



Article On the Dirichlet Problem with Corner Singularity

Viktor A. Rukavishnikov * D and Elena I. Rukavishnikova

Computing Center of Far-Eastern Branch, Russian Academy of Sciences, Kim-Yu-Chen Str. 65, 680000 Khabarovsk, Russia; rukavishnikova-55@mail.ru

* Correspondence: vark0102@mail.ru

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Abstract: We consider the Dirichlet problem for an elliptic equation with a singularity. The singularity of the solution to the problem is caused by the presence of a re-entrant corner at the boundary of the domain. We define an R_{ν} -generalized solution for this problem. This allows for the construction of numerical methods for finding an approximate solution without loss of accuracy. In this

paper, the existence and uniqueness of the R_{ν} -generalized solution in set $W_{2,\alpha}^{1}(\Omega, \delta)$ is proven. The R_{ν} -generalized solution is the same for different parameters ν .

Keywords: corner singularity; boundary-value problem; R_{ν} -generalized solution; existence and uniqueness solution

MSC: 35J25

1. Introduction

The weak solution of a boundary value problem for the elliptic equation in domain Ω ($\Omega \subset R^2$) with reentrant corners $\gamma_i (i = 1, ..., N)$ on boundary $\partial \Omega$, belongs to class $W_2^{1+k-\varepsilon}(\Omega)$, $k = \min_{i=1,...,N} \{k_i\}$, where

 $k_i = \begin{cases} \pi/\gamma_i & \text{for Dirichlet or Neuman problem,} \\ \pi/2\gamma_i & \text{for mixed boundary value problem,} \end{cases}$

and ε is an arbitrary positive number.

For two-dimensional linear elliptic boundary-value problems with homogeneous Dirichlet–Dirichlet and Dirichlet–Neumann boundary conditions posed on the adjacent sides of re-entrant corners, it was stated that $0.25 \le k_i \le 0.67$ for $3\pi/2 \le \gamma_i \le 2\pi$.

In this case, the approximate finite-element solution converges to a weak solution of the problem at an $O(h^k)$ rate (*h* is the mesh step) in the norm of Sobolev space $W_2^1(\Omega)$ [1,2].

By using meshes refined toward the singularity point or special methods for separating the singular and regular components of the solution near corner points, numerical methods of the first order of accuracy were created (see, for example, [3–12]). However, these methods lead to ill-conditioned systems of linear algebraic equations. These involve complication of computing process and affect the accuracy of the results.

For the boundary-value problem for elliptic equations with singularity, we propose to define its solutions as an R_{ν} -generalized one (see, for examples, [13]). This conception allows for investigating problems with singularities of different kinds (discontinuous or not bounded coefficients, right hands of the equation and boundary conditions; existence of the re-entrant corners on the boundary) and constructing the weighted FEMfor these problems. This method provides convergence rates O(h) and $O(h^2)$ of the approximate finite-element solution to the R_{ν} -generalized one in the norm of the Sobolev and Lebesque weighted spaces, respectively (see [14–17]).

For boundary-value problems with corner singularity, the existence and uniqueness of an R_{ν} -generalized solution was established in [18–22] assuming that the coefficient multiplying u(x) in the differential equation is sufficiently large. Below, this restrictive assumption is dropped and a weighted set of functions is defined in which a unique R_{ν} -generalized solution exists for the given class of the boundary value problems; moreover, this solution is unique for all ν .

2. Problem Formulation

Let $\Omega \subset R^2$ be a nonconvex polygon and $\partial\Omega$ its boundary, with an obtuse corner γ with the vertex at the origin O(0,0); $\overline{\Omega}$ denotes the closure of Ω , i.e. $\overline{\Omega} = \Omega \cup \partial\Omega$.

We introduce weight function

$$\rho(x) = \begin{cases} (x_1^2 + x_2^2)^{1/2}, & x \in \Omega', \\ \delta, & x \in \bar{\Omega} \setminus \Omega, \end{cases}$$

where $\Omega' = \{x \in \overline{\Omega} : (x_1^2 + x_2^2)^{1/2} \le \delta \ll 1\}$ (see Figure 1).



Figure 1. Domain Ω with re-entrant corner.

Let $W_{2\alpha}^1(\Omega, \delta)$ ($\alpha > 0$) be the set of functions subject the following conditions:

(a) $|D^k u(x)| \le c_1 (\delta/\rho(x))^{\alpha+k}$ for $x \in \Omega', \ k = 0, 1, \ c_1 > 0;$

(b)
$$||u||_{L_{2,\alpha}(\Omega \setminus \Omega')} \ge c_2 > 0$$
,

equipped with norm

$$\|u\|_{W^{1}_{2,\alpha}(\Omega)} = \left(\sum_{|\lambda| \le 1} \int_{\Omega} \rho^{2\alpha}(x) |D^{\lambda}u(x)|^{2} dx\right)^{1/2},$$
(1)

where $D^{\lambda} = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}}$, $\lambda = (\lambda_1, \lambda_2)$ with $|\lambda| = \lambda_1 + \lambda_2$ (here, λ_1 and λ_2 are non-negative integers). For k = 0, we use notation $W_{2,\alpha}^0(\Omega, \delta) = L_{2,\alpha}(\Omega, \delta)$.

Set $W_{2,\alpha}^1(\Omega, \delta) \subset W_{2,\alpha}^1(\Omega, \delta)$ is a closure in Norm (1) of set $C_{\infty}(\Omega, \delta)$ that contain functions satisfying Conditions (a) and (b). Here, $C_{\infty}(\Omega)$ is an infinitely differentiable and finite space in Ω functions.

Let $L_{\infty,-\alpha}(\Omega, c_3)$ denote the set of functions with norm

$$\|u\|_{L_{\infty,-\alpha}(\Omega,c_3)} = \operatorname{vraimax}_{x\in\Omega} |\rho^{-\alpha}(x)u(x)| \le c_3,$$

where c_3 is positive constant not depending on u(x).

We consider differential equation

$$-\sum_{l=1}^{2} \frac{\partial}{\partial x_{l}} \left(a_{ll}(x) \frac{\partial u}{\partial x_{l}} \right) + a(x)u(x) = f(x), \ x \in \Omega,$$
(2)

with boundary condition

$$u(x) = 0, \quad x \in \partial \Omega. \tag{3}$$

Definition 1. Boundary-value problem (2), (3) is called the Dirichlet problem with uncoordinated degeneration of the input data if the coefficients and right-hand side of the equation satisfy conditions

$$a_{ll} \in L_{\infty,-\beta}(\Omega, c_4), \quad a \in L_{\infty,-\beta}(\Omega, c_5),$$
(4)

$$\sum_{l=1}^{2} a_{ll}(x)\xi_{l}^{2} \ge c_{6}\rho^{\beta}(x)\sum_{l=1}^{2}\xi_{l}^{2},$$
(5)

 $a(x) > c_7 \rho^{\beta}(x)$ almost everywhere on Ω , (6)

$$f \in L_{2,\mu}(\Omega,\delta), \qquad \mu \ge 0. \tag{7}$$

Here β , μ *are real numbers,* $c_i \ge 0$, i = 4, 5, 6, 7, *while* ξ_1 *and* ξ_2 *are arbitrary real parameters.*

Define

$$a_{\Omega}(u,v) = \int_{\Omega} \left[\sum_{l=1}^{2} a_{ll} \rho^{2\nu} \frac{\partial u}{\partial x_{l}} \frac{\partial v}{\partial x_{l}} + a_{ll} \frac{\partial \rho^{2\nu}}{\partial x_{l}} \frac{\partial u}{\partial x_{l}} v \right] + a \rho^{2\nu} u v \, dx,$$
$$f_{\Omega}(v) = \int_{\Omega} \rho^{2\nu} f v \, dx.$$

Definition 2. Function u_{ν} from space $\overset{\circ}{W}^{1}_{2,\nu+\beta/2}(\Omega)$ is called an R_{ν} -generalized solution of the Dirichlet problem with uncoordinated degeneration of the input data if identity

$$a_{\Omega}(u_{\nu},v) = f_{\Omega}(v)$$

holds for all $v \in \overset{\circ}{W}^{1}_{2,\nu+\beta/2}(\Omega)$ and for any given value of ν , satisfying inequality

$$\nu \ge \mu + \beta/2. \tag{8}$$

Remark 1. The summands of bilinear form $a_{\Omega}(u, v)$ have different asymptotic orders. This is due to the fulfilment of Condition (4). In [18,19] for such problems, we established that there is no unique R_{ν} -generalized solution in space $\mathring{W}_{2,\alpha}^1(\Omega)$, since there is a sheaf of solutions in the neighborhood of the singular point. In [20–22], we proved the uniqueness of the R_{ν} -generalized solution in set $\mathring{W}_{2,\alpha}^1(\Omega, \delta)$ with the correct choice of parameters v and δ .

Examples of problems with uncoordinated degeneration of the initial data can serve as boundary-value problems for the Lamé and Maxwell systems, and Stokes problem in domains with re-entrant corners on the boundary.

Remark 2. The main idea of the proposed approach is as follows: we introduce into the generalized statement of Problems (2)–(7) a special weight function ρ raised to a certain power depending on the spaces that contain the coefficients and the right-hand side of the equation, sizes of re-entrant corners on the domain boundary.

The presence of such a function in the definition of the solution overpowers the singularity and ensures that the R_{ν} *-generalized solution of the problem belongs to set* $W_{2,\nu}^2(\Omega, \delta)$ *.*

3. Existence and Uniqueness of R_{ν} -Generalized Solution

We formulate the main result.

Theorem 1. Let Conditions (4)–(8) hold and

$$\nu + \frac{\beta}{2} > 0. \tag{9}$$

Then, for any v satisfying Conditions (8) and (9), there always exists parameter δ , such that R_v -generalized solution u_v of the Dirichlet problem with uncoordinated degeneration of the input data exists and is unique from set $\overset{\circ}{W}_{2,v+\beta/2}^1(\Omega,\delta)$ and

$$\|u_{\nu}\|_{W^{1}_{2,\nu+\beta/2}(\Omega,\delta)} \le c_{8} \|f\|_{L_{2,\mu}(\Omega,\delta)},$$
(10)

where a positive constant c_8 is independent of f and u_{ν} .

To prove this theorem, we need the lemma.

Lemma 1 ([22]). For each function u in set $W_{2,\alpha}^1(\Omega, \delta)$ and for any $\alpha^* > \alpha$, estimate

$$\|u\|_{L_{2,\alpha^*-1}(\Omega',\delta)} \le c_9 \|u\|_{L_{2,\alpha^*}(\Omega,\delta)}$$
(11)

holds, where $c_9 = c_{10} \frac{\delta^{\alpha}}{\sqrt{\alpha^* - \alpha}}$, $c_{10} = const > 0$.

Proof. For $\alpha^* > \alpha$ and Condition (a), we have

$$\|u\|_{L_{2,\alpha^*-1}(\Omega',\delta)}^2 = \int_{\Omega'} \rho^{2(\alpha^*-1)} u^2 \, dx \le c_1^2 \delta^{2\alpha} \int_{\Omega'} \rho^{2(\alpha^*-1)} \rho^{-2\alpha} \, dx \le \frac{c_1^2 \delta^{2\alpha} c_{11} \delta^{2(\alpha^*-\alpha)}}{2(\alpha^*-\alpha)},\tag{12}$$

where c_{11} is a constant dependent of mes Ω' . From Condition (b) for function u, we obtain

$$\|u\|_{L_{2,\alpha^{*}}(\Omega)}^{2} \ge \|u\|_{L_{2,\alpha^{*}}(\Omega \setminus \Omega')}^{2} = \delta^{2(\alpha^{*}-\alpha)} \|u\|_{L_{2,\alpha}(\Omega \setminus \Omega')}^{2} \ge c_{2}^{2} \delta^{2(\alpha^{*}-\alpha)}.$$
(13)

From Inequalities (12) and (13), we obtain estimate (11) with $c_{10} = \frac{c_1}{c_2} \sqrt{\frac{c_{11}}{2}}$.

Proof of Theorem 1. First, we show that forms $a_{\Omega}(u, v)$ and $f_{\Omega}(v)$ are continuous on $\overset{\circ}{W}^{1}_{2,\nu+\beta/2}(\Omega, \delta)$, and inequalities

$$a_{\Omega}(u,v) \le c_{12} \|u\|_{W^{1}_{2,\nu+\beta/2}(\Omega,\delta)} \|v\|_{W^{1}_{2,\nu+\beta/2}(\Omega,\delta)},$$
(14)

$$f_{\Omega}(v) \le c_{13} \|f\|_{L_{2,\mu}(\Omega,\delta)} \|v\|_{W^{1}_{2,\nu+\beta/2}(\Omega,\delta)}$$
(15)

hold. The proofs of Estimates (14) and (15) are established by using Conditions (4), (7), and (8), Lemma 1, and Cauchy–Schwarz inequality.

Let us show that the bilinear form is $\overset{\circ}{W}^{1}_{2,\nu+\beta/2}$ -elliptical in Ω . We have

$$a_{\Omega}(u,u) = \sum_{k=1}^{2} \int_{\Omega} \left[a_{kk} \rho^{2\nu} \left(\frac{\partial u}{\partial x_{k}} \right)^{2} + a_{kk} \frac{\partial \rho^{2\nu}}{\partial x_{k}} \frac{\partial u}{\partial x_{k}} u \right] dx + \int_{\Omega} a \rho^{2\nu} u^{2} dx$$
(16)

for any *u* from $W^1_{2,\nu+\beta/2}(\Omega, \delta)$. Considering Condition (4), we establish the inequality for the absolute value of the second term on the right-hand side in (16)

$$\begin{aligned} \left| \sum_{k=1}^{2} \int_{\Omega} a_{kk} \frac{\partial \rho^{2\nu}}{\partial x_{k}} \frac{\partial u}{\partial x_{k}} u \, dx \right| &\leq \left| \sum_{k=1}^{2} \int_{\Omega'} a_{kk} 2\nu \rho^{2\nu-1} \frac{\partial u}{\partial x_{k}} u \, dx \right| \leq \\ &\leq \varepsilon \sum_{k=1}^{2} \int_{\Omega'} \rho^{2\nu+\beta} \left(\frac{\partial u}{\partial x_{k}} \right)^{2} \, dx + \frac{2c_{4}^{2}\nu^{2}}{\varepsilon} \int_{\Omega'} \rho^{2\nu+\beta-1} u^{2} \, dx. \end{aligned}$$

From (16) and the last inequality, we obtain

$$a_{\Omega}(u,u) \geq \sum_{k=1}^{2} \int_{\Omega} a_{kk} \rho^{2\nu} \left(\frac{\partial u}{\partial x_{k}}\right)^{2} dx - \varepsilon \sum_{k=1}^{2} \int_{\Omega'} \rho^{2\nu+\beta} \left(\frac{\partial u}{\partial x_{k}}\right)^{2} dx + \int_{\Omega} a \rho^{2\nu} u^{2} dx - \frac{2c_{4}^{2}\nu^{2}}{\varepsilon} \int_{\Omega'} \rho^{2\nu+\beta-1} u^{2} dx.$$
(17)

Supposing that α^* and α equal ν and $\nu/2$ in Lemma 1, respectively, we have

$$\|u\|_{L_{2,\nu+\beta/2-1}(\Omega',\delta)}^{2} \leq \frac{2c_{10}^{2}\delta^{\nu}}{\nu}\|u\|_{L_{2,\nu+\beta/2}(\Omega,\delta)}^{2}, \quad c_{10} = \text{const} > 0.$$
(18)

Considering (5), (6) and (18) from Estimate (17) we obtain

$$a_{\Omega}(u,u) \ge (c_{5}-\varepsilon)|u|^{2}_{W^{1}_{2,\nu+\beta/2}(\Omega,\delta)} + \left(c_{6}-\frac{4c_{4}^{2}c_{10}^{2}\nu\delta^{\nu}}{\varepsilon}\right)||u||^{2}_{L_{2,\nu+\beta/2}(\Omega,\delta)}.$$

Obviously, we can always choose ε and δ , such that constants $c_5 > \varepsilon$, $c_6 > \frac{4c_4^2 c_{10}^2 v \delta^{\nu}}{\varepsilon}$, and inequality

$$a_{\Omega}(u,u) \ge c_{14} \|u\|_{W^{1}_{2,\nu+\beta/2}(\Omega,\delta)}^{2}$$
(19)

is valid with constant $c_{14} = \min\left(c_6 - \varepsilon, c_7 - \frac{4c_4^2 c_{10}^2 v \delta^v}{\varepsilon}\right)$. Therefore, bilinear form $a_{\Omega}(u, u)$ is $\overset{\circ}{W}_{2,\nu+\beta/2}^1$ -elliptical.

According to (14), (15), and (19), bilinear form $a_{\Omega}(u, u)$ is continuous, and $W_{2,\nu+\beta/2}^1$ -elliptical, and linear form $f_{\Omega}(v)$ is continuous on $W_{2,\nu+\beta/2}^1(\Omega, \delta)$; then, the existence and uniqueness of an R_{ν} -generalized solution of Problems (2)–(7) follows from the Lax–Milgram theorem (see [1]).

Considering that

$$c_{14} \|u_{\nu}\|_{W^{1}_{2,\nu+\beta/2}(\Omega,\delta)}^{2} \leq a_{\Omega}(u_{\nu},u_{\nu}) = f_{\Omega}(u_{\nu}) \leq c_{13} \|u_{\nu}\|_{W^{1}_{2,\nu+\beta/2}(\Omega,\delta)} \|f\|_{L_{2,\mu}(\Omega,\delta)}$$

we obtain Estimate (10). \Box

Theorem 2 ([23]). If for some δ there is a set of values ν , such that an R_{ν} -generalized solution of Problems (2)–(7) exists in set $\overset{\circ}{W}^{1}_{2,\nu+\beta/2}(\Omega,\delta)$, then this solution is unique for all such ν .

Remark 3. Theorems 1 and 2 are valid for the domain with a boundary containing multiple re-entrant corners.

Remark 4. For the Stokes problem, the Lamé system, and Maxwell's equations in a two-dimensional domain with a boundary containing re-entrant corners, we constructed and investigated a weighted FEM that provides convergence rate O(h), and this rate does not depend of the singularity size (value) ([14–17,23,24]). The solution for those boundary-value problems was defined as an R_{ν} -generalized one in the weighted set.

4. Conclusions

We presented the concept of an R_{ν} -generalized solution for boundary-value problems with a singularity. We proved a theorem on the existence and uniqueness of an R_{ν} -generalized solution in a special set. This definition of the solution allows for creating highly efficient numerical methods without loss of accuracy for boundary-value problems with a singularity. The established theorem helps to determine optimal parameters ν and δ in numerical methods to improve the accuracy of finding an approximate solution. We plan to determine the body of optimal parameters (BOP) for the crack problem, for boundary-value problems of the theory of elasticity in regions with re-entrant angles at the boundary, and with a change in the type of boundary conditions.

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