n_i} \rightharpoonup \bar{v}, \quad \langle \delta g(t) - \bar{V}t, j(\bar{v} - t) \rangle = \limsup_{n \rightarrow \infty} \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle. \quad (24)$$

From (21), (24) and Lemma 10, we can acquire $\bar{v} \in \text{Fix}(T)$. It follows that (22), (24) and Lemma 10, we can attain $\bar{v} \in \text{Fix}(L)$. By (14), (24), Lemma 10 and Lemma 8, we get $\bar{v} \in VI(C, \hat{B})$. So, we attain $\bar{v} \in \Psi$. From (24) and property of Q_C , we get

$$\limsup_{n \rightarrow \infty} \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle = \langle \delta g(t) - \bar{V}t, j(\bar{v} - t) \rangle \leq 0. \quad (25)$$

Step 6: Finally, from Lemma 4 and Lemma 6, we acquire

$$\begin{aligned}
\|u_n - t\|^2 &= \|\beta_n \delta g(v_n) + (I - \beta_n \bar{V})Lv_n - t\|^2 \\
&\leq \|(I - \beta_n \bar{V})(Lv_n - t)\|^2 + 2\beta_n \langle \delta g(v_n) - \bar{V}t, j(u_n - t) \rangle \\
&\leq (1 - \beta_n \bar{\delta})^2 \|v_n - t\|^2 + 2\beta_n \delta \langle g(v_n) - g(t), j(u_n - t) \rangle + 2\beta_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
&\leq (1 - \beta_n \bar{\delta})^2 \|v_n - t\|^2 + 2\beta_n \delta \|g(v_n) - g(t)\| \|u_n - t\| + 2\beta_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
&\leq (1 - \beta_n \bar{\delta})^2 \|v_n - t\|^2 + 2\beta_n \delta \eta \|v_n - t\| \|u_n - t\| + 2\beta_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
&\leq (1 - \beta_n \bar{\delta})^2 \|v_n - t\|^2 + 2\beta_n \delta \eta \|v_n - t\| (\beta_n \|\delta g(v_n) - \bar{V}t\| + (1 - \beta_n \bar{\delta}) \|Lv_n - t\|) \\
&\quad + 2\beta_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
&\leq (1 - \beta_n \bar{\delta})^2 \|v_n - t\|^2 + 2\beta_n \delta \eta \|v_n - t\| (\beta_n \delta \eta \|v_n - t\| + \beta_n \|\delta g(t) - \bar{V}t\| \\
&\quad + (1 - \beta_n \bar{\delta}) \|v_n - t\|) + 2\beta_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
&\leq (1 - \beta_n \bar{\delta})^2 \|v_n - t\|^2 + 2\beta_n^2 \delta^2 \eta^2 \|v_n - t\|^2 + 2\beta_n^2 \delta \eta \|\delta g(t) - \bar{V}t\| \|v_n - t\| \\
&\quad + 2\beta_n \delta \eta (1 - \beta_n \bar{\delta}) \|v_n - t\|^2 + 2\beta_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
&\leq (1 - \beta_n \bar{\delta}) \|v_n - t\|^2 + 2\beta_n^2 \bar{\delta}^2 \|v_n - t\|^2 + 2\beta_n^2 \bar{\delta} \|\delta g(t) - \bar{V}t\| \|v_n - t\| \\
&\quad + 2\beta_n \delta \eta \|v_n - t\|^2 + 2\beta_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle,
\end{aligned}$$

and

$$\begin{aligned}
\|v_{n+1} - t\|^2 &= \|\mu_n v_n + \nu_n T^n(\xi_n v_n + (1 - \xi_n)v_{n+1}) + \omega_n Q_C(I - \lambda \hat{B})u_n - t\|^2 \\
&\leq \mu_n \|v_n - t\|^2 + \nu_n k_n^2 \|(\xi_n v_n + (1 - \xi_n)v_{n+1}) - t\|^2 + \omega_n \|Q_C(I - \lambda \hat{B})u_n - t\|^2 \\
&\leq \mu_n \|v_n - t\|^2 + \nu_n k_n^2 \xi_n \|v_n - t\|^2 + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - t\|^2 + \omega_n \|u_n - t\|^2 \\
&\leq \mu_n \|v_n - t\|^2 + \nu_n k_n^2 \xi_n \|v_n - t\|^2 + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - t\|^2 \\
&\quad + \omega_n [(1 - \beta_n \bar{\delta}) \|v_n - t\|^2 + 2\beta_n^2 \bar{\delta}^2 \|v_n - t\|^2 + 2\beta_n^2 \bar{\delta} \|\delta g(t) - \bar{V}t\| \|v_n - t\| \\
&\quad + 2\beta_n \delta \eta \|v_n - t\|^2 + 2\beta_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle] \\
&\leq (\mu_n + \nu_n k_n^2 \xi_n + \omega_n - \omega_n \beta_n \bar{\delta} + 2\beta_n \delta \eta \omega_n) \|v_n - t\|^2 + 2\beta_n^2 \bar{\delta}^2 \omega_n \|v_n - t\|^2 \\
&\quad + 2\beta_n^2 \bar{\delta} \omega_n \|\delta g(t) - \bar{V}t\| \|v_n - t\| + 2\beta_n \omega_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
&\quad + \nu_n k_n^2 (1 - \xi_n) \|v_{n+1} - t\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1 - \nu_n k_n^2 (1 - \xi_n)) \|v_{n+1} - t\|^2 &\leq (\mu_n + \nu_n k_n^2 \xi_n + \omega_n - \omega_n \beta_n (\bar{\delta} - 2\delta \eta)) \|v_n - t\|^2 + 2\beta_n^2 \bar{\delta}^2 \omega_n \|v_n - t\|^2 \\
&\quad + 2\beta_n^2 \bar{\delta} \omega_n \|\delta g(t) - \bar{V}t\| \|v_n - t\| + 2\beta_n \omega_n \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle.
\end{aligned}$$

That is

$$\begin{aligned}
 \|v_{n+1} - t\|^2 &\leq \frac{\mu_n + \nu_n k_n^2 \xi_n + \omega_n - \omega_n \beta_n (\bar{\delta} - 2\delta\eta)}{1 - \nu_n k_n^2 (1 - \xi_n)} \|v_n - t\|^2 + \frac{2\beta_n^2 \bar{\delta}^2 \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \|v_n - t\|^2 \\
 &\quad + \frac{2\beta_n^2 \bar{\delta} \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \|\delta g(t) - \bar{V}t\| \|v_n - t\| + \frac{2\beta_n \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
 &\leq (1 - \frac{\omega_n \beta_n (\bar{\delta} - 2\delta\eta) - \nu_n (k_n^2 - 1)}{1 - \nu_n k_n^2 (1 - \xi_n)}) \|v_n - t\|^2 + \frac{2\beta_n^2 \bar{\delta}^2 \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \|v_n - t\|^2 \\
 &\quad + \frac{2\beta_n^2 \bar{\delta} \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \|\delta g(t) - \bar{V}t\| \|v_n - t\| + \frac{2\beta_n \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
 &\leq (1 - \frac{h_1 \beta_n (\bar{\delta} - 2\delta\eta) - \varepsilon \beta_n}{1 - \nu_n k_n^2 (1 - \xi_n)}) \|v_n - t\|^2 + \frac{2\beta_n^2 \bar{\delta}^2 \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \|v_n - t\|^2 \\
 &\quad + \frac{2\beta_n^2 \bar{\delta} \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \|\delta g(t) - \bar{V}t\| \|v_n - t\| + \frac{2\beta_n \omega_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
 &\leq (1 - \frac{\beta_n (h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon)}{1 - \nu_n k_n^2 (1 - \xi_n)}) \|v_n - t\|^2 + \frac{2\beta_n^2 \bar{\delta}^2}{1 - \nu_n k_n^2 (1 - \xi_n)} \|v_n - t\|^2 \\
 &\quad + \frac{2\beta_n^2 \bar{\delta}}{1 - \nu_n k_n^2 (1 - \xi_n)} \|\delta g(t) - \bar{V}t\| \|v_n - t\| + \frac{2\beta_n}{1 - \nu_n k_n^2 (1 - \xi_n)} \langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle \\
 &\leq (1 - \frac{\beta_n (h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon)}{1 - \nu_n k_n^2 (1 - \xi_n)}) \|v_n - t\|^2 + \frac{\beta_n (h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon)}{1 - \nu_n k_n^2 (1 - \xi_n)} [\frac{2\beta_n \bar{\delta}^2 \|v_n - t\|^2}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon} \\
 &\quad + \frac{2\beta_n \bar{\delta} \|\delta g(t) - \bar{V}t\| \|v_n - t\|}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon} + \frac{2\langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon}].
 \end{aligned}$$

Let

$$\sigma_n = \frac{\beta_n (h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon)}{1 - \nu_n k_n^2 (1 - \xi_n)},$$

and

$$\theta_n = \frac{2\beta_n \bar{\delta}^2 \|v_n - t\|^2}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon} + \frac{2\beta_n \bar{\delta} \|\delta g(t) - \bar{V}t\| \|v_n - t\|}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon} + \frac{2\langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon},$$

by conditions (i), (iii), (iv) in Theorem 1 and (25), we can gain

$$\sum_{n=0}^{\infty} \sigma_n = \sum_{n=0}^{\infty} \frac{\beta_n (h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon)}{1 - \nu_n k_n^2 (1 - \xi_n)} \geq \sum_{n=0}^{\infty} \beta_n (h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon) = +\infty,$$

$$\limsup_{n \rightarrow \infty} \theta_n = \limsup_{n \rightarrow \infty} [\frac{2\beta_n \bar{\delta}^2 \|v_n - t\|^2}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon} + \frac{2\beta_n \bar{\delta} \|\delta g(t) - \bar{V}t\| \|v_n - t\|}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon} + \frac{2\langle \delta g(t) - \bar{V}t, j(u_n - t) \rangle}{h_1 \bar{\delta} - 2h_1 \delta\eta - \varepsilon}] \leq 0.$$

Thus, from Lemma 5, we receive that $\lim_{n \rightarrow \infty} \|v_n - t\| = 0$. So $\{v_n\}$ converges strongly to $t = Q_{\Psi}(I - \bar{V} + \delta g)t \in \Psi$, and from Lemma 11, we have (t, s, w) is a solution of the problem (4), where $s = Q_C(I - \lambda_2 B_2)(bt + (1 - b)w)$ and $w = Q_C(I - \lambda_3 B_3)t$. The proof is completed. $\square$

As an application of our main result Theorem 1, we can prove strong convergence theorems for approximating the solution of the standard convex optimization problem.

Assume C is a convex and closed subset of X . The standard constrained convex optimization problem is to find $\tilde{v} \in C$ such that

$$\tilde{g}(\tilde{v}) = \min_{v \in C} \tilde{g}(v), \quad (26)$$

where $\tilde{g} : C \rightarrow R$ is a convex, Fréchet differentiable function. $\Phi_{\tilde{g}}$ is used to indicate the solution set of (26).

Lemma 12. ([21]) An essential condition of optimality for a point $\tilde{v} \in C$ to be a solution of (26) is that $\tilde{v}$ is a solution of the variational inequality

$$\langle \nabla \tilde{g}(\tilde{v}), v - \tilde{v} \rangle \geq 0, \forall v \in C. \quad (27)$$

Equivalently, $v \in C$ is a fixed point of the mapping $Q_C(I - \lambda \nabla \tilde{g})$, $\forall \lambda > 0$. In addition, if $\tilde{g}$ is convex, then the optimality condition (27) is also sufficient.

Theorem 2. Assume that space X , mappings Q_C , B_1 , B_2 , B_3 , L , $\bar{V}$, g and T are the same as in Theorem 1. Suppose $\tilde{g} : C \rightarrow R$ is a convex real-valued function with the gradient $\nabla \tilde{g} = \hat{B}$ is $1/L_{\tilde{g}}$ -inverse strongly accretive. Assume that $\Omega = \text{Fix}(T) \cap \text{Fix}(L) \cap \Phi_{\tilde{g}} \neq \emptyset$. Define the sequence $\{v_n\}$ in X as follows:

$$\begin{cases} v_1 \in C, \\ u_n = \beta_n \delta g(v_n) + (I - \beta_n \bar{V}) L v_n, \\ v_{n+1} = \mu_n v_n + \nu_n T^n(\xi_n v_n + (1 - \xi_n) v_{n+1}) + \omega_n Q_C(I - \lambda \nabla \tilde{g}) u_n, n \geq 1, \end{cases} \quad (28)$$

where $\lambda \in (0, 2/L_{\tilde{g}})$, β_n , μ_n , ν_n , ω_n , ξ_n and δ satisfy the same coefficient conditions as in Theorem 1. Then $\{v_n\}$ converges strongly to $t = Q_{\Omega}(I - \bar{V} + \delta g)t \in \Omega$ and (t, s, w) is a solution of (4), where $s = Q_C(I - \lambda_2 B_2)(bt + (1 - b)w)$ and $w = Q_C(I - \lambda_3 B_3)t$, $\lambda_1, \lambda_2, \lambda_3 \in (0, 2\bar{b})$ with $\bar{b} = \min\{b_1, b_2, b_3\}$.

Proof. From Lemma 12 and Theorem 1, we can gain Theorem 2, where $s = Q_C(I - \lambda_2 B_2)(bt + (1 - b)w)$ and $w = Q_C(I - \lambda_3 B_3)t$. Then, the proof is completed. $\square$

Remark 1. (1) In Theorem 2, we give the iterative approximation method and strong convergence results for the common elements of the solutions $\Phi_{\tilde{g}}$ of the standard constrained convex optimization problem (26) and the fixed points $\text{Fix}(T)$ of an asymptotically non-expansive mapping T .

(2) As we all know, a Hilbert space H is a uniformly smooth and uniformly convex Banach space and the metric projection $P_C : H \rightarrow C$ is sunny non-expansive retractive. So, when Hilbert space H takes the place of Banach space X and the sunny non-expansive retraction Q_C becomes to the metric projection P_C , the results of Theorem 1 and Theorem 2 still hold. More specifically, if $k_n \equiv 1$ and $\xi_n \equiv 1$ in Theorem 2, we can attain the major results of Keerti et al. [19].

4. Numerical Examples

Someone may question whether the coefficients in Theorem 1 are complex enough to be not obtained. Now we give a concrete numerical example to show that the coefficients in Theorem 1 can be obtained.

Let R be the real number set and $\hat{B}, B_1, B_2, B_3$ be a mapping from $[0, 20]$ to R defined by $\hat{B}v = \frac{v}{4}, B_1v = \frac{v}{2}, B_2v = \frac{v}{4}, B_3v = \frac{v}{6}$, respectively. Put $b = \frac{1}{3}, \bar{b} = 4, b_1 = 2, b_2 = 4, b_3 = 6, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, then $\bar{b} = \min\{b_1, b_2, b_3\} = 2, \lambda_1, \lambda_2, \lambda_3 \in (0, 2\bar{b}) = (0, 4)$, the mapping $Lv = \frac{5}{18}v$ defined by (5) in Lemma 11. We take $\eta = \frac{1}{3}, \delta = \frac{1}{4}, \beta_n = \frac{1}{5n}$, and let $g, \bar{V} : R \rightarrow R$ be defined by $g(v) = \frac{v}{3}, \bar{V}v = \frac{v}{4}$, respectively. So, $\{u_n\} : u_n = \frac{100n+1}{360n}v_n$ is generated by (6). Assume that T is an asymptotically non-expansive mapping defined by $T^n v = \sqrt{1 + \frac{1}{12n}}v$. Let $v_1 \in [0, 20]$, and $\{v_n\}$ is generated by (6) where $h_1 = \frac{1}{6}, h_2 = \frac{1}{2}, \lambda = 1 \in (0, 2\bar{b}) = (0, 8), \mu_n = \frac{3n-1}{12n}, \nu_n = \frac{1}{2}$ and $\omega_n = \frac{3n+1}{12n}, \xi_n = 1 - \frac{1}{3n}, \varepsilon = \frac{5}{12}, \tau = \frac{3}{4}$. From the propose definition, the all conditions of Theorem 1 are satisfied. We know $0 \in \text{Fix}(T) \cap \text{Fix}(L) \cap VI(C, \hat{B})$. Then v_n is generated by (6) and we simplify the form of (6) to get

$$v_{n+1} = \frac{1440n^2 - 480n + 480n\sqrt{3n(12n+1)} - 160\sqrt{3n(12n+1)} + 1}{5760n^2 - 160\sqrt{3n(12n+1)}}v_n \approx \frac{4320}{5760}v_n.$$

So, clearly, v_n converges strongly to $0 \in \text{Fix}(T) \cap \text{Fix}(L) \cap VI(C, \widehat{B})$.

5. Conclusions

In this paper, we present a new generalized variational inequality system problem (4) which contains many special cases. In the case when we put $B_2 \equiv 0$, then the problem (4) reduces to problem (3); If putting $b = 0$ in (4), we get

$$\begin{cases} \langle t - (I - \lambda_1 B_1)s, j(v - t) \rangle \geq 0, \forall v \in C, \\ \langle s - (I - \lambda_2 B_2)w, j(v - s) \rangle \geq 0, \forall v \in C, \\ \langle w - (I - \lambda_3 B_3)t, j(v - w) \rangle \geq 0, \forall v \in C, \end{cases} \quad (29)$$

which is a variational inequality modified system by Ceng et al. [2]. In the sense that if we take $B_3 \equiv 0$ in (29), then the problem (29) reduces to problem (2). We study the common elements of the set of solutions of an asymptotically non-expansive operation equation with the L mapping defined by (5) and the solution set of the generalized proposed system problem (4). The convergence analysis of a new way by using the generalized semi-closure principle supplied by us in Wang et al. [9] for finding the propose common elements in Banach spaces are investigated. Under the suitable conditions imposed on parameters, some strong convergence theorems are attained. We locally generalize some corresponding recent results of Keerti et al. [19] from Hilbert spaces to Banach spaces and from non-expansive mapping to asymptotically non-expansive mapping. As an application, we prove the strong convergence theorem of the standard constrained convex optimization problem by our main result. Eventually, we also give a real numerical example to show that the coefficients in Theorem 1 can be obtained and the theorems are reasonable and valid.

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