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Generalized Concentration-Compactness Principles for Variable Exponent Lebesgue Spaces with Asymptotic Analysis of Low Energy Extremals

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Abstract: In this paper, we prove two generalized concentration-compactness principles for variable exponent Lebesgue spaces and as an application study the asymptotic behaviour of low energy extremals.

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MSC: 35J30; 35J35

1. Introduction

The concentration-compactness principle (CCP) by Lions [1] has been a fundamental tool to study solutions of different kinds of elliptic PDEs with critical growth (in the sense of Sobolev embeddings), see [2–5] for some of its applications. Later on, in [6,7] Lions CCP was generalized by considering a general growth at infinity.

Consider

$$-\triangle_{p(x)}v = g(x,v), \tag{1}$$

where $\triangle_{p(x)}v := div(|\nabla v|^{p(x)-2}\nabla v)$ is known as $p(x)-$Laplacian operator. The above problem naturally arises in studying models like electroheological fluids. Many researchers studied it with different boundary conditions (Neumann, Dirichlet, nonlinear, etc.), see [8–12] and references therein.

Let Ω be a bounded sub domain of $\mathbb{R}^N$, for an exponent $p(x)$ we will use $p^- := \inf_{x\in\Omega} p(x)$, $p^+ := \sup_{x\in\Omega} p(x)$ and $p^*(x) := \frac{Np(x)}{N-p(x)}$ when $p(x) < N$. An exponent $q(x) \leq p^*(x)$ is said to be critical if $x \in \mathcal{C} := \{x \in \Omega : q(x) = p^*(x)\}$. In order to deal with the critical growth at infinity of the source function g that is

$$|g(x,s)| \leq c(1+|s|^{q(x)}), \tag{2}$$

with $q(x) \leq p^*(x)$, Bonder and Silva [13] and Yongqiang [14] extended Lions CCP to variable exponent settings, independently. Their method of proof followed the same lines as the ones that originated in Lions work.

Let $G : \mathbb{R} \to \mathbb{R}$ be an upper semicontinuous, not zero in L^1 sense and satisfying the growth condition

$$0 \leq G(s) \leq c\min\left\{|s|^{q^+}, |s|^{q^-}\right\} \text{ for } s \in \mathbb{R}, \tag{3}$$

where $p \leq q \leq p^*, 1 < p^- \leq p(x) \leq p^+ < N$. This paper aims to study the Problem (1) with a general growth at infinity by extending the work of Flucher and Müller [6] to variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. To be more precise, we study the concentration/compactness of the sequence $G(v_\epsilon)$ for $v_\epsilon \in W_0^{1,p(x)}(\Omega)$ (closure of the set of test functions in variable exponent Sobolev space), whereas, Bonder and Silva [13] studied $|v_\epsilon|^{q(x)}$. Thus, our work considerably contributes to the existing literature and it allows us to study Bernoulli's free-boundary problem, plasma problem and others in the variable exponent settings, see [7] for more details. We prove that in a extreme case either the sequence of measures concentrate to a dirac measure or have a convergent subsequence.

In addition, we analyse the asymptotic behaviour of solutions of the following variational problem, related to low energies

$$G_\epsilon^*(p(.), q(.), \Omega) := \sup \left\{ \int_\Omega \frac{G(v)}{\epsilon^{q(x)}} dx : v \in W_0^{1,p(x)}(\Omega), \|\nabla v\|_{L^{p(x)}(\Omega)} \leq \epsilon \right\}, \tag{4}$$

when $\epsilon \to 0$. Problem (4) and its other variants for a constant exponent were rigorously studied, see [6,7,15–17] and references therein. To establish the concentration or compactness of low energy extremals, another version of CCP is proved for the variable exponent Lebesgue spaces. When G is smooth i.e., $G' = g$, solutions of (4) satisfy the following Dirichlet problem

$$\begin{cases} -\triangle_{p(x)} v = g(v), & \text{in } \Omega; \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

For a detailed study on nonlinear PDEs with variable exponent, we refer [18].

Organisation of this paper: Section 2 collects some necessary primary results to be used in later sections. Section 3 deals with the proof of generalized CCP and concentration/compactness result. Section 4 is committed to the variational problem of low energy extremals. Finally, Section 5 ends the manuscript with some concluding remarks.

2. Preliminary and Known Results

We present some preliminary concepts of variable exponent Lebesgue and Sobolev spaces. Let $p : \Omega \to [1, \infty]$ be a measurable function and Ω be a bounded smooth subset of $\mathbb{R}^N$. Then $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega) = \{v \in L^1_{loc}(\Omega) : \int_\Omega |v(x)|^{p(x)} dx < \infty\},$$

endowed with the norm

$$\|v\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0 : \int_\Omega \left|\frac{v(x)}{\lambda}\right|^{p(x)} \leq 1\}.$$

In addition, $p'(x) = p(x)/(p(x) - 1)$ is known as conjugate exponent of $p(x)$, further

$$p^- := \inf_\Omega p(x), \; p^+ := \sup_\Omega p(x)$$

will be used throughout the paper and

$$p^*(x) := \begin{cases} \frac{Np(x)}{N - p(x)}, & \text{if } p(x) < N; \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

The exponent $p(x)$ is called log-Hölder continuous if

$$\sup_{x,y \in \Omega} |(p(x) - p(y)) \log(|x - y|)| < \infty. \tag{5}$$

Let $\rho(v) := \int_\Omega |v(x)|^{p(x)} dx$ then the following proposition proved in [19] is quite useful.

Proposition 1. *For $v \in L^{p(x)}(\Omega)$ and $\{v_n\}_{n\in\mathbb{N}} \subseteq L^{p(x)}(\Omega)$, we have*

$$v \neq 0 \Rightarrow \left(\|v\|_{L^{p(x)}(\Omega)} = \lambda \Leftrightarrow \rho\left(\frac{v}{\lambda}\right) = 1 \right). \tag{6}$$

$$\|v\|_{L^{p(x)}(\Omega)} < 1 (= 1; > 1) \Leftrightarrow \rho(v) < 1 (= 1; > 1). \tag{7}$$

$$\|v\|_{L^{p(x)}(\Omega)} > 1 \Rightarrow \|v\|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho(v) \leq \|v\|^{p^+}_{L^{p(x)}(\Omega)}. \tag{8}$$

$$\|v\|_{L^{p(x)}(\Omega)} < 1 \Rightarrow \|v\|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho(v) \leq \|v\|^{p^-}_{L^{p(x)}(\Omega)}. \tag{9}$$

$$\lim_{n\to\infty} \|v_n\|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho(v_n) = 0. \tag{10}$$

$$\lim_{n\to\infty} \|v_n\|_{L^{p(x)}(\Omega)} = \infty \Leftrightarrow \lim_{n\to\infty} \rho(v_n) = \infty. \tag{11}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \{v \in W^{1,1}_{loc} : u \in L^{p(x)}(\Omega) \text{ and } |\nabla v| \in L^{p(x)}(\Omega)\}.$$

Moreover, the norm for variable exponent Sobolev spaces is known as

$$\|v\|_{W^{1,p(x)}(\Omega)} = \|v\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)}.$$

$W^{1,p(x)}_0(\Omega)$ is defined to be the closure of $C^\infty_0(\Omega)$ in $W^{1,p(x)}(\Omega)$. If $1 < p^- \leq p^+ < \infty$ then all the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$ are separable and reflexive Banach spaces.

Proposition 2 (Holder-type inequality)**.** *Let $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$. Then,*

$$\int_\Omega |u(x)v(x)| dx \leq C_p \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}.$$

Proposition 3 (Sobolev embedding)**.** *Let $p, q \in C(\overline{\Omega})$ be log-Hölder continuous and $1 \leq q(x) \leq p^*(x)$ for all $x \in \overline{\Omega}$. Then,*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

also, the above embedding is compact if $\inf_\Omega (p^*(x) - q(x)) > 0$.

Proposition 4 (Poincaré inequality)**.** *For all v in $W^{1,p(x)}_0(\Omega)$ we have*

$$\|v\|_{L^{p(x)}(\Omega)} \leq C \|\nabla v\|_{L^{p(x)}(\Omega)}.$$

By the above proposition for $W^{1,p(x)}_0(\Omega)$ both norms $\|\nabla v\|_{L^{p(x)}(\Omega)}$ and $\|v\|_{W^{1,p(x)}(\Omega)}$ are equivalent. Lastly, we present a localized sobolev type inequality from [14]. By $B_r(x)$ we mean a ball of radius r centered at x in Ω.

Proposition 5. *Take x_0 in Ω. For every $\delta > 0$ there is a constant $k(\delta)$ independent of x in Ω such that if $0 < r < R$ with $\frac{r}{R} < k(\delta)$ then there is a cut-off test function ϕ_r^R in $W_0^{1,p(x)}(\Omega)$ with $\phi_r^R = 1$ in $B_r(x_0)$, $\phi_r^R = 0$ outside $B_R(x_0)$ and*

$$\int_{B_R(x_0)} |\nabla(\phi_r^R v)|^{p(x)} dx \le \int_{B_R(x_0)} |\nabla v|^{p(x)} dx + \delta \max\left\{ \|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)} \right\}, \tag{12}$$

for all v in $W_0^{1,p(x)}(\Omega)$.

3. Generalized Concentration-Compactness Principle

The exponent $q(.)$ is critical when $x \in \mathcal{C}$. The version of CCP proved in [14], only considered the critical case, whereas, in [13] $q(.)$ was allowed to be subcritical as well. Later on, in [20] CCP was refined a bit to study immersion problem for the variable exponent Sobolev space.

Now, we introduce some more notations in order to present the main results.

- Best Sobolev constant

$$S(p(.), q(.), \Omega) = \sup\left\{ \int_\Omega |v|^{q(x)} dx : v \in W_0^{1,p(x)}(\Omega), \|\nabla v\|_{L^{p(x)}(\Omega)} \le 1 \right\}. \tag{13}$$

- Generalized Sobolev constant

$$G^*(p(.), q(.), \Omega) := \sup\left\{ \int_\Omega G(v) dx : v \in W_0^{1,p(x)}(\Omega) \text{ and } \|\nabla v\|_{L^{p(x)}(\Omega)} \le 1 \right\}. \tag{14}$$

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$$\begin{aligned} G_0^+ &:= \limsup_{s\to 0} \frac{G(s)}{|s|^{q^+}}, \quad G_0^- := \liminf_{s\to 0} \frac{G(s)}{|s|^{q^+}}, \\ G_\infty^+ &:= \limsup_{|s|\to \infty} \frac{G(s)}{|s|^{q^-}}, \quad G_\infty^- := \liminf_{|s|\to \infty} \frac{G(s)}{|s|^{q^-}}, \end{aligned}$$

  moreover, denote $G_0 := G_0^+ = G_0^-$ and $G_\infty := G_\infty^+ = G_\infty^-$ in case of equality.

  By Sobolev embedding and Poincaré inequality for all $v \in W_0^{1,p(x)}(\Omega)$

$$\int_\Omega |v(x)|^{q(x)} dx \le S \max\left\{ \|\nabla v\|^{q^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{q^-}_{L^{p(x)}(\Omega)} \right\}. \tag{15}$$

By Growth condition (3) and Inequality (15), we have

$$\int_\Omega G(v) dx \le G^* \max\left\{ \|\nabla v\|^{q^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{q^-}_{L^{p(x)}(\Omega)} \right\}, \tag{16}$$

for all $v \in W_0^{1,p(x)}(\Omega)$. Now, we present the generalized CCP in form of following theorem. Let $\mathcal{M}(\overline{\Omega})$ be a set of all nonnegative finite Borel measures on $\overline{\Omega}$ and $\eta_\epsilon \overset{*}{\rightharpoonup} \eta$ in the sense of measure if $\int_{\overline{\Omega}} \phi \eta_\epsilon dx \to \int_{\overline{\Omega}} \phi \eta dx$ for all ϕ in $C(\overline{\Omega})$.

Theorem 1. *Let p and q be log-Hölder continuous exponents with*

$$1 < p^- \le p^+ < N, \ p \le q \le p^* \text{ in } \Omega \text{ and } \mathcal{C} =: \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset.$$

Let $\{v_\epsilon\}$ be a sequence in $W_0^{1,p(x)}(\Omega)$ with $\|\nabla v_\epsilon\|_{L^{p(x)}(\Omega)} \le 1$. If

- *$v_\epsilon \rightharpoonup v$ weakly in $W_0^{1,p(x)}(\Omega)$,*
- *$|\nabla v_\epsilon|^{p(x)} \overset{*}{\rightharpoonup} \eta$ in the sense of measure in $\mathcal{M}(\overline{\Omega})$,*

- $G(v_\epsilon) \overset{*}{\rightharpoonup} \zeta$ *in the sense of measure in* $\mathcal{M}(\overline{\Omega})$.

  Then, for a countable index set J

$$\eta = |\nabla v|^{p(x)} + \overline{\eta} + \sum_{j\in J} \eta_j \delta_{x_j},\ \eta(\overline{\Omega}) \leq 1, \tag{17}$$

$$\zeta = h + \sum_{j\in J} \zeta_j \delta_{x_j},\ \zeta(\overline{\Omega}) \leq G^*, \tag{18}$$

where $\{x_j\}_{j\in J} \subseteq \mathcal{C}$, $\overline{\eta}$ *is a positive nonatomic measure in* $\mathcal{M}(\overline{\Omega})$ *and* $h \in L^1(\Omega)$. *Moreover, atomic and regular parts satisfy the following generalized Sobolev type inequalities*

$$\zeta_j \leq G^* \max\left\{\eta_j^{\frac{q^+}{p^-}}, \eta_j^{\frac{q^-}{p^+}}\right\}, \tag{19}$$

$$\zeta(\overline{\Omega}) \leq G^* \max\left\{\eta(\overline{\Omega})^{\frac{q^+}{p^-}}, \eta(\overline{\Omega})^{\frac{q^-}{p^+}}\right\}, \tag{20}$$

$$\int_\Omega h dx \leq G^* \max\left\{\left(\int_\Omega |\nabla v|^{p(x)} dx + \overline{\eta}(\overline{\Omega})\right)^{\frac{q^+}{p^-}}, \left(\int_\Omega |\nabla v|^{p(x)} dx + \overline{\eta}(\overline{\Omega})\right)^{\frac{q^-}{p^+}}\right\}. \tag{21}$$

The strategy of the proof is analogous to that of [6,7], adapted to the variable exponent settings. In order to prove generalized CCP, first we prove two types of local generalized Sobolev inequalities, given in the following lemma.

Lemma 1. *Take* $\delta > 0$ *and* $r < R$ *satisfying* $\frac{r}{R} \leq k(\delta)$ *as in the Proposition 5. For* $x_0 \in \Omega$ *and* G *satisfying the growth condition (3) following inequalities hold*

$$\begin{aligned}\int_{B_r(x_0)} G(v)dx \leq G^* \max\Bigg\{&\left(\int_{B_R(x_0)} |\nabla v|^{p(x)} dx + \delta \max\left\{\|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^+}{p^-}}, \\ &\left(\int_{B_R(x_0)} |\nabla v|^{p(x)} dx + \delta \max\left\{\|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^-}{p^+}}\Bigg\},\end{aligned} \tag{22}$$

$$\begin{aligned}\int_{\Omega\setminus B_R(x_0)} G(v)dx \leq G^* \max\Bigg\{&\left(\int_{\Omega\setminus B_r(x_0)} |\nabla v|^{p(x)} dx + \delta \max\left\{\|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^+}{p^-}}, \\ &\left(\int_{\Omega\setminus B_r(x_0)} |\nabla v|^{p(x)} dx + \delta \max\left\{\|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^-}{p^+}}\Bigg\}.\end{aligned} \tag{23}$$

Proof. Without loss, assume $x_0 = 0$ and let ϕ_r^R be a cutoff test function as in Proposition 5 i.e., $\phi_r^R = 1$ in $B_r(0)$ and $\phi_r^R = 0$ outside $B_R(0)$. Then, for v in $W_0^{1,p(x)}(\Omega)$

$$\int_{B_r(0)} G(v)dx \leq \int_{B_R(0)} G(\phi_r^R v)dx \leq G^* \max\left\{\|\nabla(\phi_r^R v)\|^{q^+}_{L^{p(x)}(B_R(0))}, \|\nabla(\phi_r^R v)\|^{q^-}_{L^{p(x)}(B_R(0))}\right\}.$$

If $\|\nabla(\phi_r^R v)\|_{L^{p(x)}(B_R(0))} \geq 1$ then

$$\|\nabla(\phi_r^R v)\|^{q^+}_{L^{p(x)}(B_R(0))} \leq \left(\int_{B_R(0)} |\nabla(\phi_r^R v)|^{p(x)} dx\right)^{q^+/p^-}$$
$$\leq \left(\int_{B_R(x_0)} |\nabla v|^{p(x)} dx + \delta \max\left\{\|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^+}{p^-}}.$$

Similarly, for $\|\nabla(\phi_r^R v)\|_{L^{p(x)}(B_R(0))} < 1$ we have

$$\|\nabla(\phi_r^R v)\|^{q^-}_{L^{p(x)}(B_R(0))} \leq \left(\int_{B_R(x_0)} |\nabla v|^{p(x)} dx + \delta \max\left\{\|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^-}{p^+}}.$$

For the cutoff function $\psi = (1 - \phi_r^R)\phi_{R_1}^{R2}$ with $0 < r < R < R_1 < R_2$

$$\int_{B_{R_1}(0)\setminus B_R(0)} G(v)dx \leq \int_{B_{R_2}(0)\setminus B_r(0)} G(\psi v)dx$$
$$\leq G^* \max\left\{\left(\int_{B_{R_2}(0)\setminus B_r(0)} |\nabla v|^{p(x)} dx + \delta \max\left\{\|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^+}{p^-}},\right.$$
$$\left.\left(\int_{B_{R_2}(0)\setminus B_r(0)} |\nabla v|^{p(x)} dx + \delta \max\left\{\|\nabla v\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^-}{p^+}}\right\}.$$

Inequality (23) follows by letting $R_1 \to \infty$, $R_2 \to \infty$ in a way that $R_2/R_1 \to \infty$ and extending v as zero outside Ω. □

Now, we proceed to prove generalized CCP.

Proof of Theorem 1. Step 1: (Estimations of atomic part)
Let $\{x_j\}_{j\in J}$ be atoms of ζ. For $x \in \overline{\Omega}$ by Lemma 1

$$G(v_\epsilon)(\overline{B}_r(x)) \leq G^* \max\left\{\left(|\nabla v_\epsilon|^{p(x)}(\overline{B}_R(x)) + \delta\right)^{\frac{q^+}{p^-}}, \left(|\nabla v_\epsilon|^{p(x)}(\overline{B}_R(x)) + \delta\right)^{\frac{q^-}{p^+}}\right\},$$

passing the limits

$$\zeta(\{x\}) \leq \zeta(\overline{B}_r(x)) \leq G^* \max\left\{(\eta(\overline{B}_R(x)) + \delta)^{\frac{q^+}{p^-}}, (\eta(\overline{B}_R(x)) + \delta)^{\frac{q^-}{p^+}}\right\},$$

taking $r \to 0$, $R \to 0$ and $\delta \to 0$

$$\zeta(\{x\}) \leq G^* \max\left\{\eta(\{x\})^{\frac{q^+}{p^-}}, \eta(\{x\})^{\frac{q^-}{p^+}}\right\}.$$

In particular

$$\zeta_j \leq G^* \max\left\{\eta_j^{\frac{q^+}{p^-}}, \eta_j^{\frac{q^-}{p^+}}\right\},$$

where $\zeta_j = \zeta(\{x_j\})$ and $\eta_j = \eta(\{x_j\})$. Therefore, all the atoms of ζ are atoms of η. By Inequality (16)

$$\begin{aligned}\int_\Omega G(v_\epsilon)dx &\leq G^* \max\left\{\|\nabla v_\epsilon\|^{q^+}_{L^{p(x)}(\Omega)}, \|\nabla v_\epsilon\|^{q^-}_{L^{p(x)}(\Omega)}\right\}, \\ &\leq G^* \max\left\{\left(\int_\Omega |\nabla v_\epsilon|^{p(x)}dx\right)^{\frac{q^+}{p^-}}, \left(\int_\Omega |\nabla v_\epsilon|^{p(x)}dx\right)^{\frac{q^-}{p^+}}\right\}.\end{aligned}$$

Taking ϵ goes to zero

$$\zeta(\overline{\Omega}) \leq G^* \max\left\{\eta(\overline{\Omega})^{\frac{q^+}{p^-}}, \eta(\overline{\Omega})^{\frac{q^-}{p^+}}\right\}.$$

Step 2: (Decomposition of η)

Consider a functional $Q : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$

$$Q(v) = \int_\Omega |\nabla v|^{p(x)}\phi dx,$$

for a fix test function ϕ. It is differentiable and convex and hence weakly semicontinuous. Thus,

$$\int_\Omega |\nabla v|\phi dx \leq \liminf_{\epsilon\to 0} \int_\Omega |\nabla v_\epsilon|^{p(x)}\phi dx.$$

Therefore, $\eta \geq |\nabla v|^{p(x)}$ and $\overline{\eta} := \eta - |\nabla v|^{p(x)} - \sum_{j\in J}\eta_j\delta_{x_j}$ is a positive nonatomic measure.

Step 3: (Decomposition of ζ)

There is a subsequence such that $|v_\epsilon|^{q(x)} \overset{*}{\rightharpoonup} \zeta^*$ in $\mathcal{M}(\overline{\Omega})$. By the CCP ([13], Theorem 1.1), we know that

$$\zeta^* = |v|^{q(x)} + \sum_{j\in J}\zeta_j^*\delta_{x_j},$$

with $x_j \in \mathcal{C}$ for all $j \in J$. By Growth condition (3) $\zeta \leq c\zeta^*$ and ζ is absolutely continuous with respect ζ^*. Thus, by Radon-Nikodym theorem there is h in $L^1(\Omega)$ such that

$$\zeta = h + \sum_{j\in J}\zeta_j\delta_{x_j}.$$

Step 4: (Estimation of regular part)

Fix $\delta > 0$, choose j_0 such that $\sum_{j_0<j}\eta_j < \delta$ and take R in a way that $B_R(x_j)$ are disjoints for all $j \leq j_0$. Consider,

$$\psi = \phi_{R_1}^{R2}\prod_{j\leq j_0}(1 - \phi_r^R(. - x_j)),$$

where ϕ_r^R is a cutoff test function having support in $B_R(0)$, as in Proposition 5 and $0 < R_1 < R_2$. Then, support of ψ is in $B_{R_2}(0)\backslash\bigcup_{j\leq j_0}B_R(x_j)$. By Lemma 1

$$\begin{aligned}
\int_{B_{R_1}(0)\setminus\bigcup_{j\le j_0}B_R(x_j)} h\,dx &\le \zeta\left(\overline{B_{R_1}(0)\setminus\bigcup_{j\le j_0}B_R(x_j)}\right)\\
&\le G^*\max\left\{\left(\eta\left(\overline{B_{R_2}(0)\setminus\bigcup_{j\le j_0}B_R(x_j)}\right)+\delta\right)^{\frac{q^+}{p^-}},\right.\\
&\qquad\left.\left(\eta\left(\overline{B_{R_2}(0)\setminus\bigcup_{j\le j_0}B_R(x_j)}\right)+\delta\right)^{\frac{q^-}{p^+}}\right\}\\
&\le G^*\max\left\{\left(\int_\Omega|\nabla v|^{p(x)}dx+\overline{\eta}(\overline{\Omega})+\delta\right)^{\frac{q^+}{p^-}},\right.\\
&\qquad\left.\left(\int_\Omega|\nabla v|^{p(x)}dx+\overline{\eta}(\overline{\Omega})+\delta\right)^{\frac{q^-}{p^+}}\right\}.
\end{aligned}$$

Taking $R\to 0$, $R_1\to\infty$ and $\delta\to 0$, we get our desired generalized Sobolev type inequality for regular part. □

Like in [13,14], generalized CCP can be used to prove the existence of solutions of different kinds of PDEs, but here we focus on the concentration/compactness of the maximizing sequence of generalized Sobolev constant G^* i.e.,

$$\int_\Omega G(v_\epsilon)dx\to G^* \text{ when } \epsilon\to 0.$$

As we know $\zeta(\overline{\Omega})\le G^*$, but when we have equality, then there are two possibilities, either the limit measure is non-atomic, or the sequence concentrates to a single point, see the following result. Our idea is to use a type of convexity argument to prove it.

Theorem 2. *In addition to assumptions of Theorem 1, if $\zeta(\overline{\Omega})=G^*$ then $\eta(\overline{\Omega})=1$ and one of the following statements are true.*

(a) Concentration: for some x_0 in Ω $\eta=\delta_{x_0}$, $\zeta=G^*\delta_{x_0}$ and $v=0$.
(b) Compactness: v is an extremal of G^*, $\eta=|\nabla v|^{p(x)}$, $v_\epsilon\to v$ in $W_0^{1,p(x)}(\Omega)$ and $G(v_\epsilon)\to G(v)$ in $L^1(\Omega)$.

Proof. Let $\eta_0:=\int_\Omega|\nabla v|^{p(x)}dx+\overline{\eta}(\overline{\Omega})$ and $\zeta_0=\int_\Omega h dx$. By Inequality (20)

$$G^*=\zeta(\overline{\Omega})\le G^*\max\left\{\eta(\overline{\Omega})^{\frac{q^+}{p^-}},\eta(\overline{\Omega})^{\frac{q^-}{p^+}}\right\}\le G^*.$$

Thus, $\eta(\overline{\Omega})=\eta_0+\sum_{j\in J}\eta_j=1$, further, if more than one η_j for $j\in J':=J\cup\{0\}$ are less than one, then due to strict convexity of function s^n for $n>1$ and $s\in[0,1]$

$$\begin{aligned}
1=\max\left\{\left(\sum_{j\in J'}\eta_j\right)^{\frac{q^+}{p^-}},\left(\sum_{j\in J'}\eta_j\right)^{\frac{q^-}{p^+}}\right\}&=\frac{\zeta(\overline{\Omega})}{G^*}=\frac{1}{G^*}\sum_{j\in J'}\zeta_j\\
\le\sum_{j\in J'}\max\left\{\eta_j^{\frac{q+}{p^-}},\eta_j^{\frac{q-}{p^+}}\right\}<\sum_{j\in J'}\eta_j=1,&
\end{aligned}$$

therefore, only one of the η_j is equal one and the rest of them are zeros. Statement (a) follows if $\eta_0 = 0$ otherwise, we prove statement (b). Let $\eta_0 \neq 0$ then $\eta_j = \zeta_j = 0$ for all $j \in J$, $\int_\Omega |\nabla v|^{p(x)} dx + \overline{\eta}(\overline{\Omega}) = 1$ and $\int_\Omega h dx = G^*$. There is a subsequence $v_\epsilon \xrightarrow{pointwise} v$ a.e in Ω and $|v|^{q(x)} \overset{*}{\rightharpoonup} \zeta^*$ in $\mathcal{M}(\overline{\Omega})$. As G is upper semicontinuous, we have

$$G(v) = \limsup_{\epsilon \to 0} G(v_\epsilon) \; a.e. \tag{24}$$

Fatou's lemma applied to sequence $c|v_\epsilon|^{q(x)} - G(v_\epsilon)$ gives us

$$\begin{aligned} G^* = \lim_{\epsilon \to 0} \int_\Omega G(v_\epsilon) dx \leq \int_\Omega \limsup_{\epsilon \to 0} G(v_\epsilon) dx \leq \int_\Omega G(v) dx \\ \leq G^* \max\left\{ \|\nabla v\|^{q^+}_{L^{p(x)}(\Omega)}, \|\nabla v\|^{q^-}_{L^{p(x)}(\Omega)} \right\} \leq G^*. \end{aligned}$$

Hence, $\|\nabla v\|_{L^{p(x)}(\Omega)} = 1$, $\overline{\eta} = 0$ and $\int_\Omega G(v) = G^*$. As, we have already discussed $\zeta^* = |v|^{q(x)}$. So,

$$\int_\Omega |\nabla v_\epsilon|^{p(x)} dx \to \int_\Omega |\nabla v|^{p(x)} dx \text{ and } \int_\Omega |v_\epsilon|^{q(x)} dx \to \int_\Omega |v|^{q(x)} dx.$$

Then $v_\epsilon \to v$ strongly in $W_0^{1,p(x)}(\Omega)$ and $L^{q(x)}(\Omega)$ due to the fact that $v_\epsilon \rightharpoonup v$ in $W_0^{1,p(x)}(\Omega)$. In other wards, $|v_\epsilon|^{q(x)} \to |v|$ in $L^1(\Omega)$ and Growth condition (3) implies the sequence $G(v_\epsilon)$ is equi-integrable and hence weakly compact due to the Dunford–Pettis compactness theorem, whereas $G(v_\epsilon) \overset{*}{\rightharpoonup} h$ implies there is a subsequence such that

$$G(v_\epsilon) \rightharpoonup h \text{ weakly in } L^1(\Omega). \tag{25}$$

Upper semicontinuity of G gives us $h \leq G(v)$, but on the other hand $\int_\Omega G(w) dx = G^* = \int_\Omega h dx$ which implies equality $h = G(v)$ a.e. in Ω. Moreover, (24) implies $[G(v_\epsilon) - G(v)]^+ \to 0$ a.e., $G(v_\epsilon) \leq c|v_\epsilon|^{q(x)}$ due to Growth condition (3) and $|v_\epsilon|^{q(x)} \to |v|^{q(x)}$ in $L^1(\Omega)$. So, by Lebesgue dominated convergence theorem

$$\int_\Omega [G(v_\epsilon) - G(v)]^+ dx \to 0.$$

In addition, due to the weak convergence (25)

$$\int_\Omega [G(v_\epsilon) - G(v)]^- dx = \int_\Omega [G(v_\epsilon) - G(v)] dx - \int_\Omega [G(v_\epsilon) - G(v)]^+ dx \to 0.$$

Together

$$\int_\Omega |G(v_\epsilon) - G(v)| dx \to 0,$$

the proof is complete. □

4. Generalized Concentration Compactness Principle for Low Energies

In a model without external energy source, internal energies will run out eventually. We deal with possible limit of low energy extremals of (4) and determine its shape. For v in $W_0^{1,p(x)}(\Omega)$ with $\|\nabla v_\epsilon\|_{L^{p(x)}(\Omega)} \leq \epsilon$, consider $w := v/\epsilon$ then by Growth condition (3) and Sobolev embedding (15)

$$\int_\Omega \frac{G(v)}{\epsilon^{q(x)}} dx \leq G^*_\epsilon \max\{ \|\nabla w\|^{q^+}_{L^{p(x)}(\Omega)}, \|\nabla w\|^{q^-}_{L^{p(x)}(\Omega)} \}. \tag{26}$$

Theorem 3. *Let p and q be log-Hölder continuous exponents with*

$$1 < p^- \leq p^+ < N, \; p \leq q \leq p^* \text{ in } \Omega \text{ and } \mathcal{C} =: \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset.$$

Let $\{v_\epsilon\}$ be a sequence in $W_0^{1,p(x)}(\Omega)$ with $\|\nabla v_\epsilon\|_{L^{p(x)}(\Omega)} \leq \epsilon$. Take $w_\epsilon := v_\epsilon/\epsilon$, If

- *$w_\epsilon \rightharpoonup w$ weakly in $W_0^{1,p(x)}(\Omega)$,*
- *$|\nabla w_\epsilon|^{p(x)} \overset{*}{\rightharpoonup} \eta$ in the sense of measure in $\mathcal{M}(\overline{\Omega})$,*
- *$\epsilon^{-q(x)}G(v_\epsilon) \overset{*}{\rightharpoonup} \zeta$ in the sense of measure in $\mathcal{M}(\overline{\Omega})$.*

Then, for a countable index set J

$$\eta = |\nabla w|^{p(x)} + \overline{\eta} + \sum_{j\in J} \eta_j \delta_{x_j},\ \eta(\overline{\Omega}) \leq 1, \tag{27}$$

$$\zeta = h + \sum_{j\in J} \zeta_j \delta_{x_j},\ \zeta(\overline{\Omega}) \leq G^*, \tag{28}$$

where $\{x_j\}_{j\in J} \subseteq \mathcal{C}$, $\overline{\eta}$ is a positive nonatomic measure in $\mathcal{M}(\overline{\Omega})$ and $h \in L^1(\Omega)$. Moreover, atomic and regular parts satisfy the following generalized Sobolev type inequalities

$$\zeta_j \leq G^* \max\left\{ \eta_j^{\frac{q^+}{p^-}}, \eta_j^{\frac{q^-}{p^+}} \right\}, \tag{29}$$

$$\zeta(\overline{\Omega}) \leq G^* \max\left\{ \eta(\overline{\Omega})^{\frac{q^+}{p^-}}, \eta(\overline{\Omega})^{\frac{q^-}{p^+}} \right\}, \tag{30}$$

$$\int_\Omega h dx \leq G^* \max\left\{ \left(\int_\Omega |\nabla w|^{p(x)} dx + \overline{\eta}(\overline{\Omega}) \right)^{\frac{q^+}{p^-}}, \left(\int_\Omega |\nabla w|^{p(x)} dx + \overline{\eta}(\overline{\Omega}) \right)^{\frac{q^-}{p^+}} \right\}, \tag{31}$$

$$h \leq G_0^+ |w|^{q(x)} \text{ a.e. in } \Omega, \tag{32}$$

$$\int_\Omega h dx \leq G_0^+ S \max\left\{ \left(\int_\Omega |w|^{p(x)} dx \right)^{\frac{q^+}{p^-}}, \left(\int_\Omega |w|^{p(x)} dx \right)^{\frac{q^-}{p^+}} \right\}. \tag{33}$$

In order to prove generalized CCP of low energies, first we prove couple of auxiliary lemmas.

Lemma 2. *Consider $G^*(p(.), q(.), \mathbb{R}^N)$ and $\epsilon < 1$. Then:*

(a) $G^*_\epsilon(p(.), q(.), \Omega) \leq G^*$;
(b) $G^* = \lim_{\epsilon\to 0} G^*_\epsilon(p(.), q(.), \Omega)$.

Proof. Take $v \in W_0^{1,p(x)}(\Omega)$ with $\|v\|_{L^{p(x)}(\Omega)} \leq \epsilon$. Consider $w = v(\epsilon^{q^+/N}x)$, then w is admissible for G^* and we have

$$G^* \geq \int_{\mathbb{R}^N} G(w)dx = \epsilon^{-q^+} \int_\Omega G(v)dx \geq \int_\Omega \frac{G(v)}{\epsilon^{q(x)}} dx,$$

taking supremum for all such v gives us (a) and

$$\limsup_{\epsilon\to 0} G^*_\epsilon(p(.), q(.), \Omega) \leq G^*.$$

On the other hand fix $\delta > 0$, x_0 in Ω. There exists w in $W_0^{1,p(x)}(\Omega)$ such that

$$\int_{\mathbb{R}^N} G(w)dx \geq G^* - \delta.$$

For sufficiently large $r > 0$

$$\int_{B_r(0)} G(w)dx \geq G^* - 2\delta.$$

By Proposition 5 there exist a cutoff test function ϕ_r^R, supported in $B_R(0)$ with

$$\int_{B_R(0)} |\nabla(\phi_r^R w)|^{p(x)}dx \leq 1 + \delta.$$

Define

$$v := (\phi_r^R w)\left(\left(\frac{(1+\delta)^{1/(N-p^+)}}{\epsilon^{q^-/N}}\right)(x - x_0)\right),$$

for sufficiently small ϵ, v is supported in Ω and $\|\nabla v\|_{L^{p(x)}(\Omega)} \leq \epsilon$. Now,

$$\begin{aligned} G_\epsilon^*(p(), q(), \Omega) \geq &\int_\Omega \frac{G(v)}{\epsilon^{q(x)}}dx \geq \frac{1}{\epsilon^{q^-}}\int_\Omega G(v) \\ =&(1+\delta)^{\frac{-N}{N-p^+}} \int_{B_R(0)} G(\phi_r^R w)dx \geq (1+\delta)^{\frac{-N}{N-p^+}}(G^* - 2\delta). \end{aligned}$$

Hence,

$$\liminf_{\epsilon \to 0} G_\epsilon^*(p(), q(), \Omega) \geq G^*.$$

□

Lemma 3. *Let v in $W_0^{1,p(x)}(\Omega)$ with $\|\nabla v\|_{L^{p(x)}(\Omega)} \leq \epsilon$. Take $w = v/\epsilon$, $\delta > 0$ and $r < R$ satisfying $\frac{r}{R} \leq k(\delta)$ as in the Proposition 5. For $x_0 \in \Omega$ and G satisfying the growth condition (3), following inequalities hold*

$$\begin{aligned} \int_{B_r(x_0)} \frac{G(v)}{\epsilon^{q(x)}}dx \leq G_\epsilon^* \max\Bigg\{ &\left(\int_{B_R(x_0)} |\nabla w|^{p(x)}dx + \delta \max\left\{\|\nabla w\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla w\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^+}{p^-}}, \\ &\left(\int_{B_R(x_0)} |\nabla w|^{p(x)}dx + \delta \max\left\{\|\nabla w\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla w\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^-}{p^+}} \Bigg\}, \end{aligned} \tag{34}$$

$$\begin{aligned} \int_{\Omega\setminus B_R(x_0)} \frac{G(v)}{\epsilon^{q(x)}}dx \leq G_\epsilon^* \max\Bigg\{ &\left(\int_{\Omega\setminus B_r(x_0)} |\nabla w|^{p(x)}dx + \delta \max\left\{\|\nabla w\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla w\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^+}{p^-}}, \\ &\left(\int_{\Omega\setminus B_r(x_0)} |\nabla w|^{p(x)}dx + \delta \max\left\{\|\nabla w\|^{p^+}_{L^{p(x)}(\Omega)}, \|\nabla w\|^{p^-}_{L^{p(x)}(\Omega)}\right\}\right)^{\frac{q^-}{p^+}} \Bigg\}. \end{aligned} \tag{35}$$

Proof. The proof is similar as of Lemma 1. □

The generalized CCP for low energies is proved in the same manner of Theorem 1.

Proof of Theorem 3. Steps 1–4 are analogous with the use of Lemmas 2 and 3, as in proof of Theorem 1, we just need to prove pointwise estimate (32) and Inequality (33) for the regular part. Indeed, there exists a subsequence such that

$$|w_\epsilon|^{q(x)} \overset{*}{\rightharpoonup} \zeta^* = |w|^{q(x)} + \sum_{j\in J} \zeta_j^* \delta_{x_j} \text{ or } |w_\epsilon - w|^{q(x)} \overset{*}{\rightharpoonup} \sum_{j\in J} \zeta_j^* \delta_{x_j}.$$

For $A \subseteq \Omega$ and $s > 0$

$$\begin{aligned}\int_A h dx \leq & \zeta(A) \leq \liminf_{\epsilon \to 0} \int_A \frac{G(v_\epsilon)}{\epsilon^{q(x)}} dx \\ \leq & \limsup_{\epsilon \to 0} \int_{A \cap \{|w_\epsilon| < s\}} \frac{G(\epsilon w_\epsilon)}{\epsilon^{q(x)} |w_\epsilon|^{q(x)}} |w_\epsilon|^{q(x)} + c \limsup_{\epsilon \to 0} \int_{A \cap \{|w_\epsilon| \geq s\}} |w_\epsilon|^{q(x)} dx \\ \leq & (G_0^+ + o(1)) \zeta^*(\overline{A}) + c \limsup_{\epsilon \to 0} \left(\int_{A \cap \{|w_\epsilon| \geq s\}} |w|^{q(x)} dx + \int_A |w_\epsilon - w|^{q(x)} dx \right).\end{aligned}$$

However, as we know

$$\lim_{s \to \infty} |\{x \in A : |w_\epsilon| \geq s\}| = 0,$$

it yields that

$$\int_A h dx \leq G_0^+ \zeta^*(\overline{A}) + c \sum_{x_j \in \overline{A}} \zeta_j^* \leq G_0^+ \int_A |w|^{q(x)} dx + (1 + c) \sum_{x_j \in \overline{A}} \zeta_j^*.$$

By Radon–Nikodym theorem, we deduce that $h \leq G_0^+ |w|^{q(x)}$ a.e. in Ω. Lastly, Inequality (33) follows from integration and Sobolev inequality (15). □

Lastly, we know $\zeta(\overline{\Omega}) \leq G^*$, but in case of equality and $G_0^- = G_0^+$, in comparison to Theorem 2 compactness of low energies results into an approximation of Sobolev constant S i.e.,

$$\int_\Omega \left| \frac{v_\epsilon}{\epsilon} \right|^{q(x)} dx \to S \text{ when } \epsilon \to 0.$$

Theorem 4. *In addition to assumptions of Theorem 3, if $\zeta(\overline{\Omega}) = G^*$ then, $\eta(\overline{\Omega}) = 1$ and one of the following statements hold.*

(a) Concentration: for some x_0 in $\overline{\Omega}$ $\eta = \delta_{x_0}$, $\zeta = G^* \delta_{x_0}$ and $w = 0$.
(b) Compactness:

$$\eta = |\nabla w|^{p(x)} + \overline{\eta}, \tag{36}$$

$$w_\epsilon \to w \text{ in } L^{q(x)}(\Omega),\ G^* \leq c \int_\Omega |w|^{q(x)} dx, \tag{37}$$

$$\epsilon^{-q(x)} F(v_\epsilon) \rightharpoonup h \text{ weakly in } L^1(\Omega), G^* = \int_\Omega h dx \leq G_0^+ S. \tag{38}$$

If in addition $G_0^+ = G_0^-$ then $w_\epsilon \to w$ in $W_0^{1,p(x)}(\Omega)$, $G^* = G_0 S$ and w is an extremal for S.

Proof. Let $\eta_0 := \int_\Omega |\nabla v|^{p(x)} dx + \overline{\eta}(\overline{\Omega})$ and $\zeta_0 = \int_\Omega h dx$. By similar arguments in proof of Theorem 2, $\eta(\overline{\Omega}) = 1$ and only one of the η_j in $J' = J \cup \{0\}$ is one, and rest of them are zeros. If $\eta_0 = 0$ then we deduce (a). Let $\eta_0 \neq 0$ then $\eta_j = \zeta_j = 0$ for all j in J. $\int_\Omega |\nabla v|^{p(x)} dx + \overline{\eta}(\overline{\Omega}) = 1$ and $\int_\Omega h dx = G^*$. There is a subsequence $v_\epsilon \xrightarrow{pointwise} v$ a.e in Ω and $|v|^{q(x)} \overset{*}{\rightharpoonup} \zeta^*$ in $\mathcal{M}(\overline{\Omega})$. Then, $\zeta^* = |w|^{q(x)}$ and

$$w_\epsilon \to w \text{ strongly in } L^{q(x)}(\Omega).$$

By Growth condition (3)

$$G^* = \int_\Omega h dx \leq c \int_\Omega |w|^{q(x)} dx \text{ and sequence } \epsilon^{-q(x)} G(v_\epsilon) \text{ is equi-integrable.}$$

Therefore, by Dunford–Pettis compactness theorem for a subsequence

$$\epsilon^{-q(x)} G(v_\epsilon) \rightharpoonup h \text{ weakly in } L^1(\Omega),$$

in comparison with Theorem 2, convergence is not strong, the main reason for which is L^1 norm is not strictly convex.

Let that $G_0^+ = G_0^- = G_0 > 0$, G_0 cannot be zero. By Fatou's lemma, we have $G_0^- S \leq G^*$ and together with Inequality (33) yield

$$G^* = \zeta_0 \leq G^* \max \left\{ \left(\int_\Omega |w|^{p(x)} dx \right)^{\frac{q^+}{p^-}}, \left(\int_\Omega |w|^{p(x)} dx \right)^{\frac{q^-}{p^+}} \right\} \leq G^*.$$

The equality in above implies $\|\nabla w\|_{L^{p(x)}(\Omega)} = 1$, therefore $\eta = |w|^{p(x)}$ and $w_\epsilon \to w$ strongly in $W_0^{1,p(x)}(\Omega)$. Fix $s > 0$

$$\begin{aligned} \int_\Omega \frac{G(v_\epsilon)}{\epsilon^{q(x)}} dx &\leq \int_{\{|w_\epsilon| < s\}} \frac{G(\epsilon w_\epsilon)}{\epsilon^{q(x)}} dx + c \int_{\{|w_\epsilon| \geq s\}} |w_\epsilon|^{q(x)} dx \\ &\leq (G_0^+ + o(1)) \int_\Omega |w_\epsilon|^{q(x)} dx + \int_\Omega g_s(w_\epsilon) dx, \end{aligned}$$

where for v in $W_0^{1,p(x)}(\Omega)$

$$g_s(v) = \begin{cases} |v|^{q(x)}, & \text{if } |v| \geq s; \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, g_s is upper semicontinuous, in a way that if $v_\epsilon \xrightarrow{pointwise} v$ a.e in Ω then

$$g_s(v) = \limsup_{\epsilon \to 0} g_s(v_\epsilon) \text{ a.e. in } \Omega.$$

Applying Fatou's lemma to $|w_\epsilon|^{q(x)} - g_s(w_\epsilon)$ results into

$$\begin{aligned} \limsup_{\epsilon \to 0} \int_\Omega g_s(w_\epsilon) dx &\leq \int_\Omega \limsup_{\epsilon \to 0} g_s(w_\epsilon) dx \\ &\leq \int_\Omega g_s(w) dx = \int_{\{|w| \geq s\}} |w|^{q(x)} dx. \end{aligned}$$

Therefore, taking $s \to \infty$

$$G_0^- S \leq G^* = \lim_{\epsilon \to 0} \int_\Omega \frac{G(v_\epsilon)}{\epsilon^{q(x)}} dx \leq G_0^+ \lim_{\epsilon \to 0} \int_\Omega |w_\epsilon|^{q(x)} dx \leq G_0^+ S.$$

Hence,

$$\lim_{\epsilon \to 0} \int_\Omega |w_\epsilon|^{q(x)} dx = S.$$

□

5. Conclusions and Future Work

The main work of this paper is to study a class of elliptic equations with general growth at infinity for variable exponent Lebesgue spaces. The main results nicely determine the limit measures. Therefore, with the proposed work, we can study several models in variable exponent settings, for many fields like plasma physics, fluid mechanics and control systems. As future lines of research, one can also explored concentration compactness principles for fractional Sobolev spaces and fractional PDEs, we refer [21–26] for basic theory. One can also study the convergence of low energy extremals with variational methods.

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