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Linear Convergence of Split Equality Common Null Point Problem with Application to Optimization Problem

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Abstract: The purpose of this paper is to propose an iterative algorithm for solving the split equality common null point problem (SECNP), which is to find an element of the set of common zero points for a finite family of maximal monotone operators in Hilbert spaces. We introduce the concept of bounded linear regularity for the SECNP and construct several sufficient conditions to ensure the linear convergence of the algorithm. Moreover, some numerical experiments are given to test the validity of our results.

Keywords: split equality problem; split equality common null point problem; bounded linear regularity; linear convergence; split equality optimization problem

1. Introduction

Let H_1, H_2, H_3 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Moudafi [1] introduced the following split equality problem (SEP), which is formulated as finding

$$x \in C \text{ and } y \in Q \text{ such that } Ax = By, \tag{1}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. When $B = I$, the SEP reduces to the split feasibility problem (SFP) which was introduced by Censor and Elfving [2]. The SEP allows asymmetric and partial relations between the variables x and y. It has also received much attention due to the application in many disciplines such as medical image reconstruction, game theory, decomposition methods for PDEs and radiation therapy treatment planning; see [3–6].

In [7], Moudafi introduced and studied the following split equality null point problem (SENP): given two set-valued maximal monotone operators $F : H_1 \to 2^{H_1}$ and $K : H_2 \to 2^{H_2}$, the SENP is formulated as finding

$$x^* \in F^{-1}(0), y^* \in K^{-1}(0) \text{ such that } Ax^* = By^*, \tag{2}$$

where $F^{-1}(0) = \{x \in H_1 : 0 \in Fx\} = Fix(J_r^F)$ is closed and convex, F is set-valued maximal monotone operators [8]. We note that if $B = I$, this problem reduces to the well-known split common null point problem which was originally introduced by Byrne et al. [9]. For $i = 1, 2, \cdots, m$, let $\{F_i\}_{i=1}^m : H_1 \to 2^{H_1}$ and $\{K_i\}_{i=1}^m : H_2 \to 2^{H_2}$ be two families of set-valued maximal monotone operators. The SECNP is formulated as finding

$$x^* \in F^{-1}(0) = \bigcap_{i=1}^m F_i^{-1}(0), y^* \in K^{-1}(0) = \bigcap_{i=1}^m K_i^{-1}(0) \text{ such that } Ax^* = By^*. \tag{3}$$

In [7], Moudafi proposed the following algorithm for solving SENP and obtained a weak convergence theorem:

$$\begin{cases} x_{n+1} = J_r^F\Big(x_n - \gamma_n A^*(Ax_n - By_n)\Big), \\ y_{n+1} = J_r^K\Big(y_n + \gamma_n B^*(Ax_n - By_n)\Big), \quad \forall n \geq 0, \end{cases} \tag{4}$$

We note that in the above algorithm, the step-size γ_n depends on the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B, where A^* and B^* are the adjoint operators of A and B, respectively). To implement the alternating algorithm (4) for solving SENP (2), we need to compute $\|A\|$ and $\|B\|$, which is generally not an easy task in practice.

To overcome this difficulty, Eslamian [10] considered an algorithm for solving SECNP for a finite family of maximal monotone operators which does not require any knowledge of the operator norms. In addition, they presented a strong convergence theorem which is more desirable than weak convergence. The algorithm is as follows:

$$\begin{cases} x_{n+1} = \beta_{n,0}\vartheta + \sum_{i=1}^m \beta_{n,i} J_{r_{n,i}}^{F_i}\Big(x_n - \gamma_n A^*(Ax_n - By_n)\Big), \quad \forall x_0, \vartheta \in H_1, \\ y_{n+1} = \beta_{n,0}v + \sum_{i=1}^m \beta_{n,i} J_{s_{n,i}}^{K_i}\Big(y_n + \gamma_n B^*(Ax_n - By_n)\Big), \quad \forall y_0, v \in H_2, n \geq 0, \end{cases} \tag{5}$$

where $\sum_{i=0}^m \beta_{n,i} = 1$ and $\gamma_n = \Big(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2} - \epsilon\Big), n \in \Pi$, the index set $\Pi = \{n : Ax_n - By_n \neq 0\}$. In addition, the sequences $\{\beta_{n,i}\}$, $\{r_{n,i}\}$ and $\{s_{n,i}\}$ satisfy the following conditions: (i) $\liminf_n r_{n,i} > 0, \liminf_n s_{n,i} > 0$ and $\liminf_n \beta_{n,i} > 0$, $\forall i \in \{1, 2 \cdots, m\}$, (ii) $\lim_{n\to\infty} \beta_{n,0} = 0$ and $\sum_{n=0}^\infty \beta_{n,0} = \infty$. It is proved that the sequence $\{(x_n, y_n)\}$ generated by algorithm (5) converges strongly to a solution (x^*, y^*) of SECNP (3).

Without loss of generality, let $H_1 \times H_2 =: H$, $U = F \times K : H \to 2^H$, $U_i = F_i \times K_i : H \to 2^H$ is a family of set-valued maximal monotone operators. Define an operator $G : H \to H_3$ by $G(x, y) = Ax - By$, $\forall (x, y) \in H$. Let G^* denote the adjoint operator of G, then G and G^*G have the following matrix form

$$G = \begin{bmatrix} A & -B \end{bmatrix} \text{ and } G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}.$$

Then the SENP (2) and SECNP (3) can be reformulated as

$$\text{Finding } w^* = (x^*, y^*) \in U^{-1}(0) \text{ such that } Gw^* = 0, \tag{6}$$

and

$$\text{Finding } w^* = (x^*, y^*) \in U^{-1}(0) = \bigcap_{i=1}^m U_i^{-1}(0) \text{ such that } Gw^* = 0, \tag{7}$$

respectively. In addition, the algorithm (5) can be expressed as:

$$w_{n+1} = \beta_{n,0}\alpha + \sum_{i=1}^m \beta_{n,i} J_{\mu_{n,i}}^{U_i}(I - \gamma_n G^*G)w_n, \forall n \geq 0, w_0 \in H, \tag{8}$$

where $\alpha = (\vartheta, v) \in H$, $\sum_{i=0}^m \beta_{n,i} = 1$, $\liminf_n \mu_{n,i} > 0$ and $\liminf_n \beta_{n,i} > 0$, $\forall i \in \{1, 2 \cdots, m\}$. In [10], it has been proved that the sequence $\{w_n\}$ generated by algorithm (8) converges strongly to a solution w^* of the SECNP (3), and $w^* = P_\Gamma(\alpha)\Big(\Gamma$ is the solution set of SECNP (7)$\Big)$. However, as with most algorithms, the convergence rate of the iterative sequence (8) is not taken into account.

Recently, the notion of bounded linear regularity has been used to explore the linear convergence of the split equality problems in [11]. In the present paper, we introduce the bounded linear regularity property of SECNP to consider the linear convergence of the algorithm (8).

The structure of this paper is as follows. In Section 2, we mainly propose the definition of bounded linear regularity and introduce some lemmas which are very useful in the proof of the main result. In Section 3, we propose an iterative algorithm and prove its linear convergence in detail, we also use our result to research the split equality optimization problem. In Section 4, some numerical experiments are given to test the validity of our results.

2. Preliminaries

Throughout this paper, we will denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the unit open ball and unit closed ball with center at origin by $\mathbb{B}$ and $\overline{\mathbb{B}}$, respectively. Let S be a subset of H, we denote the interior and relative interior of S by $intS$, and riS, respectively. For $w \in H$, the classical metric projection of w onto S and the distance of w from S, denoted by $P_s(w)$ and $d_s(w)$, respectively, and defined by

$$P_S(w) := \arg\min\{\|w - v\| : v \in S\} \text{ and } d_S(w) := \inf\{\|w - v\| : v \in S\}.$$

Let U be a mapping of H into 2^H, the effective domain of U is denoted by $dom(U)$, i.e., $dom(U) = \{x \in H : Ux \neq \emptyset\}$. The single-valued operator $J_\mu^U = (I + \mu U)^{-1} : H \to dom(U)$, which is called the resolvent of U for $\mu(\mu > 0)$ and the resolvent J_μ^U is firmly nonexpansive [12]. It is known that $U^{-1}(0) = Fix(J_\mu^U)$, for all $\mu > 0$, and if $U^{-1}(0) \neq \emptyset$, then

$$\langle x - J_\mu^U x, J_\mu^U x - w \rangle \geq 0, \ \forall x \in H, w \in U^{-1}(0). \tag{9}$$

Let $G : H \to H_3$ be a bounded linear operator. The kernel of G is denoted by $kerG = \{x \in H : Gx = 0\}$ and the orthogonal complement of $kerG$ is denoted by $(kerG)^\perp = \{y \in H : \langle x, y \rangle = 0, \ \forall x \in kerG\}$. Both $kerG$ and $(kerG)^\perp$ are closed subspaces of H.

Recall that a sequence $\{w_n\}$ in H is said to converge linearly to its limit w^*(with rate $\sigma \in [0, 1)$) if there exist $\lambda > 0$ and a positive integer N such that

$$\|w_n - w^*\| \leq \lambda \sigma^n, \ \forall n \geq N.$$

Definition 1 ([13])**.** *Let $\{E_i\}_{i \in I}$ be a family of closed convex subsets of a real Hilbert space H, where I is an arbitrary set and $E = \bigcap_{i \in I} E_i \neq \emptyset$. The family $\{E_i\}_{i \in I}$ is said to be bounded linearly regular if $\forall r > 0$, there exists a constant $\gamma_r > 0$ such that*

$$d_E(w) \leq \gamma_r \sup\{d_{E_i}(w) : i \in I\}, \ \forall w \in r\mathbb{B}.$$

Lemma 1 ([14])**.** *Let $\{E_i\}_{i \in I}$ be a family of closed convex subsets of a real Hilbert space H, where I is an arbitrary set. If $E_i \bigcap int(\bigcap_{j \in I \setminus \{i\}} E_j) \neq \emptyset$, the family $\{E_i\}_{i \in I}$ is boundedly linearly regular.*

As we know, $U^{-1}(0)$ is closed and convex. Throughout this paper, we use Γ to denote the solution set of SECNP (7), i.e.,

$$\Gamma := \{w^* \in U^{-1}(0) : Gw^* = 0\}.$$

And assume that the SECNP is consistent, thus, Γ is also a closed, convex and nonempty set.

Definition 2. *The SECNP is said to satisfy the bounded linear regularity property if $\forall r > 0$, there exists $\gamma_r > 0$ such that*

$$\gamma_r d_\Gamma(w) \leq \|Gw\|, \ \forall w \in r\mathbb{B} \cap U^{-1}(0).$$

Lemma 2 ([15])**.** *Let $G : H \to H_3$ be a bounded linear operator on H. Then G is injective and has closed range if and only if G is bounded below(i.e., there exists a constant $\gamma > 0$ such that $\|Gw\| \geq \gamma \|w\|, \forall w \in H$).*

Lemma 3. *Let $\{U^{-1}(0),\ kerG\}$ be bounded linearly regular and G has closed range. Then the SECNP (7) satisfies the bounded linear regularity property.*

Proof. Since $\{U^{-1}(0), kerG\}$ is bounded linearly regular, $\forall r > 0$, there exists $\gamma_r > 0$ such that

$$d_\Gamma(w) = d_{U^{-1}(0)\cap kerG}(w) \leq \gamma_r \max\{d_{U^{-1}(0)}(w), d_{kerG}(w)\},\ \forall w \in r\mathbb{B}.$$

Hence,

$$d_\Gamma(w) \leq \gamma_r d_{kerG}(w),\ \forall w \in r\mathbb{B} \cap U^{-1}(0).$$

Since G restricted to $(kerG)^\perp$ is injective and has closed range, it follows from Lemma 2 that there exists $v > 0$,

$$\|G(\widetilde{w})\| \geq v\|\widetilde{w}\|,\ \forall \widetilde{w} \in (kerG)^\perp.$$

It follows that

$$d_{kerG}(w) \leq \frac{1}{v}\|Gw\|,\ \forall w \in H.$$

Therefore,

$$d_\Gamma(w) \leq \frac{\gamma_r}{v}\|Gw\|,\ \forall w \in U^{-1}(0) \cap r\mathbb{B}.$$

This completes the proof. □

Lemma 4 ([16])**.** *$\forall x_1, \cdots, x_m \in H$ and $\alpha_1, \cdots, \alpha_m \in [0,1]$ with $\Sigma_{i=1}^m \alpha_i = 1$ the equality*

$$\|\sum_{i=1}^m \alpha_i x_i\|^2 = \sum_{i=1}^m \alpha_i\|x_i\|^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j\|x_i - x_j\|^2,$$

holds.

Lemma 5 ([13])**.** *Let E and F be closed convex subsets of H. Then $\{E, F\}$ is bounded linearly regular provided that at least one of the following conditions holds:*

(a) *$riE \cap F\cap \neq \emptyset$ and F is a polyhedron;*
(b) *$riE \cap riF\cap \neq \emptyset$ and E is finite dimensional;*
(c) *$riE \cap riF\cap \neq \emptyset$ and E is finite codimensional.*

3. Main Results

Throughout this section we assume that: (1) H_1, H_2, H_3 are real Hilbert spaces, $H := H_1 \times H_2$; (2) for $i = 1, 2\cdots, m$, $\{F_i\}_{i=1}^m : H_1 \to 2^{H_1}$, $\{K_i\}_{i=1}^m : H_2 \to 2^{H_2}$ and $\{U_i\}_{i=1}^m : H \to 2^H$ are three families of set-valued maximal monotone operators, where $U_i = F_i \times K_i$, $F^{-1}(0) = \bigcap_{i=1}^m F_i^{-1}(0)$, $K^{-1}(0) = \bigcap_{i=1}^m K_i^{-1}(0)$ and $U^{-1}(0) = \bigcap_{i=1}^m U_i^{-1}(0)$; (3) $J_\mu^{U_i} = (J_r^{F_i}, J_s^{K_i})$, where μ, r, s are any positive real numbers.

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Lemma 6. *For $\gamma > 0$ and $\mu > 0$, $w^* := (x^*, y^*) \in H_1 \times H_2$ is a solution of SECNP (7) if and only if $\forall i \geq 1$,*

$$w^* = J_\mu^{U_i}(I - \gamma G^*G)w^*. \tag{10}$$

Proof. As we know, $J_\mu^{U_i} = (J_r^{F_i}, J_s^{K_i})$, and μ, r, s are positive real numbers. If $w^* := (x^*, y^*) \in H_1 \times H_2$ is a solution of SECNP (7), then $\forall i \geq 1$, any $\gamma > 0$ we have

$$x^* \in F_i^{-1}(0) = Fix(J_r^{F_i}),\ y^* \in K_i^{-1}(0) = Fix(J_s^{K_i}) \text{ and } Ax^* = By^*$$

$$\Leftrightarrow x^* = J_r^{F_i} x^*,\ y^* = J_s^{K_i} y^* \text{ and } Ax^* = By^*.$$

Hence we have $G(w^*) = Ax^* - By^* = 0$, and so

$$J_\mu^{U_i}(I - \gamma G^* G)(w^*) = J_\mu^{U_i}(w^*) = (J_r^{F_i} x^*, J_s^{K_i} y^*) = (x^*, y^*) = w^*.$$

This implies that (10) is true.

Conversely, if $w^* := (x^*, y^*) \in H_1 \times H_2$ satisfies (10), then we have

$$\begin{cases} x^* = J_r^{F_i}\Big(x^* - \gamma A^*(Ax^* - By^*)\Big), \\ y^* = J_s^{K_i}\Big(y^* + \gamma B^*(Ax^* - By^*)\Big). \end{cases} \tag{11}$$

By (9) and (11), we have

$$\Big\langle J_r^{F_i} x^* - J_r^{F_i}\Big(x^* - \gamma A^*(Ax^* - By^*)\Big),\ J_r^{F_i} x - J_r^{F_i} x^* \Big\rangle \geq 0,\ \forall x \in F_i^{-1}(0).$$

That is

$$\Big\langle x^* - \Big(x^* - \gamma A^*(Ax^* - By^*)\Big),\ x - x^* \Big\rangle \geq 0,\ \forall x \in F_i^{-1}(0).$$

And we can get

$$\langle Ax^* - By^*,\ Ax - Ax^* \rangle \geq 0,\ \forall x \in F_i^{-1}(0). \tag{12}$$

Similarly, we have

$$\langle Ax^* - By^*,\ By^* - By \rangle \geq 0,\ \forall y \in K_i^{-1}(0). \tag{13}$$

Adding up (12) and (13), one gets

$$\langle Ax^* - By^*,\ Ax - Ax^* + By^* - By \rangle \geq 0,\ \forall x \in F_i^{-1}(0),\ y \in K_i^{-1}(0).$$

Simplifying it, we have

$$\|Ax^* - By^*\|^2 \leq \langle Ax^* - By^*,\ Ax - By \rangle,\ \forall x \in F_i^{-1}(0),\ y \in K_i^{-1}(0). \tag{14}$$

Since $\Gamma \neq \emptyset$, taking $\widetilde{w} = (\widetilde{x}, \widetilde{y}) \in \Gamma$, we have $\widetilde{x} \in F_i^{-1}(0), \widetilde{y} \in K_i^{-1}(0)$ and $A\widetilde{x} = B\widetilde{y}, \forall i \geq 1$. Let $x = \widetilde{x},\ y = \widetilde{y}$, according to (14) we have

$$\|Ax^* - By^*\| = 0, \text{ that is, } Ax^* = By^*. \tag{15}$$

From (11) and (15)

$$\begin{cases} x^* = J_r^{F_i}(x^*), \\ y^* = J_s^{K_i}(y^*). \end{cases} \Leftrightarrow x^* \in F_i^{-1}(0),\ y^* \in K_i^{-1}(0),\ \forall i \geq 1.$$

So we get

$$x^* \in F_i^{-1}(0),\ y^* \in K_i^{-1}(0) \text{ such that } Ax^* = By^*.$$

That is $w^* := (x^*,\ y^*)$ is a solution of SECNP (7). This completes the proof. □

Lemma 7. *If $\gamma \in (0, \frac{2}{L})$, where $L = \|G\|^2$, then $J_\mu^{U_i}(I - \gamma G^* G) : H \to H$ is a nonexpansive mapping.*

Proof. Since $J_{\mu}^{U_i}$ is firmly nonexpansive, $\forall a, b \in H$, we have

$$\begin{aligned}\|J_{\mu}^{U_i}(I-\gamma G^*G)a - J_{\mu}^{U_i}(I-\gamma G^*G)b\|^2 &\le \|(I-\gamma G^*G)a-(I-\gamma G^*G)b\|^2\\ &= \|(a-b)-\gamma G^*G(a-b)\|^2\\ &= \|a-b\|^2+\gamma^2\|G^*G(a-b)\|^2-2\gamma\langle a-b, G^*G(a-b)\rangle\\ &\le \|a-b\|^2+\gamma^2 L\|G(a-b)\|^2-2\gamma\langle G(a-b), G(a-b)\rangle\\ &= \|a-b\|^2+\gamma^2 L\|G(a-b)\|^2-2\gamma\|G(a-b)\|^2\\ &= \|a-b\|^2-\gamma(2-\gamma L)\|G(a-b)\|^2\\ &\le \|a-b\|^2\end{aligned}$$

This completes the proof. □

Corollary 1. *The SECNP (7) satisfies the bounded linear regularity property if one of the following conditions holds:*

(a) $F^{-1}(0)$ *and* $K^{-1}(0)$ *are polyhedrons, and* G *has closed range;*
(b) $riU^{-1}(0)\cap kerG \neq \emptyset$, $kerU^{-1}(0)$ *is finite dimensional;*
(c) $riU^{-1}(0)\cap kerG \neq \emptyset$, $kerU^{-1}(0)$ *is finite codimensional;*
(d) $riU^{-1}(0)\cap kerG \neq \emptyset$, G *has closed range and* $U^{-1}(0)$ *is finite dimensional;*
(e) $riU^{-1}(0)\cap kerG \neq \emptyset$, G *has closed range and* $U^{-1}(0)$ *is finite codimensional.*

Next, we establish the linear convergence property for the iterative algorithm under the assumption of bounded linear regularity property for SECNP.

Theorem 1. *Assume that the SECNP (7) satisfies the bounded linear regularity property, let* $\{w_n\}$ *be a sequence generated by*

$$w_{n+1} = \beta_{n,0}\alpha + \sum_{i=1}^{m}\beta_{n,i}J_{\mu_{n,i}}^{U_i}(I-\gamma_n G^*G)w_n,\ \forall n\ge 0,\ w_0\in U^{-1}(0), \tag{16}$$

with $\gamma_n \in (0,\infty)$, *where* $\alpha \in U^{-1}(0)$, $\sum_{i=0}^{m}\beta_{n,i} = 1$, $\liminf_n \beta_{n,i} > 0$ *and* $\liminf_n \mu_{n,i} > 0$ *for* $i = 1,\cdots,m$, *then* $\{w_n\}$ *converges to a solution* w^* *of SECNP (7) such that*

$$\|w_n - w^*\| \le \lambda\sigma^n \text{ and } w^* = P_\Gamma(\alpha) \tag{17}$$

for $\lambda \ge 1$ *and* $0<\sigma<1$, *under one of the following conditions:*

(a) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < \frac{2}{\|G\|^2}$;
(b) $\gamma_n = \begin{cases} 0, & w_n\in\Gamma\\ \frac{\rho_n\|Gw_n\|^2}{\|G^*Gw_n\|^2}, & \text{otherwise}\end{cases}$ *and* $0<\liminf_{n\to\infty}\rho_n \le \limsup_{n\to\infty}\rho_n < 2$.

Proof. Without loss of generality, we assume that w_n is not in Γ, $\forall n\ge 1$. We now show that $\{w_n\}$ converges to a solution w^* of SECNP (7) and (17) holds. From $\sum_{i=0}^{m}\beta_{n,i} = 1$ and Lemma 4, we get

$$\begin{aligned}\|w_{n+1}-w^*\|^2 &= \|\beta_{n,0}\alpha + \sum_{i=1}^{m} \beta_{n,i} J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^*\|^2 \\ &= \|\beta_{n,0}(\alpha - w^*) + \sum_{i=1}^{m} \beta_{n,i}\Big(J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^*\Big)\|^2 \\ &\le \beta_{n,0}\|\alpha - w^*\|^2 + \sum_{i=1}^{m} \beta_{n,i}\|J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^*\|^2 \\ &\quad - \beta_{n,0}\sum_{i=1}^{m} \beta_{n,i}\|J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^* - (\alpha - w^*)\|^2 \\ &= (\beta_{n,0} - \beta_{n,0}\sum_{i=1}^{m} \beta_{n,i})\|\alpha - w^*\|^2 \\ &\quad + \sum_{i=1}^{m} \beta_{n,i}(1-\beta_{n,0})\|J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^*\|^2 \\ &\quad + 2\beta_{n,0}\sum_{i=1}^{m} \beta_{n,i}\|J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^*\| \cdot \|\alpha - w^*\|\end{aligned}$$

As we know, $w^* = P_\Gamma(\alpha)$, then $\|\alpha - w^*\| \le \|w_n - w^*\|, \forall n > 0$. According to condition (a) and Lemma 7, $J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)$ is nonexpansive. In addition, as $w^* \in \Gamma$, by Lemma 6, we have $w^* = J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w^*$. So we can get

$$\begin{aligned}\|w_{n+1}-w^*\|^2 &\le (\beta_{n,0} - \beta_{n,0}\sum_{i=1}^{m} \beta_{n,i})\|w_n - w^*\|^2 \\ &\quad + \sum_{i=1}^{m} \beta_{n,i}(1-\beta_{n,0})\|J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^*\|^2 + 2\beta_{n,0}\sum_{i=1}^{m} \beta_{n,i}\|(w_n - w^*)\|^2 \\ &= (\beta_{n,0} + \beta_{n,0}\sum_{i=1}^{m} \beta_{n,i})\|w_n - w^*\|^2 \\ &\quad + \sum_{i=1}^{m} \beta_{n,i}(1-\beta_{n,0})\|J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^*\|^2 \end{aligned} \tag{18}$$

For $w^* \in \Gamma$, since $J^{U_i}_{\mu_{n,i}}$ is firmly nonexpansive and $Gw^* = 0$, we get

$$\begin{aligned}\|J^{U_i}_{\mu_{n,i}}(I-\gamma_n G^*G)w_n - w^*\|^2 &= \|J^{U_i}_{\mu_{n,i}}(w_n - \gamma_n G^*Gw_n) - J^{U_i}_{\mu_{n,i}}w^*\|^2 \\ &\le \|w_n - w^* - \gamma_n G^*Gw_n\|^2 \\ &= \|w_n - w^*\|^2 - 2\gamma_n\langle w_n - w^*, G^*Gw_n\rangle + \gamma_n^2\|G^*Gw_n\|^2 \\ &= \|w_n - w^*\|^2 - 2\gamma_n\|Gw_n\|^2 + \gamma_n^2\|G^*Gw_n\|^2 \\ &= \|w_n - w^*\|^2 - \gamma_n(2 - \gamma_n\tfrac{\|G^*Gw_n\|^2}{\|Gw_n\|^2})\|Gw_n\|^2 \end{aligned} \tag{19}$$

Now, we substitute (19) in (18) so we have

$$\begin{aligned}\|w_{n+1}-w^*\|^2 &\le (\beta_{n,0} + \beta_{n,0}\sum_{i=1}^{m} \beta_{n,i})\|w_n - w^*\|^2 \\ &\quad + \sum_{i=1}^{m} \beta_{n,i}(1-\beta_{n,0})\Big(\|w_n - w^*\|^2 - \gamma_n(2 - \gamma_n\tfrac{\|G^*Gw_n\|^2}{\|Gw_n\|^2})\|Gw_n\|^2\Big) \\ &= (\beta_{n,0} + \beta_{n,0}\sum_{i=1}^{m} \beta_{n,i})\|w_n - w^*\|^2 + (\sum_{i=1}^{m} \beta_{n,i} - \beta_{n,0}\sum_{i=1}^{m} \beta_{n,i})\|w_n - w^*\|^2 \\ &\quad - \sum_{i=1}^{m} \beta_{n,i}(1-\beta_{n,0})\gamma_n(2 - \gamma_n\tfrac{\|G^*Gw_n\|^2}{\|Gw_n\|^2})\|Gw_n\|^2 \\ &= \|w_n - w^*\|^2 - \sum_{i=1}^{m} \beta_{n,i}(1-\beta_{n,0})\gamma_n(2 - \gamma_n\tfrac{\|G^*Gw_n\|^2}{\|Gw_n\|^2})\|Gw_n\|^2\end{aligned}$$

Since SECNP (7) satisfies the bounded linear regularity property and $w_n \in U^{-1}(0), \forall n \ge 1$, so there exists $\delta > 0$ such that $\delta d_\Gamma(w_n) \le \|Gw_n\|$, $\forall n \ge 1$. It follows that

$$\|w_{n+1}-w^*\|^2 \le \|w_n - w^*\|^2 - \delta^2\sum_{i=1}^{m} \beta_{n,i}(1-\beta_{n,0})\gamma_n(2 - \gamma_n\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2})d_\Gamma(w_n)^2,\ \forall w^* \in \Gamma.$$

Hence,

$$d_\Gamma(w_{n+1})^2 \leq \Big[1-\delta^2\sum_{i=1}^{m}\beta_{n,i}(1-\beta_{n,0})\gamma_n(2-\gamma_n\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2})\Big]d_\Gamma(w_n)^2.$$

Please note that if (a) or (b) holds, then

$$\liminf_{n\to\infty}\Big(2-\gamma_n\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2}\Big)>0.$$

Since $\liminf_n \beta_{n,i}>0$, $\forall i=1,\cdots,m$, so there exists N such that $\sum_{i=1}^{m}\beta_{n,i}(1-\beta_{n,0})>0$ for $n\geq N$. And,

$$\phi=\inf_{n\geq N}\delta^2\sum_{i=1}^{m}\beta_{n,i}(1-\beta_{n,0})(2-\gamma_n\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2})>0.$$

Therefore,

$$d_\Gamma(w_{n+1})^2\leq(1-\phi\gamma_n)d_\Gamma(w_n)^2\leq d_\Gamma(w_N)^2\prod_{k=N+1}^{n}(1-\phi\gamma_k),\ \forall n\geq K.$$

Observe that $\forall w^*\in\Gamma$, $\|w_{n+1}-w^*\|$ is monotone decreasing for n, hence

$$\begin{aligned}\|w_m-w_n\| &\leq \|w_m-P_\Gamma(w_n)\|+\|w_n-P_\Gamma(w_n)\|\\ &\leq 2\|w_n-P_\Gamma(w_n)\|\\ &=2d_\Gamma(w_n), \qquad m\geq n\end{aligned}$$

It follows that

$$\|w_m-w_{n+1}\|\leq 2d_\Gamma(w_N)\prod_{k=N+1}^{n}\sqrt{1-\phi\gamma_k},\ \forall m\geq N+1.$$

Let $p:=e^{-\frac{\phi}{2}}\in(0,1)$, then

$$\prod_{k=N+1}^{n}\sqrt{1-\phi\gamma_k}=exp\Big\{\frac{1}{2}\sum_{k=N+1}^{n}\ln(1-\phi\gamma_k)\Big\}\leq p^{\sum_{k=N+1}^{n}\gamma_k}.$$

Therefore,

$$\|w_m-w_{n+1}\|\leq 2d_\Gamma(w_N)p^{\sum_{k=N+1}^{n}\gamma_k},\ \forall m\geq n+1.$$

As one of (a) and (b) holds, it follows that $\{w_n\}$ is a Cauchy sequence and converges to a solution w^* of SECNP (7) satisfying

$$\|w_{n+1}-w^*\|\leq 2d_\Gamma(w_N)p^{\sum_{k=N+1}^{n}\gamma_k},\ \forall n\geq N.$$

Let

$$m=\max\Big\{2d_\Gamma(w_N)p^{-\sum_{k=1}^{N}\gamma_k},\max\{\|w_k-w^*\|p^{-\sum_{j=1}^{k}\gamma_j}:k=1,2,\cdots,N\}\Big\}.$$

Then

$$\|w_n-w^*\|\leq mp^{\sum_{k=1}^{n}\gamma_k}.$$

Moreover, if (a) or (b) is assumed, then $\liminf_{n\to\infty}\gamma_n>0$. Let $\liminf_{n\to\infty}\gamma_n=\theta>0$, then $\exists N_1>0$, such that $\theta_n>\theta$ for $n\geq N_1$. It follows that

$$\|\ w_n-w^*\ \|\leq mp^{\sum_{i=1}^{N_1}\theta_i}p^{(n-N_1)\theta}=\lambda\sigma^n,\ \forall n\geq max\{N_1,N\},$$

where $\lambda = mp^{\sum_{i=1}^{N_1}(\theta_i-\theta)} > 0, \sigma = p^{\theta} \in (0,1)$. Hence, $\{w_n\}$ converges to w^* linearly.

This completes the proof. $\square$

3.2. The Application of Split Equality Optimization Problem

The so-called split equality optimization problem (SEOP) is formulated as finding $(x^*, y^*) \in (H_1, H_2)$ such that

$$f(x^*) = \min_{x\in H_1} f(x), \;\; k(y^*) = \min_{y\in H_2} k(y) \text{ and } Ax^* = By^*, \tag{20}$$

where $f : H_1 \to \mathbb{R}$ and $k : H_2 \to \mathbb{R}$ are two proper, lower semicontinuous, and convex functionals. Let $u = (f, k) : H \to \mathbb{R}$ be a proper, lower semicontinuous, and convex functional. Then SEOP (20) can be reformulated as finding $w^* \in H$ such that

$$u(w^*) = \min_{w\in H} u(w) \text{ and } Gw^* = 0. \tag{21}$$

The subdifferential of u at w is the set

$$\partial u(w) := \{a \in H : u(v) \geq u(w) + \langle a, v - w\rangle, \; \forall v \in H\}.$$

Denote by $\partial u = U$. It is know that $U : H \to 2^H$ is maximal monotone operator, so we can define the resolvent J_{μ}^{U} where $\mu > 0$, and

$$u(w^*) = \min_{w\in H} u(w) \quad \Leftrightarrow \quad 0 \in \partial u(w^*) = U(w^*) \quad \Leftrightarrow \quad w^* \in U^{-1}(0).$$

Therefore SEOP (21) is equivalent to the SENP (6), then the following corollary can be obtained from Theorem 1 immediately.

Corollary 2. *Assume that the solution of SEOP (21) $\Gamma_1 = \{w^* \in H, \; s.t. \; u(w^*) = \min_{w\in H} u(w)$ and $Gw^* = 0\}$ is nonempty where $\partial u = U$. In addition, the statements (a) and (b) are consist with Theorem 1. Let the SEOP (21) satisfies the bounded linear regularity property and $\{w_n\}$ be a sequence generated by*

$$w_{n+1} = \beta_n \alpha + (1 - \beta_n) J_{\mu_n}^{U} (I - \gamma_n G^* G) w_n, \; \forall n \geq 0, \; w_0 \in U^{-1}(0),$$

with $\gamma_n \in (0, \infty)$, where $\alpha \in U^{-1}(0)$, $\liminf_n \beta_n > 0$ and $\liminf_n \mu_n > 0$, then $\{w_n\}$ converges linearly to a solution w^* of SEOP (21).

4. Numerical Experiments

Let $H_1 = \mathbb{R}, H_2 = \mathbb{R}^2, H_3 = \mathbb{R}^3$. Let

$$f(x) = \begin{cases} x + 1, & x \geq 0, \\ x - 1, & x \leq 0. \end{cases}$$

and

$$F_i(x) = [f(x - 0), f(x + 0)], \; \forall i = 1, 2, \cdots, m, \; x \in \mathbb{R}.$$

Then $F_i(x)$, $i = 1, 2, \cdots, m$ are set-valued maximal monotone operators and $F_i^{-1}(0) = \{x \in \mathbb{R}, |x| \leq 1\}$.

Let $\forall i = 1, 2, \cdots, m$

$$K_i(x) = \begin{cases} \begin{pmatrix} -1 \\ 0 \end{pmatrix}, & x < -1, y \in \mathbb{R} \\ \begin{pmatrix} [-1,0] \\ 0 \end{pmatrix}, & x = -1, y \in \mathbb{R} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & -1 < x < 1, y \in \mathbb{R} \\ \begin{pmatrix} [0,1] \\ 0 \end{pmatrix}, & x = 1, y \in \mathbb{R} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & x > 1, y \in \mathbb{R} \end{cases}$$

Then $K_i(x)(i = 1, 2, \cdots, m)$ are set-valued maximal monotone operators and $K_i^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : |x| \le 1, y \in \mathbb{R}\}$.

Let $A : H_2 \to H_3$, $B : H_1 \to H_3$ are defined by $A(x, y) = (x, y, 0)$ and $B(z) = (z, 0, 0), \forall x, y, z \in \mathbb{R}$, respectively. Let $U^{-1}(0) = \cap_{i=1}^{m} U_i^{-1}(0) = U_i^{-1}(0) = F_i^{-1}(0) \times K_i^{-1}(0) = \{(x, y, z) : |x| \le 1, |y| \le 1, z \in \mathbb{R}\}$ and $G = [A, -B] : H_3 \to H_3$ be defined by

$$G(x, y, z) = (x - z, y, 0), \ \forall (x, y, z) \in \mathbb{R}^3.$$

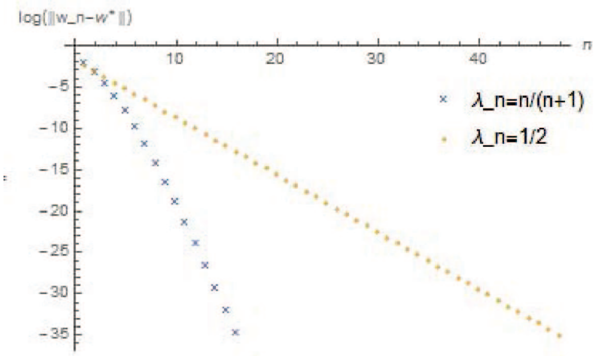
Then $riU^{-1}(0) \cap kerG = \{(x, 0, x), x \in \mathbb{R}\} \neq \emptyset$, $U^{-1}(0)$ is finite codimensional, the range of G is closed, and the solution set of SECNP is $\Gamma = \bigcap_{i=1}^{m} (F_i^{-1}(0) \times K_i^{-1}(0)) \cap kerG = \{(x, 0, x), \ x \in \mathbb{R}\}$. By Corollary 1 we can get that SECNP satisfies the bounded linear regularity property.

Let $w_0 = (x_0, y_0, z_0) \in \bigcap_{i=1}^{m} F_i^{-1}(0) \times K_i^{-1}(0)$. From the algorithm (16), we have

$$\begin{cases} x_{n+1} = (1 - \gamma_n) x_n + \gamma_n z_n, \\ y_{n+1} = (1 - \gamma_n) y_n, \\ z_{n+1} = (1 - \gamma_n) z_n + \gamma_n x_n. \end{cases}$$

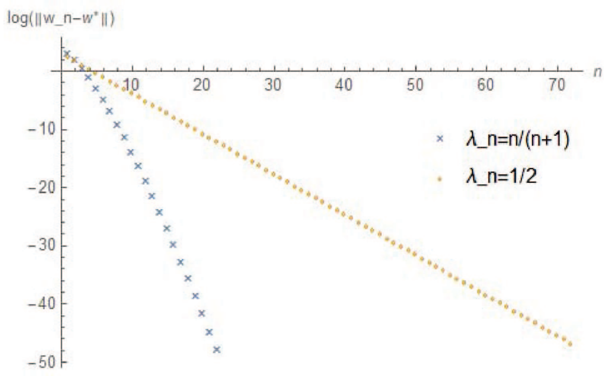
In algorithm (16), we take $\gamma_n = \frac{1}{2}, \frac{n}{n+1}$, respectively. Then we have the following numerical results (the x-coordinate denotes the iteration times, and the y-coordinate denotes the logarithm of the error). The whole program was written in Wolfram Mathematica (version 10.3). All the numerical results were performed on a personal Dell computer with Inter(R) Core(TM) i5-7200 U CPU 2.50 GHz and RAM 4.00 GB.

We choose error to be 10^{-15} , the initial value $w_0 = (0.5, 0.2, 0.4)$ and $w_0 = (0.8, 0.8, 0.5)$, according to the algorithm (16), they converges to $w^* = (0.45, 0, 0.45) \in \Gamma$ and $w^* = (0.65, 0, 0.65) \in \Gamma$, respectively (See Figure 1).



(a)

Figure 1. *Cont.*



(**b**)

Figure 1. Numerical Results. (**a**) error $= 10^{-15}, w_0 = (0.5, 0.2, 0.4), w^* = (0.45, 0, 0.45)$, (**b**) error = 10^{-15}, $w_0 = (0.8, 0.8, 0.5), w^* = (0.65, 0, 0.65)$.

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