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Binary (k, k)-Designs

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Abstract: We introduce and investigate binary (k, k)-designs, a special case of T-designs. Our combinatorial interpretation relates (k, k)-designs to the binary orthogonal arrays. We derive a general linear programming bound and propose as a consequence a universal bound on the minimum possible cardinality of (k, k)-designs for fixed k and n. Designs which attain our bound are investigated.

Keywords: binary (k, k)-designs; orthogonal arrays; linear programming

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1. Introduction

Let $F = \{0, 1\}$ be the alphabet of two symbols and F_2^n the set of all binary vectors $x = (x_1, x_2, \ldots, x_n)$ over F. The Hamming distance $d(x, y)$ between points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ from F_2^n is equal to the number of coordinates in which they differ.

In considerations of F_2^n as a polynomial metric space (cf. [1–3]) it is convenient to use the "inner product"

$$\langle x, y \rangle := 1 - \frac{2d(x, y)}{n} \tag{1}$$

instead of the distance $d(x, y)$. The geometry in F_2^n is then related to the properties of the Krawtchouk polynomials $\{Q_i^{(n)}(t)\}_{i=0}^n$ satisfying the following three-term recurrence relation

$$ntQ_i^{(n)}(t) = (n - i)Q_{i+1}^{(n)}(t) + iQ_{i-1}^{(n)}(t),$$

$i = 1, 2, \ldots, n - 1$, with initial conditions $Q_0^{(n)}(t) = 1$ and $Q_1^{(n)}(t) = t$.

Any nonempty subset $C \subseteq F_2^n$ is called a code. Given a code $C \subset F_2^n$, the quantities

$$\begin{aligned} M_i(C) \quad &:= \quad \sum_{x,y \in C} Q_i^{(n)}(\langle x, y \rangle) \\ &= \quad |C| + \sum_{x,y \in C, x \neq y} Q_i^{(n)}(\langle x, y \rangle),\ i = 1, 2, \ldots, n, \end{aligned} \tag{2}$$

are called *moments* of C, where $|C|$ denotes the cardinality of C.

The well known positive definiteness of the Krawtchouk polynomials (see [1,3,4]) implies that $M_i(C) \geq 0$ for every $i = 1, 2, \ldots, n$. The case of equality is quite important.

Definition 1. *[5] Let $T \subset \{1, 2, \ldots, n\}$. A code $C \subset F_2^n$ is called a T-design if*

$$M_i = 0 \text{ for all } i \in T.$$

If $T = \{1, 2, \ldots, m\}$ for some $m \leq n$, then C is known as an m-design (see [1,3,4]), or a (binary) orthogonal array of strength m (cf. [1,3–6]), or an m-wise independent set [7].

The case of T consisting of even integers was introduced and considered by Bannai et al. in [5] (Section 6.2) but (to the best of our knowledge) the special case of the next definition is not claimed yet. The Euclidean analogs of the (k, k)-designs on $\mathbb{S}^{n-1}$ were considered earlier [8–12]. Further analogs in polynomial metric spaces such as q-ary Hamming spaces and infinite projective spaces could be interesting and will be considered elsewhere.

Orthogonal arrays have nice combinatorial properties which imply, in particular, a divisibility condition for (k, k)-designs (Corollary 1 below). Our approach allows a combinatorial interpretation (Theorem 1) which reveals relations with the binary orthogonal arrays and implies a divisibility condition. Note that the notion of T-designs seems to be too general for arbitrary T and even for most specific T, so we do not find any combinatorial interpretation in [5].

Definition 2. *If $C \subset F_2^n$ is a T-design with $T = \{2, 4, \ldots, 2k\}$, where $k \leq n/2$ is a positive integer, then C is called a (k, k)-design. In other words, C is a (k, k)-design if and only if*

$$M_i(C) = 0 \text{ for all } i = 2, 4, \ldots, 2k.$$

Thus, in this paper we focus on the special case when T consists of several consecutive even integers beginning with 2. It is clear from the definition that any (k, k)-design is also an (ℓ, ℓ)-design for every $\ell = 1, 2, \ldots, k - 1$.

We also derive and investigate general and specific linear programming (Delsarte) bounds. After recalling general linear programming techniques, we will derive and investigate an universal (in sense of Levenshtein [3]) bound. More precisely, we obtain a lower bound on the quantity

$$\mathcal{M}(n, k) := \min\{|C| : C \subset F_2^n \text{ is a } (k, k)\text{-design}\}, \tag{3}$$

the minimum possible cardinality of a (k, k)-design in F_2^n, as follows:

$$\mathcal{M}(n, k) \geq \sum_{i=0}^{k} \binom{n-1}{i}.$$

The paper is organized as follows. In Section 2 we derive a relation between (k, k)-designs and antipodal $(2k + 1)$-designs implying a strong divisibility condition. Section 3 reviews the general linear programming bound and recalls the definition of so-called adjacent (to Krawtchouk) polynomials which will be important ingredients in our approach. Section 4 is devoted to our new universal bound. In Section 5 we discuss (k, k)-designs which attain this bound.

2. Relations to Antipodal $(2k + 1)$-Designs

Classical binary m-designs have nice combinatorial properties.

Definition 3. *Let $C \subseteq F_2^n$ be a code and M be a codeword matrix consisting of all vectors of C as rows. Then C is called an m-design, $1 \leq m \leq n$, if any set of m columns of M contains any m-tuple of F_2^m the same number of times (namely, $\lambda := |C|/2^m$). The largest positive integer m such that C is an m-design is called the strength of C. The number λ is called the index of C.*

It follows from Definition 3 that the cardinality of any m-design is divisible by 2^m. This property implies a strong divisibility condition for a basic type of (k, k)-designs.

Definition 4. *A code $C \subseteq F_2^n$ is called* antipodal *if for every $x \in C$ the unique point $y \in F_2^n$ such that $d(x,y) = n$ (equivalently, $\langle x,y \rangle = -1$) also belongs to C. The point y is denoted also by $-x$ and is called* antipodal to x.

If $C \subset F_2^n$, then the set of the points, which are antipodal to points of C is denoted as usually by $-C$. A strong relation between antipodal $(2k+1)$-designs and (k,k)-designs is given as follows.

Theorem 1. *Let $D \subset F_2^n$ be an antipodal $(2k+1)$-design. Let the code $C \subset F_2^n$ be formed by the following rule: from each pair $(x,-x)$ of antipodal points of D exactly one of the points x and $-x$ belongs to C. Then C is a (k,k)-design. Conversely, if $C \subset F_2^n$ is a (k,k)-design which does not possess a pair of antipodal points, then $D = C \cup -C$ is an antipodal $(2k+1)$-design in F_2^n.*

Proof. For the first statement we use in (2) the antipodality of D, the relation $|C| = |D|/2$, and the fact that the polynomials $Q_{2i}^{(n)}(t)$ are even functions; i.e., $Q_{2i}^{(n)}(t) = Q_{2i}^{(n)}(-t)$ for every t, to see that

$$M_{2i}(C) = \frac{M_{2i}(D)}{2} = 0$$

for every $i = 1,2,\ldots,k$. Therefore C is a (k,k)-design (whatever is the way of choosing one of the points in pairs of antipodal points).

The second statement follows similarly. □

Corollary 1. *If $C \subset F_2^n$ is a (k,k)-design which does not possess a pair of antipodal points, then $|C|$ is divisible by 2^{2k}.*

Proof. By Definition 3 it follows that 2^{2k+1} divides the cardinality of the antipodal $(2k+1)$-design D constructed from C as in Theorem 1. Thus $|C| = |D|/2$ is divisible by 2^{2k}. □

Example 1. *For even $n = 2\ell$, the even weight code $D \subset F_2^n$ is an antipodal $(2\ell-1)$-design. Therefore, any code C obtained as in Theorem 1 is an $(\ell-1,\ell-1)$-design. Obviously, $|C| = |D|/2 = 2^{n-2} = 2^{2\ell-2}$. We will come back to this example in Section 5.*

We note that Definition 2 shows that any $(2k)$- or $(2k+1)$-design is also a (k,k)-design. For small k, this relation gives some examples of (k,k)-designs with relatively small cardinalities (see Section 5).

The m-designs in F_2^n possess further nice combinatorial properties. For example, if a column of the codeword matrix in Definition 3 is deleted, the resulting matrix is still an m-design in F_2^{n-1} with the same cardinality (possibly with repeating rows). Moreover, the rows with 0 in that column determine an $(m-1)$-design in F_2^{n-1} of twice less cardinality. It would be interesting to have analogs of these properties for (k,k)-designs.

3. General Linear Programming Bounds

Linear programming methods were introduced in coding theory by Delsarte (see [4,13]). The case of T-designs in F_2^n was recently considered by Bannai et al. [5] (see also [14] (Sections 4–6)).

The transformation (1) means that all numbers $\langle x,y \rangle$ are rational and belong to the set

$$T_n := \{-1 + 2i/n : i = 0,1,\ldots,n\}.$$

We will be interested in values of polynomials in T_n.

For any real polynomial $f(t)$ we consider its expansion in terms of Krawtchouk polynomials

$$f(t) = \sum_{j=0}^{n} f_j Q_j^{(n)}(t)$$

(if the degree of the polynomial $f(t)$ exceeds n, then $f(t)$ is taken modulo $\prod_{i=0}^{n}(t-t_i)$, where $t_i=-1+2i/n \in T_n$, $i=0,1,\dots,n$). We define the following set of polynomials

$$F_{n,k} := \{f(t) \geq 0 \ \forall t \in T_n : f_0 > 0, f_j \leq 0, j=1,3,\dots,2k-1 \text{ and } j \geq 2k+1\}.$$

The next theorem was proved (in slightly different setting) in [5]. We provide a proof here in order to make the paper self-contained.

Theorem 2. *[5] [Proposition 6.8] If $f \in F_{n,k}$, then*

$$\mathcal{M}(n,k) \geq \frac{f(1)}{f_0}.$$

If a (k,k)-design $C \subset F_2^n$ attains this bound, then all inner products $\langle x,y\rangle$ of distinct $x,y \in C$ are among the zeros of $f(t)$ and $f_i M_i(C)=0$ for every positive integer i.

Proof. Bounds of this kind follow easily from the identity

$$|C|f(1) + \sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) = |C|^2 f_0 + \sum_{i=1}^{m} f_i M_i(C) \tag{4}$$

(see, for example, [2] [Equation (1.20)], [15] [Equation (26)]), which is true for every code $C \subset F_2^n$ and every polynomial $f(t)=\sum_{j=0}^{m} f_j Q_j^{(n)}(t)$.

Let C be a (k,k)-design and $f \in F_{n,k}$. We apply (4) for C and f. Since $M_{2j}(C)=0$ for $j=1,2,\dots,k$, $M_i \geq 0$ for all i, and $f_j \leq 0$ for all odd j and for all even $j>2k$, the right hand side of (4) does not exceed $f_0|C|^2$. The sum in the left hand side is non-negative because $f(t)\geq 0$ for every $t \in T_n$. Thus the left hand side is at least $f(1)|C|$ and we conclude that $|C| \geq f(1)/f_0$. Since this inequality follows for every C, we have $\mathcal{M}(n,k) \geq f(1)/f_0$.

If the equality is attained by some (k,k)-design $C \subset F_2^n$ and a polynomial $f \in F_{n,k}$, then

$$\sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) = \sum_{i=1}^{m} f_i M_i(C) = 0.$$

Since $f(t)\geq 0$ for every $t\in T_n$, we conclude that $f(\langle x,y\rangle)=0$ whenever $x,y\in C$ are distinct. Finally, $M_i(C)\geq 0$ for every i and $f_i \leq 0$ for $i \notin \{2,4,\dots 2k\}$ yield $f_i M_i(C)=0$ for every positive integer i. □

We will propose suitable polynomials $f(t)\in F_{n,k}$ in the next section. Key ingredients are certain polynomials $\{Q_i^{1,1}(t)\}_{i=0}^{n-2}$ (*adjacent* to the Krawtchouk ones) which were first introduced as such and investigated by Levenshtein (cf. [3] and references therein). In what follows in this section we describe the derivation of these polynomials.

The definition of the adjacent polynomials $\{Q_i^{1,1}(t)\}_{i=0}^{n-2}$ requires a few steps as follows (cf. [3]). Let

$$T_i(u,v) := \sum_{j=0}^{i} \binom{n}{i} Q_i^{(n)}(u) Q_i^{(n)}(v)$$

be the Christoffel-Darboux kernel (cf. [16]) for the Krawtchouk polynomials as defined in the Introduction. Then one defines $(1,0)$-adjacent polynomials [3] (Equation (5.65)) by

$$Q_i^{1,0}(t) := \frac{T_i(t,1)}{T_i(1,1)}, \quad i=0,1,\dots,n-1. \tag{5}$$

For the final step, denote

$$T_i^{1,0}(x,y) := \sum_{j=0}^{i} \frac{\left(\sum_{u=0}^{j} \binom{n}{u}\right)^2}{\binom{n-1}{j}} Q_j^{1,0}(x) Q_j^{1,0}(y) \tag{6}$$

(the Christoffel-Darboux kernel for the $(1,0)$-adjacent polynomials) and define [3] (Equation (5.68))

$$Q_i^{1,1}(t) := \frac{T_i^{1,0}(t,-1)}{T_i^{1,0}(1,-1)}, \quad i = 0, 1, \ldots, n-2. \tag{7}$$

The first few $(1,1)$-adjacent polynomials are

$$Q_0^{1,1}(t) = 1, \ \ Q_1^{1,1}(t) = t,$$

$$Q_2^{1,1}(t) = \frac{n^2t^2 - n + 2}{n^2 - n + 2}, \ \ Q_3^{1,1}(t) = \frac{n^2t^3 - (n-8)t}{n^2 - n + 8}.$$

Equivalently, the polynomials $\{Q_i^{1,1}(t)\}_{i=0}^{n-2}$ can be defined as the unique series of normalized (to have value 1 at 1) polynomials orthogonal on T_n with respect to the discrete measure

$$\frac{nq^{2-n}(1-t)(1+t)}{4(n-1)(q-1)} \sum_{i=0}^{n} r_{n-i} \delta_{t_i}, \tag{8}$$

where δ_{t_i} is the Dirac-delta measure at $t_i \in T_n$ [3] (Section 6.2).

Finally, we note the explicit formula (cf. [3] (Section 6.2), [17] (p. 281))

$$Q_i^{1,1}(t) = \frac{K_i^{(n-2)}(z-1)}{\sum_{j=0}^{i} \binom{n-1}{j}}, \tag{9}$$

where $z = n(1-t)/2$, which relates the $(1,1)$-adjacent polynomials and the usual (binary) Krawtchouk polynomials

$$K_i^{(n)}(z) := \sum_{j=0}^{i} (-1)^j \binom{z}{j} \binom{n-z}{i-j}.$$

It follows from (9) that the polynomials $Q_i^{1,1}(t)$ are odd/even functions for odd/even i (this also follows from the fact that the measure (8) is symmetric in $[-1,1]$, therefore on T_n). We will use this fact when we deal with our proposal for a polynomial in Theorem 2.

4. A Universal Lower Bound for $\mathcal{M}(n,k)$

Using suitable polynomials in Theorem 2 we obtain the following universal bound.

Theorem 3. *We have*

$$\mathcal{M}(n,k) \geq \sum_{i=0}^{k} \binom{n-1}{i}.$$

If a (k,k)-design $C \subset F_2^n$ attains this bound, then all inner products $\langle x,y \rangle$ of distinct $x,y \in C$ are among the zeros of $Q_k^{1,1}(t)$ and $|C| = \sum_{i=0}^{k} \binom{n-1}{i}$ is divisible by 2^{2k}.

Proof. We use Theorem 2 with the polynomial $f(t) = \left(Q_k^{1,1}(t)\right)^2$ of degree $2k$ (so we have $f_i = 0$ for $i \geq 2k+1$) and arbitrary (k,k)-design in F_2^n. It is obvious that $f(t) \geq 0$ for every $t \in [-1,1]$. Since $Q_k^{1,1}(t)$ is an odd or even function, its square is an even function. Then $f_i = 0$ for every odd i and thus $f \in F_{n,k}$. The calculation of the ratio $f(1)/f_0$ gives the desired bound.

If a (k,k)-design $C \subset F_2^n$ attains the bound, then equality in (4) follows (for C and the above $f(t)$). Since $f_i M_i(C) = 0$ for every i, the equality $|C| = f(1)/f_0$ is equivalent to

$$\sum_{x,y \in C, x \neq y} \left(Q_k^{1,1}(\langle x,y \rangle) \right)^2 = 0,$$

whence $Q_k^{1,1}(\langle x,y \rangle) = 0$ whenever x and y are distinct points from C. The divisibility condition follows from Corollary 1. $\square$

Remark 1. *Linear programming bounds (cf. (7)–(9) and Theorem 4.3 in [15]) with the polynomial* $(t+1)\left(Q_k^{1,1}(t)\right)^2$ *give the Rao [18] bound (see also [3,6] and references therein) for the minimum possible cardinality of* $(2k+1)$*-designs in* F_2^n*, that is* $2\sum_{i=0}^{k}\binom{n-1}{i}$*. Thus our calculation of* $f(1)/f_0$ *quite resembles (and in fact follows from) the classical one [4] (see also [3] (Section 2)) by noting that, obviously, the value in one is two times less and the coefficient* f_0 *is the same because of the symmetric measure (equivalently, since* $(t+1)\left(Q_k^{1,1}(t)\right)^2$ *is equal to the sum of the odd function* $t\left(Q_k^{1,1}(t)\right)^2$ *and our polynomial).*

Remark 2. *The bound of Theorem 3 can be proved also via the relation from Theorem 1 if we allow consideration of multisets and apply the Rao bound for orthogonal arrays with (possibly) repeating points. However, we prefer to keep the linear programming framework as more general and as giving information for the structure of designs which attain the linear programming bounds (to be used in the next section).*

5. On Tight (k,k)-Designs

Following Bannai et al. [5] we call *tight* every (k,k)-design in F_2^n with cardinality $\sum_{i=0}^{k}\binom{n-1}{i}$. Example 1 provides tight $(\ell-1,\ell-1)$-designs for any even $n = 2\ell$. Indeed, we have

$$\sum_{i=0}^{\ell-1}\binom{2\ell-1}{i} = \frac{1}{2}\sum_{i=0}^{2\ell}\binom{2\ell-1}{i} = 2^{2\ell-2}.$$

Theorem 1 allows us to relate the existence of tight (k,k)-designs and tight $(2k+1)$-designs.

Theorem 4. *For fixed n and k, tight* (k,k)*-designs exist if and only if tight* $(2k+1)$*-designs exist.*

Proof. If $C \subset F_2^n$ is a tight (k,k)-design, it cannot possess a pair of antipodal points since -1 is not a zero of $Q_k^{1,1}(t)$. Thus we may construct an antipodal $(2k-1)$-design $D \subset F_2^n$ with cardinality

$$2|C| = 2\sum_{i=0}^{k}\binom{n-1}{i};$$

i.e., attaining Rao bound.

Conversely, any tight $(2k+1)$-design in F_2^n has cardinality $2\sum_{i=0}^{k}\binom{n-1}{i}$ and is antipodal. By Theorem 1 it produces a tight (k,k)-design. $\square$

We proceed with consideration of the tight (k,k)-designs with $k \leq 3$. The tight $(1,1)$-designs coexist with the Hadamard matrices due to a well known construction. We recall for completeness the definition of a Hadamard matrix—it is a square matrix whose entries are either $+1$ or -1 and whose rows are mutually orthogonal.

The next result was also obtained in [14] (Proposition 2) for the classification of tight index 2 designs.

Theorem 5. *Tight* $(1,1)$*-designs exist if and only if n is divisible by 4 and there exists a Hadamard matrix of order n.*

Proof. Let $C \subset F_2^n$ be a (k,k)-design with

$$1+\binom{n-1}{1}=n$$

points. Then n is divisible by 4 and, moreover, since $Q_1^{1,1}(t)=t$, the only possible inner product is 0, meaning that the only possible distance is $n/2$. Therefore C is a $(n,n,n/2)$ binary code. Changing $0 \to -1$ we obtain a Hadamard matrix of order n. Clearly, this works in the other direction as well. □

Doubling a tight $(1,1)$-design gives a tight 3-design which is clearly related to a Hadamard code $(n,2n,n/2)$. It is also worth noting that a Hadamard matrix of order $n+1$ defines a tight 2-design in F_2^n, which is a $(1,1)$-design with cardinality $n+1$ [6] (Theorem 7.5); i.e., exceeding our bound by 1. The divisibility condition now shows that this is the minimum possible cardinality for length $n \equiv 3 \pmod 4$. Further examples of $(1,1)$-designs can be extracted from the examples in [19,20], where linear programming bounds for codes with given minimum and maximum distances are considered.

The classification of tight (k,k)-designs, $k \geq 2$, will be already as difficult combinatorial problem as the analogous problems for classical designs in Hamming spaces (see, for example [5,8,21,22] and references therein). We present here the direct consequences of the linear programming approach combined with the divisibility condition of Corollary 1.

Theorem 6. *Tight $(2,2)$-designs could possibly exist only for $n=m^2+2$, where $m \geq 3$ is a positive integer, $m \equiv 2,5,6,10,11$ or $14 \pmod{16}$.*

Proof. Let $C \subset F_2^n$ be a tight $(2,2)$-design. For $k=2$, we have

$$\mathcal{M}(n,2) \geq 1+\binom{n-1}{1}+\binom{n-1}{2}=(n^2-n+2)/2,$$

which means that n^2-n+2 is divisible by 32. This yields $n \equiv 6$ or $27 \pmod{32}$.

Looking at the zeros of $Q_2^{1,1}(t)$, we obtain $\pm\sqrt{n-2}/n \in T_n$, whence it follows that $n-2$ has to be a perfect square. Setting $n=m^2+2$, we obtain $m \equiv 2,5,6,10,11$ or $14 \pmod{16}$. □

The classification of tight 4-designs was recently completed by Gavrilyuk, Suda, and Vidali [21] (see also [22]). The only tight 4-design is the unique even-weight code of length 5 (see Example 1). It has cardinality 16, which is the minimum possibility for a $(2,2)$-design of length 5 since in this case our bound is 11 and the cardinality must be divisible by $2^4=16$.

Theorem 7. *Tight $(3,3)$-designs could possibly exist only for $n \equiv 8 \pmod{16}$ or $n \equiv 107 \pmod{128}$, where $n=(m^2+8)/3$, $m \geq 4$ is a positive integer, divisible by 4 and not divisible by 3, or $m \equiv 43 \pmod{64}$. The code obtained as in Theorem 1 from the binary Golay code $[24,12,8]$ is a tight $(3,3)$-design.*

Proof. Let $C \subset F_2^n$ be a tight $(3,3)$-design. Then 2^6 divides

$$|C|=1+\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}=\frac{n(n^2-3n+8)}{6},$$

i.e., $n(n^2-3n+8)$ is divisible by 2^7. This gives $n \equiv 0 \pmod 8$ or $n \equiv 107 \pmod{128}$. Since $Q_3^{1,1}(t)$ has roots 0 and $\pm\sqrt{3n-8}/n$ (the later necessarily belonging to T_n; otherwise C would be an equidistant code with the only allowed distance $n/2$), it follows that $3n-8$ is a perfect square. Setting $n=8u$ and $3n-8=m^2$, we easily see that u has to be odd and m cannot be multiple of 3. If $n \equiv 107 \pmod{128}$, we obtain $m \equiv 43 \pmod{64}$.

The necessary conditions are fulfilled for $n=24$, where the Golay code, which is a tight 7-design, produces as in Theorem 1 a tight $(3,3)$-design of $2^{11}=2048$ points. □

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