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Some Economic Dynamics Problems for Hybrid Models with Aftereffect

Eugene Bravyi [1] **, Vladimir Maksimov** [2,*] **and Pyotr Simonov** [2]

[1] Scientific Center of Functional Differential Equations, Perm National Research Polytechnic University, 614990 Perm, Russia; bravyi@perm.ru

[2] Department of Information Systems and Mathematical Methods in Economics, Perm State University, 614990 Perm, Russia; simpm@mail.ru

[*] Correspondence: maksimov@econ.psu.ru; Tel.: +7-908-259-8494

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Abstract: In this paper, we consider a class of economic dynamics models in the form of linear functional differential systems with continuous and discrete times (hybrid models) that covers many kinds of dynamic models with aftereffect. The focus of attention is periodic boundary value problems with deviating argument, control problems with respect to general on-target vector-functional and questions of stability to solutions. For boundary value problems, some sharp sufficient conditions of the unique solvability are obtained. The attainability of on-target values is under study as applied to control problems with polyhedral constraints with respect to control, some estimates of the attainability set as well as estimates to a number of switch-points of programming control are presented. For a class of hybrid systems, a description of asymptotic properties of solutions is given.

Keywords: economic dynamics; boundary value problems; control problems; attainability; stability

1. Introduction

Currently, in the context of the digitalization of the economy, strengthening the scientific validity of decisions is becoming more and more relevant. The basis of such strengthening is economic and mathematical methods that implement a targeted approach to solving actual applied problems. This approach allows one to find ways and means to achieve strategic goals, balance the goals and means to achieve them. Here we consider some of the typical problems of economic dynamics in relation to a wide class of hybrid dynamic models with aftereffect. For certainty, the focus is on systems that model the interaction of production processes and procedures of financing them. Recall that an essential feature of any economic processes is the presence of a lag which means a period of time between the moment of external action and a reply of the system, for instance, between capital investments moment and an actual growth in output. Thus a model governing the dynamics of the system under consideration can be written in the form of functional differential system (FDS) with continuous and discrete times (Hybrid FDS = HFDS, for short). The term "hybrid" is deeply embedded in the literature in different senses, that is why we follow the authors employing the more definite name "continuous-discrete systems" (CDS), see, for instance, References [1–5] and references therein. In the works mentioned, the reader can find a detailed motivation for studying CDS and examples of applications to problems of controllability, observability and stability. It should be noted that in most cases dynamics in continuous time is governed by ordinary differential systems. Contrary to those we consider a quite general case of continuous subsystems.

After introductory Sections 1 and 2 with necessary preliminaries, we consider three types of problems separately. Boundary value problems (BVPs) are the problems on the solutions to CDS that satisfy some additional (boundary) conditions. From the view-point of Economics, the solvability of a BVP means the existence of a model trajectory with certain property, say, with the periodicity in development. Some sharp sufficient conditions of the solvability for a wide class of periodic BVPs for hybrid systems with deviating argument are presented in Section 3. The study of control problems (CPs) in the case of the solvability answers the questions of attainable values of on-target functionals and allows one to construct a corresponding programming control with taking into account prescribed constraints. The results on CPs for CDS with polyhedral constraints with respect to control are given in Section 4. Section 5 focuses on questions of solutions stability to HFDS as the property of great importance for trajectories defined on unbounded period of time. For a class of hybrid systems, a description of asymptotic properties of solutions is presented.

2. Preliminaries

Our consideration is based in essence on the ideas and results of the theory of Abstract Functional Differential Equation (AFDE) constructed by N.V. Azbelev and L.F. Rakhmatullina in Reference [6]. AFDE is the equation $\mathcal{L}x = f$ with an operator $\mathcal{L}$ acting from a Banach space $\mathbf{D}$ isomorphic to the direct product $\mathbf{B} \times R^n$ with a Banach space $\mathbf{B}$. The key idea of the AFDE theory applications is in the appropriate choice of $\mathbf{D}$ while studying each new problem. Such a choice allows one to apply standard constructions and statements to the problem needed before an one-off approach and special constructions. This concept has demonstrated the high efficiency as applied to wide classes of problems, see Reference [7]. Here we follow this concept while studying some urgent problems with respect to hybrid models of economic dynamics.

To give a description of the general form to the model, let us start with the main spaces where the equations are considered.

In description of the systems under consideration we follow Reference [8].

Fix a segment $[0, T] \subset R$. By $L^n = L^n[0, T]$ we denote the space of summable functions $v : [0, T] \to R^n$ under the norm $||v||_{L^n} = \int_0^T |v(s)|_n \, ds$, where $| \cdot |_n$ stands for an arbitrary fixed norm of R^n (we omit the subscript when it does not cause confusion); $L_2^n = L_2^n[0, T]$ is the space of square summable functions $u : [0, T] \to R^r$ with the inner product $(u, v) = \int_0^T u'(s)v(s) \, ds$, where $\cdot'$ stands for transposition.

The space $AC^n = AC^n[0, T]$ is the space of absolutely continuous functions $x : [0, T] \to R^n$ with the norm

$$||x||_{AC^n} = ||\dot{x}||_{L^n} + |x(0)|_n.$$

Let us fix a set $J = \{t_0, t_1, ..., t_\mu\}, 0 = t_0 < t_1 < ... < t_\mu = T$. $FD^\nu = FD^\nu\{t_0, t_1, ..., t_\mu\}$ denotes the space of functions $z : J \to R^\nu$ under the norm

$$||z||_{FD^\nu} = \sum_{i=0}^{\mu} |z(t_i)|_\nu.$$

Below we use the symbol z for elements of FD^ν keeping in mind $z = \mathrm{col}(z(t_0), \ldots, z(t_\mu))$ and $\rho z = \mathrm{col}(z(t_1), \ldots, z(t_\mu))$. The space of all $\rho z, z \in FD^\nu$ embedded in FD^ν by $\rho z \to \mathrm{col}(0, z(t_1), \ldots, z(t_\mu)) \in FD^\nu$ is denoted by FD_1^ν.

We consider the hybrid system

$$\begin{aligned}
\dot{x} &= \mathcal{T}_{11}x + \mathcal{T}_{12}z + f, \\
\rho z &= \mathcal{T}_{21}x + \mathcal{T}_{22}z + g,
\end{aligned} \tag{1}$$

where the linear bounded operators $\mathcal{T}_{ij}$, $i,j = 1,2$ act as follows: $\mathcal{T}_{11} : AC^n \to L^n$, $\mathcal{T}_{12} : FD^\nu \to L^n$, $\mathcal{T}_{21} : AC^n \to FD^\nu$, $\mathcal{T}_{22} : FD^\nu \to FD^\nu$; $f \in L^n$, $g \in FD^\nu$. Additionally, operator $\mathcal{T}_{11} : AC^n \to L^n$ is assumed to be compact one (compactness conditions can be found in References [7] (pp. 43, 284), [9] (p. 77)). As for the representation of all operators, it will be given more in detail in each section below.

The system (1) is a particular case of the abstract functional differential equation (AFDE) [6,7]. In the sequel we use some fundamental results from the Theory of AFDE. Let us recall some of basic results as those given in Reference [7].

2.1. From the AFDE Theory

Let **D** and **B** be Banach spaces such that **D** is isomorphic to the direct product $\mathbf{B} \times R^n$ (in the sequel we write $\mathbf{D} \simeq \mathbf{B} \times R^n$ for short).

A linear *abstract functional differential equation* (AFDE) is the equation

$$\mathcal{L}x = f, \tag{2}$$

with a linear operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$.

We denote by $\mathcal{J} = \{\Lambda, Y\} : \mathbf{B} \times R^n \to \mathbf{D}$ a linear isomorphism with $J^{-1} = [\delta, r]$. Define the norms in $\mathbf{B} \times R^n$ and **D** by the equalities

$$\| \{z, \beta\} \|_{\mathbf{B} \times R^n} = \| z \|_{\mathbf{B}} + | \beta |, \quad \| x \|_{\mathbf{D}} = \| \delta x \|_{\mathbf{B}} + | rx | .$$

The operator $Q = \mathcal{L}\Lambda : \mathbf{B} \to \mathbf{B}$ is called the *principal part* of $\mathcal{L} : \mathbf{D} \to \mathbf{B}$, $A = \mathcal{L}Y : R^n \to \mathbf{B}$ is the so-called *finite-dimensional part*.

Theorem 1 ([7]). *An operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is a Noether one if and only if the principal part $Q : \mathbf{B} \to \mathbf{B}$ of $\mathcal{L}$ is a Noether operator. In this case $\mathrm{ind}\mathcal{L} = \mathrm{ind}Q + n$.*

It follows from Theorem 1 that in the case $\mathrm{ind}\,\mathcal{L} = n$ the operator Q is Fredholm (i.e., it can be represented as a sum of an invertible and a compact operators).

Let $(x_1, \ldots, x_n)$ be a basis for the kernel of $\mathcal{L}$. The vector $X = (x_1, \ldots, x_n)$ is called the *fundamental vector* to the equation $\mathcal{L}x = 0$ with the fundamental system of solutions $x_1, \ldots, x_n$.

To define a *linear boundary value problem*, we introduce a linear bounded vector-functional $\ell = \mathrm{col}(\ell^1, \ldots, \ell^m) : \mathbf{D} \to R^m$ with linearly independent components $\ell^i : \mathbf{D} \to R$ and consider the system

$$\mathcal{L}x = f, \ lx = \alpha \in R^m. \tag{3}$$

In the case $R(\mathcal{L}) = \mathbf{B}$, $\dim \ker \mathcal{L} = n$, the unique solvability of (3) takes place if and only if $\det \ell X \neq 0$, where $\ell X = (\ell^i x_j)_{i,j=1,\ldots,n}$.

Theorem 2 ([7]). *The problem (3) is a Noether one if and only if the principal part $Q : \mathbf{B} \to \mathbf{B}$ of $\mathcal{L}$ is a Noether operator and also $\mathrm{ind}\,[\mathcal{L}, l] = \mathrm{ind}\,Q + n - m$.*

Corollary 1 ([7]). *The problem (3) is a Fredholm one if and only if $\mathrm{ind}Q = m - n$.*

In the sequel we consider only the case Q is Fredholm. Thus, for the problem (3) with $m = n$, the assertions "the problem has a unique solution for a certain right-hand part $\{f, \alpha\}$ (the problem is

uniquely solvable)", "the problem is solvable for each $\{f, \alpha\}$ (the problem is solvable everywhere)", "the problem is everywhere and uniquely solvable" are equivalent.

Theorem 3 ([7]). *The following assertions are equivalent.*

(a) $R(\mathcal{L}) = \mathbf{B}$.

(b) $\dim \ker \mathcal{L} = n$.

(c) *There exists a vector-functional* $l : \mathbf{D} \to \mathbf{R}^n$ *such that the problem* (3) *is uniquely solvable for each* $f \in \mathbf{B}$, $\alpha \in \mathbf{R}^n$.

Consider the uniquely solvable problem

$$\mathcal{L}x = f, \ lx = \alpha \in R^n. \tag{4}$$

The solution to (4) has the representation

$$x = Gf + X\alpha,$$

where the linear operator $G : \mathbf{B} \to \mathbf{D}$ is called the *Green operator*.

2.2. Hybrid System as an AFDE

From the view-point of the AFDE Theory, the system (1) is the Equation (2) with $\mathcal{L}$ defined by the equality

$$\mathcal{L}y = \begin{pmatrix} \dot{x} - \mathcal{T}_{11}x - \mathcal{T}_{12}z \\ \rho z - \mathcal{T}_{21}x - \mathcal{T}_{22}z \end{pmatrix}, \tag{5}$$

where $y = \mathrm{col}(x, z) \in AC^n \times FD^\nu$. Thus we have here $\mathbf{D} = \mathbf{B} \times \mathbf{Z}$ with $\mathbf{B} = L^n \times FD_1^\nu$ and $\mathbf{Z} = R^n \times R^\nu$. Note also that the operators δ and r are defined by the equalities

$$\delta y = \mathrm{col}(\dot{x}, \rho z), \ \ ry = \mathrm{col}(x(0), z(0)).$$

To define the principal part $Q : \mathbf{B} \to \mathbf{B}$ for this case, introduce operators $V : L^n \to AC^n$ and $U : FD_1^\nu \to FD^\nu$ by the equalities $(Vv)(t) = \int_0^t v(s) \, ds$ and $Uu = \mathrm{col}(0, u)$. Now we have the representation

$$Q \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} v - \mathcal{T}_{11}Vv + \mathcal{T}_{12} Uu \\ u - \mathcal{T}_{21}Vv + \mathcal{T}_{22} Uu \end{pmatrix}. \tag{6}$$

Due to the assumption on the compactness of $\mathcal{T}_{11}$ this Q is a Fredholm operator.

3. Periodic Boundary Value Problems

Recently, in a variety of mathematical models (including economic ones), both time lags and advances have begun to be used. Here our review partially follows the review from Reference [10]. Anticipated backward stochastic equations with a generator depended not only on the solution in the present, but also in the future time were considered in Reference [11]. In Reference [12], there were introduced backward equations, the coefficients of which depended on the solution in present and past times. It was natural to consider backward equations, the coefficients of which depend on both the past and future states [10,13,14]. Such equations have been applied in many fields, from finance to stochastic management. In References [15–17], mixed functional differential equations were considered in the

deterministic case, and some economic applications of such equations were indicated. As a rule, to find conditions for the existence of solutions, the principle of contraction maps or general theorems on fixed points were used.

Economic mathematical models giving rise to mixed functional differential equations are also considered in References [18,19]: in production cycle theory the following equation arises

$$\dot{u}(t) = F(u(t), u(t - \tau), u(t + \tau)), \quad t \geq 0, \quad \tau > 0.$$

It is clear that this equation is an idealization, and delays and advances are not constant, they are not located in a set of points. More realistic formulations lead to functional differential equations with a general deviation of the argument, the exact laws of change to which cannot be known, as a rule.

We will consider systems of two functional differential equations with general delays and advances. The first unknown variable is an absolutely continuous function $x \in AC^1$ defined on a segment of the real axis (a component with continuous time), the second is a component with discrete time, represented as an element of space R^{n+1}. Boundary value problems for such systems often arise when considering problems of economic dynamics. We restrict ourselves to considering a periodic boundary value problem that arises, in particular, in the study of business cycles in economics.

Here the space $C = C[0, T]$ is the space of continuous functions $x : [0, T] \to R$ with the norm $||x||_C = \max_{t \in [0,T]} |x(t)|$; R^n_+ is the set of vectors from R^n with nonnegative components.

We consider the continuous–discrete system

$$\dot{x}(t) = (\mathcal{A}x)(t) + (\mathcal{C}z)(t) + f(t), \quad t \in [0, T], \tag{7}$$

$$z_i - z_{i-1} = (\mathcal{D}x)_i + (\mathcal{B}z)_i + g_i, \quad i = 1, \ldots, n, \tag{8}$$

subjected to the periodic-like boundary conditions

$$x(0) - x(T) = \alpha_x, \quad z_0 - z_n = \alpha_z, \tag{9}$$

where $x \in AC^1$ is a continuous component of a solution; $z = \mathrm{col}(z_0, z_1, \ldots, z_n) \in R^{n+1}$ is a discrete component of a solution; $\mathcal{A} : C \to L^1, \mathcal{C} : R^{n+1} \to L^1, \mathcal{B} : R^{n+1} \to R^n, \mathcal{D} : C \to R^n$ are linear bounded operators; $f \in L^1$, the numbers $g_i \in R, i = 1, \ldots, n$, and $\alpha_x, \alpha_z \in R$ are given.

We suppose that the linear operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are differences of two positive linear operators:

$$\mathcal{A} = \mathcal{A}^+ - \mathcal{A}^-, \quad \mathcal{B} = \mathcal{B}^+ - \mathcal{B}^-, \quad \mathcal{C} = \mathcal{C}^+ - \mathcal{C}^-, \quad \mathcal{D} = \mathcal{D}^+ - \mathcal{D}^-,$$

that means the operators $\mathcal{A}^{+/-} : C \to L^1$ map nonnegative continuous functions to almost everywhere nonnegative functions, the operators $\mathcal{C}^{+/-} : R^{n+1} \to L^1$ map elements of R^{n+1}_+ to almost everywhere nonnegative functions, the operators $\mathcal{D}^{+/-} : C \to R^n$ map nonnegative continuous functions to elements of R^n_+, the operators $\mathcal{B}^{+/-} : R^{n+1} \to R^n$ map R^{n+1}_+ into R^n_+.

Problems (7)–(9) can always be rewritten in the following way

$$\dot{x}(t) = (\mathcal{A}x)(t) + \sum_{j=1}^{n} c_j(t)z_j + f(t), \quad t \in [0, T], \tag{10}$$

$$z_i - z_{i-1} = \mathcal{D}_i x + \sum_{j=0}^{n} b_{ij} z_j + g_i, \quad i = 1, \ldots, n, \tag{11}$$

$$x(0) - x(T) = \alpha_x, \quad z_0 - z_n = \alpha_z, \tag{12}$$

where $\mathcal{D}_i : C \to R, i = 1, \ldots, n$, are linear bounded functionals; the coefficients $c_i \in L^1, i = 1, \ldots, n; b_{ij} \in R$, $i = 1, \ldots, n, j = 0, \ldots, n$ are given. So, the operators $\mathcal{C}^{+/-}, \mathcal{B}^{+/-}$ are defined by the equalities

$$(\mathcal{C}^+ z)(t) = \sum_{j=0}^{n} c_j^+(t) z_j, \quad (\mathcal{C}^- z)(t) = \sum_{j=0}^{n} c_j^-(t) z_j, \quad t \in [0, T],$$

$$(\mathcal{B}^+ z)_i = \sum_{j=0}^{n} b_{ij}^+ z_j \quad (\mathcal{B}^- z)_i = \sum_{j=0}^{n} b_{ij}^- z_j, \quad i = 1, \ldots, n,$$

where for $a \in R$ we use notation

$$a^+ = \begin{cases} a & \text{if } a \geq 0; \\ 0 & \text{if } a < 0, \end{cases} \qquad a^- = \begin{cases} -a & \text{if } a < 0; \\ 0 & \text{if } a \geq 0. \end{cases}$$

Together with system (7)–(9) (or (10)–(12)) we consider a system with continuous time:

$$\dot{x}(t) = (\mathcal{A}x)(t) + (\widehat{\mathcal{C}}w)(t) + f(t), \quad t \in [0, T], \tag{13}$$

$$\dot{w}(t) = (\widehat{\mathcal{D}}x)(t) + (\widehat{\mathcal{B}}w)(t) + \widehat{g}(t), \quad t \in [0, T], \tag{14}$$

$$x(0) - x(T) = \alpha_x, \quad w(0) - w(T) = \alpha_z, \tag{15}$$

where

$$(\widehat{\mathcal{C}}w)(t) = \sum_{j=0}^{n} c_j(t) w(\theta_j), \quad t \in [0, T], \tag{16}$$

$$(\widehat{\mathcal{D}}x)(t) = \mathcal{D}_i x, \quad (\widehat{\mathcal{B}}w)(t) = \sum_{j=0}^{n} b_{ij} w(\theta_i), \quad t \in e_i, \quad i = 1, \ldots, n, \tag{17}$$

with

$$\theta_i = (i-1)T/n, \quad i = 0, \ldots, n; \quad e_i = [\theta_{i-1}, \theta_i), \quad i = 1, \ldots, n-1; \quad e_n = [\theta_{n-1}, T].$$

Obviously, there is a one-to-one correspondence between solutions (x, z) of problem (10)–(12) and solutions (x, w) to (13)–(15):

$$w(t) = (z_{i-1}(\theta_i - t) + z_i(t - \theta_{i-1}))n/T, \quad t \in e_i, \quad i = 1, \ldots, n, \tag{18}$$
$$z_i = w(\theta_i) \quad i = 0, \ldots, n.$$

Moreover, the equalities

$$\mathcal{C}^{+/-}\mathbb{1}(t) = (\widehat{\mathcal{C}}^{+/-}\mathbf{1})(t), \quad t \in [0, T];$$
$$(\mathcal{D}^{+/-}\mathbf{1})_i = (\widehat{\mathcal{D}}^{+/-}\mathbf{1})(t), \quad (\mathcal{B}^{+/-}\mathbb{1})_i = (\widehat{\mathcal{B}}^{+/-}\mathbf{1})(t), \quad t \in e_i, \quad i = 1, \ldots, n,$$
$$\frac{T}{n} \sum_{i=1}^{n} (\mathcal{D}^{+/-}\mathbf{1})_i = \int_0^T (\widehat{\mathcal{D}}^{+/-}\mathbf{1})(t) \, dt, \quad \frac{T}{n} \sum_{i=1}^{n} (\mathcal{B}^{+/-}\mathbb{1})_i = \int_0^T (\widehat{\mathcal{B}}^{+/-}\mathbf{1})(t) \, dt \tag{19}$$

hold, where $\mathbb{1} = \mathrm{col}(1, \ldots, 1)$ is the unit vector from R^{n+1}, $\mathbf{1}(t) \equiv 1, t \in [0, T]$, is the unit function from C.

Further we use only the functions $\mathcal{A}^{+/-}\mathbf{1}, \mathcal{C}^{+/-}\mathbb{1}$ and vectors $\mathcal{D}^{+/-}\mathbf{1}, \mathcal{B}^{+/-}\mathbb{1}$ for formulating our assertions. It follows that we may consider in this section problem (13)–(15) instead of problem (7)–(9).

3.1. Integral Restrictions on Functional Operators

Denote in this section

$$\mathfrak{A}^+ \equiv \int_0^T (\mathcal{A}^+ \mathbb{1})(s)\, ds, \quad \mathfrak{A}^- \equiv \int_0^T (\mathcal{A}^- \mathbb{1})(s)\, ds,$$

$$\mathfrak{B}^+ \equiv \frac{T}{n}\sum_{i=1}^n (\mathcal{B}^+ \mathbb{1})_i = \frac{T}{n}\sum_{i=1}^n \sum_{j=1}^n b_{ij}^+, \quad \mathfrak{B}^- \equiv \frac{T}{n}\sum_{i=1}^n (\mathcal{B}^- \mathbb{1})_i = \frac{T}{n}\sum_{i=1}^n \sum_{j=1}^n b_{ij}^-,$$

$$\mathfrak{C}^+ \equiv \sum_{i=1}^n (\mathcal{C}^+ \mathbb{1})_i = \sum_{i=1}^n \int_0^T (c_i(s))^+\, ds, \quad \mathfrak{C}^- \equiv \sum_{i=1}^n (\mathcal{C}^- \mathbb{1})_i = \sum_{i=1}^n \int_0^T (c_i(s))^-\, ds,$$

$$\mathfrak{D}^+ \equiv \frac{T}{n}\sum_{i=1}^n \mathcal{D}_i^+ \mathbb{1}, \quad \mathfrak{D}^- \equiv \frac{T}{n}\sum_{i=1}^n \mathcal{D}_i^- \mathbb{1}. \tag{20}$$

Theorem 4. *Periodic boundary value problem* (7)–(9) *is uniquely solvable if the conditions*

$$\triangle_1 \equiv \begin{vmatrix} \mathfrak{A}^+ - \mathfrak{A}^- & y_a + x_a & \mathfrak{C}^+ - \mathfrak{C}^- & y_c + x_c \\ -(a_1^+ - a_1^-) & 1 - y_a & -(c_1^+ - c_1^-) & -y_c \\ \mathfrak{D}^+ - \mathfrak{D}^- & y_d + x_d & \mathfrak{B}^+ - \mathfrak{B}^- & y_b + x_b \\ -(d_1^+ - d_1^-) & -y_d & -(b_1^+ - b_1^-) & 1 - y_b \end{vmatrix} \neq 0, \tag{21}$$

$$\triangle_2 \equiv \begin{vmatrix} \mathfrak{A}^+ - \mathfrak{A}^- & y_a + x_a & \mathfrak{C}^+ - \mathfrak{C}^- & y_c + x_c \\ -(a_1^+ - a_1^-) & 1 - y_a & -(c_1^+ - c_1^-) & -y_c \\ \mathfrak{D}^+ - \mathfrak{D}^- & y_d + x_d & \mathfrak{B}^+ - \mathfrak{B}^- & y_b + x_b \\ d_1^+ - d_1^- & y_d & b_1^+ - b_1^- & 1 + y_b \end{vmatrix} \neq 0, \tag{22}$$

are fulfilled for all real numbers x_a, x_b, x_c, x_d satisfying the conditions

$$x_a \in [-\mathfrak{A}^- + a_1^-, \quad \mathfrak{A}^+ - a_1^+], \quad x_b \in [-\mathfrak{B}^- + b_1^-, \quad \mathfrak{B}^+ - b_1^+],$$

$$x_c \in [-\mathfrak{C}^- + c_1^-, \quad \mathfrak{C}^+ - c_1^+], \quad x_d \in [-\mathfrak{D}^- + d_1^-, \quad \mathfrak{D}^+ - d_1^+],$$

all real numbers y_a, y_b, y_c, y_d satisfying the conditions

$$y_a \in [-a_1^-, a_1^+], \quad y_b \in [-b_1^-, b_1^+], \quad y_c \in [-c_1^-, c_1^+], \quad y_d \in [-d_1^-, d_1^+],$$

whenever the real numbers $a_1^{+/-}$, $b_1^{+/-}$, $c_1^{+/-}$, $d_1^{+/-}$ satisfy the conditions

$$a_1^{+/-} \in [0, \mathfrak{A}^{+/-}], \quad b_1^{+/-} \in [0, \mathfrak{B}^{+/-}], \quad c_1^{+/-} \in [0, \mathfrak{C}^{+/-}], \quad d_1^{+/-} \in [0, \mathfrak{D}^{+/-}]$$

For any given nonnegative numbers $\mathfrak{A}^{+/-}$, $\mathfrak{B}^{+/-}$, $\mathfrak{C}^{+/-}$, $\mathfrak{D}^{+/-}$, the conditions of Theorem 4 can always be verified by numerical calculations.

Remark 1. *Theorem 4 and Theorems 5–8 presented below give sufficient conditions of the unique solvability. In fact, some kind of the necessity takes place. Namely, the proofs implies that these conditions are necessary for the unique solvability of all problems for given numbers $\mathfrak{A}^{+/-}$, $\mathfrak{B}^{+/-}$, $\mathfrak{C}^{+/-}$, $\mathfrak{D}^{+/-}$.*

If diagonal operators $\mathcal{A}$, $\mathcal{B}$ are zero or all operators $\mathcal{A}, \mathcal{C}, \mathcal{B}, \mathcal{D}$ are positive, we can formulate solvability conditions of problems (10)–(12) in closed analytical form (in Theorems 5 and 6). After that we will prove all these theorem together.

Theorem 5. *Let* $\mathfrak{A}^+ = 0$, $\mathfrak{A}^- = 0$, $\mathfrak{B}^+ = 0$, $\mathfrak{B}^- = 0$. *Then periodic problems* (7)–(9) *is uniquely solvable if*

$$\mathfrak{C}^+ \neq \mathfrak{C}^-, \quad \mathfrak{D}^+ \neq \mathfrak{D}^-, \tag{23}$$

and

$$\frac{\overline{C} \max\left(\overline{C}, 4\underline{C}\right)}{\overline{C} - \underline{C}} \cdot \frac{\overline{D} \max\left(\overline{D}, 4\underline{D}\right)}{\overline{D} - \underline{D}} < 16, \tag{24}$$

where $\overline{D} = \max(\mathfrak{D}^+, \mathfrak{D}^-)$, $\overline{C} = \max(\mathfrak{C}^+, \mathfrak{C}^-)$, $\underline{D} = \min(\mathfrak{D}^+, \mathfrak{D}^-)$, $\underline{C} = \min(\mathfrak{C}^+, \mathfrak{C}^-)$.

Theorem 6. *Let* $\mathfrak{A}^- = 0$, $\mathfrak{B}^- = 0$, $\mathfrak{C}^- = 0$, $\mathfrak{D}^- = 0$. *The periodic boundary value problems* (7)–(9) *is uniquely solvable if*

$$0 < \mathfrak{A}^+ < 4,\ 0 < \mathfrak{B}^+ < 4,$$
$$\mathfrak{C}^+ \mathfrak{D}^+ < \mathfrak{A}^+ \mathfrak{B}^+ \min\left(\frac{1}{1 + \mathfrak{A}^+}, 1 - \frac{\mathfrak{A}^+}{4}\right) \min\left(\frac{1}{1 + \mathfrak{B}^+}, 1 - \frac{\mathfrak{B}^+}{4}\right) \tag{25}$$

or

$$0 \leq \mathfrak{A}^+ < 1, \quad 0 \leq \mathfrak{B}^+ < 1, \quad \frac{\mathfrak{A}^+ \mathfrak{B}^+}{(1 - \mathfrak{A}^+)(1 - \mathfrak{B}^+)} < \mathfrak{C}^+ \mathfrak{D}^+,$$
$$\mathfrak{C}^+ \mathfrak{D}^+ < \min_{t \in (0,1)} \frac{1 - t^2\mathfrak{A}^+ - (1-t)^2\mathfrak{B}^+ + \sqrt{(1 - t^2\mathfrak{A}^+ - (1-t)^2\mathfrak{B}^+)^2 - 4\mathfrak{A}^+\mathfrak{B}^+t^2(1-t)^2}}{2t^2(1-t)^2}. \tag{26}$$

Proofs of Theorems 4–6. These theorems have been proved by a unified scheme. For the corresponding problems (7)–(9) with continuous time, these assertions are proved in References [20,21] for Theorems 4 and 6 and in Reference [22] for Theorem 5. There are one-to-one correspondences between solutions of problem (13)–(15) with continuous time and the corresponding problems (10)–(12) from Theorems 4–6. In References [20–22], we used only the functions $\mathcal{A}^{+/-}\mathbf{1}$, $\widehat{\mathcal{C}}^{+/-}\mathbf{1}$, $\widehat{\mathcal{D}}^{+/-}\mathbf{1}$, $\widehat{\mathcal{B}}^{+/-}\mathbf{1}$ for proving of sufficiency. As for proving of necessity, it was used only operators of the form (16) and (17) in References [20–22]. From equalities (19), it follows that the assertions of the theorems and Remark 1 are valid for hybrid systems too. □

Remark 2. *Condition* (26) *is fulfilled if*

$$0 \leq \mathfrak{A}^+, \mathfrak{B}^+ < 1, \quad \frac{\mathfrak{A}^+ \mathfrak{B}^+}{(1 - \mathfrak{A}^+)(1 - \mathfrak{B}^+)} < \mathfrak{C}^+ \mathfrak{D}^+ < 4\left(1 + \sqrt{1 - \max(\mathfrak{A}^+, \mathfrak{B}^+)}\right)^2. \tag{27}$$

In the symmetric case $\mathfrak{A}^+ = \mathfrak{B}^+ \in [0,1]$, *the minimum in* (26) *occurs at* $t = 1/2$. *Therefore, condition* (26) *means that*

$$0 \leq \mathfrak{A}^+ = \mathfrak{B}^+ < 3/4, \quad \left(\frac{\mathfrak{A}^+}{1 - \mathfrak{A}^+}\right)^2 < \mathfrak{C}^+ \mathfrak{D}^+ < 4\left(1 + \sqrt{1 - \mathfrak{A}^+}\right)^2. \tag{28}$$

In the case $\mathfrak{A}^+ = 1 - \mathfrak{B}^+ \in [0,1]$, *it is easy to check that the minimum in* (26) *occurs at*

$$t = \frac{1 + \sqrt{-4\mathfrak{B}^{+2} + 4\mathfrak{B}^+ + 1}}{2\mathfrak{B}^+ + \sqrt{-4\mathfrak{B}^{+2} + 4\mathfrak{B}^+ + 1} + \sqrt{3 + 2\sqrt{-4\mathfrak{B}^{+2} + 4\mathfrak{B}^+ + 1}}}.$$

So, if $\mathfrak{A}^+ = 1 - \mathfrak{B}^+$, *we can obtain the minimum in* (26) *analytically. In particular, we have the following sufficient condition:* (26) *is fulfilled if*

$$1 < \mathfrak{C}^+\mathfrak{D}^+ < 6 + 4\sqrt{2} - (2 + 16\sqrt{2} - 10\sqrt{5})(\mathfrak{B}^+ - 1/2)^2. \tag{29}$$

Conditions (27), (28) *are obtained in Reference* [21]. *Because the norms of the functions* $\mathcal{C}^+\mathbb{1}$ *and* $\widehat{\mathcal{C}}^+\mathbf{1}$, $\mathcal{B}^+\mathbb{1}$ *and* $\widehat{\mathcal{B}}^+\mathbf{1}$, $\mathcal{D}^+\mathbb{1}$ *and* $\widehat{\mathcal{D}}^+\mathbf{1}$ *coincide, we can use these results from Reference* [21] *for hybrid systems. To derive condition* (29) *from condition* (26), *we need only to check the elementary inequality with the polynomial of degree* 4 *in* $\mathfrak{B}^+$ *and t:*

$$F^2 t^2 (1-t)^2 - F(1 - t^2(1 - \mathfrak{B}^+) - (1-t)^2\mathfrak{B}^+) + (1 - \mathfrak{B}^+)\mathfrak{B}^+ < 0,$$

for all $\mathfrak{B}^+ \in [0,1)$, $t \in [0,1]$, *where F is the right-hand side in* (29).

Remark 3. *From Reference* [21] *it follows that the assertion of Theorem* 6 *is valid for every problem of the form*

$$\dot{x}(t) = \varepsilon_{11}(\mathcal{A}^+ x)(t) + \varepsilon_{12}(\mathcal{C}^+ z)(t) + f(t), \quad t \in [0,T],$$

$$z_i - z_{i-1} = \varepsilon_{21}(\mathcal{D}^+ x)_i + \varepsilon_{22}(\mathcal{B}^+ z)_i + g_i, \quad i = 1,\ldots,n,$$

$$x(0) - x(T) = \alpha_x, \quad z_0 - z_n = \alpha_z,$$

where $\varepsilon_{ij} \in \{-1,1\}$, $i,j = 1,2$, $\varepsilon_{11}\varepsilon_{12}\varepsilon_{21}\varepsilon_{22} = 1$.

3.2. Point-Wise Restrictions on Functional Operators

Let nonnegative numbers $\mathfrak{A}^{+/-}, \mathfrak{B}^{+/-}, \mathfrak{C}^{+/-}, \mathfrak{D}^{+/-}$ be given.

First we consider problems (7)–(9) only with positive operators and under more general point-wise restrictions than conditions (40) in the next theorem, where, however, the operators can be differences of positive operators.

We suppose (without loss of generality) that there exist a positive function $p \in L^1$ and a vector $q \in R^n$ with positive components such that

$$\int_0^T p(s)\, ds = 1, \quad \sum_{i=1}^n q_i = 1, \tag{30}$$

and there exist nonnegative numbers $\mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{C}^+, \mathfrak{D}^+$ such that

$$(\mathcal{A}^+\mathbf{1})(t) = \mathfrak{A}^+ p(t), \quad (\mathcal{C}^+\mathbb{1})(t) = \mathfrak{C}^+ p(t), \quad t \in [0,T],$$
$$(\mathcal{D}^+\mathbf{1})_i(t) = \mathfrak{D}^+ q_i, \quad (\mathcal{B}^+\mathbb{1})_i = \mathfrak{B}^+ q_i, \quad i = 1,\ldots,n. \tag{31}$$

Denote $\widehat{\mathfrak{D}}^+ = T/n\, \mathfrak{D}^+, \widehat{\mathfrak{B}}^+ = T/n\, \mathfrak{B}^+$.

Theorem 7. *Let* $\mathfrak{A}^- = 0$, $\mathfrak{B}^- = 0$, $\mathfrak{C}^- = 0$, $\mathfrak{D}^- = 0$ *and conditions* (30), (31) *be fulfilled. If*

$$\mathfrak{A}^+ < 4, \quad \widehat{\mathfrak{B}}^+ < 4, \quad \mathfrak{C}^+ \mathfrak{D}^+ \neq \mathfrak{A}^+ \mathfrak{B}^+, \quad \mathfrak{C}^+ \widehat{\mathfrak{D}}^+ < \left(4 - \mathfrak{A}^+\right)\left(4 - \widehat{\mathfrak{B}}^+\right), \tag{32}$$

then periodic boundary value problem (7)–(9) *is uniquely solvable.*

Proof. It is more convenient to deal with continuous systems. Thus we consider the system

$$\dot{x}(t) = (\mathcal{T}_{11}x)(t) + (\mathcal{T}_{12}w)(t) + f(t), \quad t \in [0, T], \tag{33}$$

$$\dot{w}(t) = (\mathcal{T}_{21}x) + (\mathcal{T}_{22}w) + g(t), \quad t \in [0, T], \tag{34}$$

$$x(0) = x(T), \quad w(0) = w(T), \tag{35}$$

with positive operators $\mathcal{T}_{ij} : C \to L^1$ such that

$$\begin{aligned}
(\mathcal{T}_{11}\mathbf{1})(t) &\equiv \mathfrak{A}^+ p(t), \quad (\mathcal{T}_{12}\mathbf{1})(t) \equiv \mathfrak{C}^+ p(t), \quad t \in [0, T], \\
(\mathcal{T}_{21}\mathbf{1})(t) &\equiv \widehat{\mathfrak{D}}^+ q(t), \quad (\mathcal{T}_{22}\mathbf{1})(t) \equiv \widehat{\mathfrak{B}}^+ q(t), \quad t \in [0, T],
\end{aligned} \tag{36}$$

where $q(t) = q_i\, n/T$ for $t \in e_i$, $i = 1, \dots, n$.

By all the above arguments, it suffices to prove the statement only for problems (33)–(35) with continuous time.

Let the homogeneous problems (33)–(35) (for which $f \equiv 0$, $g \equiv 0$) have a non-trivial solution (x, w) and

$$\max_{t \in [0,T]} x(t) = x(t_1) \equiv x_1, \quad \min_{t \in [0,T]} x(t) = x(t_2) \equiv x_2,$$

$$\max_{t \in [0,T]} w(t) = w(\theta_1) \equiv w_1, \quad \min_{t \in [0,T]} w(t) = w(\theta_2) \equiv w_2.$$

Suppose $t_1 \leq t_2$, $\theta_1 < \theta_2$ (if it is needed, we renumber x_i and w_i, $i = 1, 2$).

By Lemma 3.3 [20] (p. 216) (or Lemma 1 [21]) we have

$$\begin{aligned}
\dot{x}(t) &= p_1(t)\mathfrak{A}x_1 + (p(t) - p_1(t))\mathfrak{A}x_2 + p_2(t)\mathfrak{C}w_1 + (p(t) - p_2(t))\mathfrak{C}w_2, \quad t \in [0, T], \\
\dot{w}(t) &= q_1(t)\widehat{\mathfrak{D}}x_1 + (q(t) - q_1(t))\widehat{\mathfrak{D}}x_2 + q_2(t)\widehat{\mathfrak{B}}w_1 + (q(t) - q_2(t))\widehat{\mathfrak{B}}w_2, \quad t \in [0, T], \\
x(0) &= x(T), \quad w(0) = w(T)
\end{aligned} \tag{37}$$

for some functions $p_i, q_i \in L^1$, $i = 1, 2$, such that $p_i(t) \in [0, p(t)]$, $q_i(t) \in [0, p(t)]$, $i = 1, 2$, $t \in [0, T]$.

From (37) it follows that the numbers x_1, x_2, w_1, w_2 satisfy the following algebraic system:

$$x_1 \int_0^T p_1(s)ds\mathfrak{A} + x_2 \int_0^T (p(s) - p_1(s))ds\mathfrak{A} + w_1 \int_0^T p_2(s)ds\mathfrak{C} + w_2 \int_0^T (p(s) - p_2(s))ds\mathfrak{C} = 0,$$

$$x_1 \int_0^T q_1(s)ds\widehat{\mathfrak{D}} + x_2 \int_0^T (q(s) - q_1(s))ds\widehat{\mathfrak{D}} + w_1 \int_0^T q_2(s)ds\widehat{\mathfrak{B}} + w_2 \int_0^T (q(s) - q_2(s))ds\widehat{\mathfrak{D}} = 0,$$

$$x_1 = x(0) + x_1 \int_0^{t_1} p_1(s)ds\mathfrak{A} + x_2 \int_0^{t_1} (p(s) - p_1(s))ds\mathfrak{A} + w_1 \int_0^{t_1} p_2(s)ds\mathfrak{C} + w_2 \int_0^{t_1} (p(s) - p_2(s))ds\mathfrak{C},$$

$$x_2 = x(0) + x_1 \int_0^{t_2} p_1(s)ds\mathfrak{A} + x_2 \int_0^{t_2} (p(s) - p_1(s))ds\mathfrak{A} + w_1 \int_0^{t_2} p_2(s)ds\mathfrak{C} + w_2 \int_0^{t_2} (p(s) - p_2(s))ds\mathfrak{C}, \qquad (38)$$

$$w_1 = w(0) + x_1 \int_0^{\theta_1} p_1(s)ds\mathfrak{A} + x_2 \int_0^{\theta_1} (p(s) - p_1(s))ds\mathfrak{A} + w_1 \int_0^{\theta_1} p_2(s)ds\mathfrak{C} + w_2 \int_0^{\theta_1} (p(s) - p_2(s))ds\mathfrak{C},$$

$$w_2 = w(0) + x_1 \int_0^{\theta_2} p_1(s)ds\mathfrak{A} + x_2 \int_0^{\theta_2} (p(s) - p_1(s))ds\mathfrak{A} + w_1 \int_0^{\theta_2} p_2(s)ds\mathfrak{C} + w_2 \int_0^{\theta_2} (p(s) - p_2(s))ds\mathfrak{C}.$$

We can exclude variable $x(0)$ and $w(0)$ and obtain a system of four equations in four unknowns:

$$x_1 \int_0^T p_1(s)ds\mathfrak{A} + x_2 \int_0^T (p(s) - p_1(s))ds\mathfrak{A} + w_1 \int_0^T p_2(s)ds\mathfrak{C} + w_2 \int_0^T (p(s) - p_2(s))ds\mathfrak{C} = 0,$$

$$x_1 \int_0^T q_1(s)ds\widehat{\mathfrak{D}} + x_2 \int_0^T (q(s) - q_1(s))ds\widehat{\mathfrak{D}} + w_1 \int_0^T q_2(s)ds\widehat{\mathfrak{B}} + w_2 \int_0^T (q(s) - q_2(s))ds\widehat{\mathfrak{B}} = 0,$$

$$x_1\left(1 + \int_{t_1}^{t_2} p_1(s)ds\mathfrak{A}\right) + x_2\left(\int_{t_1}^{t_2} (p(s) - p_1(s))ds\mathfrak{A} - 1\right) + w_1 \int_{t_1}^{t_2} p_2(s)ds\mathfrak{C} + w_2 \int_{t_1}^{t_2} (p(s) - p_2(s))ds\mathfrak{C} = 0,$$

$$x_1 \int_{\theta_1}^{\theta_2} q_1(s)ds\widehat{\mathfrak{D}} + x_2 \int_{\theta_1}^{\theta_2} (q(s) - q_1(s))ds\widehat{\mathfrak{D}} + w_1\left(1 + \int_{\theta_1}^{\theta_2} q_2(s)ds\widehat{\mathfrak{B}}\right) + w_2\left(\int_{\theta_1}^{\theta_2} (q(s) - q_2(s))ds\widehat{\mathfrak{B}} - 1\right) = 0.$$

Further we use the notation: $e = [t_1, t_2]$, $\delta = [\theta_1, \theta_2]$, $e' = [0, T] \setminus [t_1, t_2]$, $\delta' = [0, T] \setminus [\theta_1, \theta_2]$. Obviously, system (38) has a non-trivial solution if and only if the following determinant Δ vanishes:

$$\Delta \equiv \begin{vmatrix} \int_0^T p_1(s)ds\mathfrak{A} & (1 - \int_0^T p_1(s)ds)\mathfrak{A} & \int_0^T p_2(s)ds\mathfrak{C} & (1 - \int_0^T p_2(s)ds)\mathfrak{C} \\ \int_0^T q_1(s)ds\widehat{\mathfrak{D}} & (1 - \int_0^T q_1(s)ds)\widehat{\mathfrak{D}} & \int_0^T q_2(s)ds\widehat{\mathfrak{B}} & (1 - \int_0^T q_2(s)ds)\widehat{\mathfrak{B}} \\ 1 + \int_e p_1(s)ds\mathfrak{A} & -1 + \int_e (p(s) - p_1(s))ds\mathfrak{A} & \int_e p_2(s)ds\mathfrak{C} & \int_e (p(s) - p_2(s))ds\mathfrak{C} \\ \int_\delta q_1(s)ds\widehat{\mathfrak{D}} & \int_\delta (q(s) - q_1(s))ds\widehat{\mathfrak{D}} & 1 + \int_\delta q_2(s)ds\widehat{\mathfrak{B}} & -1 + \int_\delta (q(s) - q_2(s))ds\widehat{\mathfrak{B}} \end{vmatrix} =$$

$$\begin{vmatrix} -1 + \int_{e'} p_1(s)ds\mathfrak{A} & \int_{e'} p(s)ds)\mathfrak{A} & \int_{e'} p_2(s)ds\mathfrak{C} & \int_{e'} p(s)ds\mathfrak{C} \\ \int_{\delta'} q_1(s)ds\widehat{\mathfrak{D}} & \int_{\delta'} q(s)ds\widehat{\mathfrak{D}} & -1 + \int_{\delta'} q_2(s)ds\widehat{\mathfrak{B}} & \int_{\delta'} q(s)ds\widehat{\mathfrak{B}} \\ 1 + \int_e p_1(s)ds\mathfrak{A} & \int_e p(s)ds\mathfrak{A} & \int_e p_2(s)ds\mathfrak{C} & \int_e p(s)ds\mathfrak{C} \\ \int_\delta q_1(s)ds\widehat{\mathfrak{D}} & \int_\delta q(s)ds\widehat{\mathfrak{D}} & 1 + \int_\delta q_2(s)ds\widehat{\mathfrak{B}} & \int_\delta q(s)ds\widehat{\mathfrak{B}} \end{vmatrix} =$$

$$
\begin{vmatrix}
-1+z_p\mathfrak{A} & (1-P_e)\mathfrak{A} & w_p\mathfrak{C} & (1-P_e)\mathfrak{C} \\
z_p\widehat{\mathfrak{D}} & (1-Q_\delta)\widehat{\mathfrak{D}} & -1+w_q\widehat{\mathfrak{B}} & (1-Q_\delta)\widehat{\mathfrak{B}} \\
1+x_p\mathfrak{A} & P_e\mathfrak{A} & y_p\mathfrak{C} & P_e\mathfrak{C} \\
x_q\widehat{\mathfrak{D}} & Q_\delta\widehat{\mathfrak{D}} & 1+y_q\widehat{\mathfrak{B}} & Q_\delta\widehat{\mathfrak{B}}
\end{vmatrix} = 0,
$$

where

$$
P_e \equiv \int_e p(s)ds, \quad Q_\delta \equiv \int_\delta q(s)ds, \quad z_p \equiv \int_{e'} p_1(s)ds, \quad w_p \equiv \int_{e'} p_2(s)ds, \quad x_p \equiv \int_e p_1(s)ds,
$$

$$
y_p \equiv \int_e p_2(s)ds, \quad z_q \equiv \int_{\delta'} q_1(s)ds, \quad w_q \equiv \int_{\delta'} q_2(s)ds, \quad x_q \equiv \int_\delta q_1(s)ds, \quad y_q \equiv \int_\delta q_2(s)ds.
$$

Thus, the variables x_p, y_p may take any value from the interval $[0, P_e]$; z_p, w_p from the interval $[0, 1 - P_e]$; z_q, w_q from $[0, Q_\delta]$; y_q, x_q from $[0, 1 - Q_\delta]$ independently. Here P_e and Q_δ take all values from the interval $[0, 1]$ also independently.

Now to prove the assertion of the theorem we have to show that $\Delta \neq 0$ for all admissible values of parameters x_p, y_p, x_q, y_q, w_p, z_p, w_q, z_q, P_e, Q_δ. Then the homogeneous problem has no non-trivial solutions. Therefore, by the Fredholm property (see Section 2.2) problem (33)–(35) is uniquely solvable.

For admissible values $x_p = y_p = x_q = y_q = w_p = z_p = w_q = z_q = P_e = Q_\delta = 0$, we have $\Delta = \mathfrak{C}\widehat{\mathfrak{D}} - \mathfrak{A}\widehat{\mathfrak{B}}$. Therefore, the condition

$$
\mathfrak{C}\widehat{\mathfrak{D}} \neq \mathfrak{A}\widehat{\mathfrak{B}} \tag{39}
$$

is the first necessary condition of the unique solvability.

Since $\Delta \neq 0$ and Δ is continuous in all its parameters, we see that Δ must preserve its sign for all admissible values of parameters. Since Δ is linear in every variable x_p, y_p, x_q, y_q, w_p, z_p, w_q, z_q, we have to consider Δ for all these variables at the ends of admissible intervals. There are 8 such variables, therefore, we have to consider 256 cases. However, all these cases can be reduced to only 16: the ratio $\frac{\Delta}{\mathfrak{A}\widehat{\mathfrak{B}} - \mathfrak{C}\widehat{\mathfrak{D}}}$ may take the following values

$$
-1, \; \pm A - 1, \; \pm B - 1, \; (1 - \pm A)(\pm B - 1), \; \pm Y - 1, \; \pm Y - 1, \; \pm Y \pm A - 1, \; \pm Y \pm B - 1,
$$

$$
-X - Y - A + B - 1, \; -X - Y - A - B - 1, \; -X + Y + A + B - 1, \; -X + Y - A - B - 1,
$$

$$
X - Y + A - B - 1, \; X - Y - A + B - 1, \; X + Y + A - B - 1, \; X + Y - A + B - 1,
$$

where

$$
X = \mathfrak{A}\widehat{\mathfrak{B}}r(1-r)s(1-s) \in [0, \mathfrak{A}\widehat{\mathfrak{B}}/16], \quad Y = \mathfrak{C}\widehat{\mathfrak{D}}r(1-r)s(1-s) \in [0, \mathfrak{C}\widehat{\mathfrak{D}}/16],
$$

$$
A = \mathfrak{A}r(1-r) \in [0, \mathfrak{A}/4], \quad B = \widehat{\mathfrak{B}}s(1-s) \in [0, \widehat{\mathfrak{B}}/4],
$$

$$
r = P_e \in [0, 1], \quad s = Q_\delta \in [0, 1].
$$

Therefore, if and only if the problem is uniquely solvable, all these values are negative. From inequalities $A < 1$, $B < 1$, we obtain that $\mathfrak{A} < 4$ and $\widehat{\mathfrak{B}} < 4$. As to the rest inequalities, it is enough to verify the inequalities

$$
X + Y - A + B - 1 < 0, \quad X + Y + A - B - 1 < 0, \quad -X + Y + A + B - 1 < 0.
$$

These inequalities are fulfilled for all $r, s \in [0, 1]$ if and only if

$$
\mathfrak{C}\widehat{\mathfrak{D}} \leq (4 - \mathfrak{A})(4 - \widehat{\mathfrak{B}}).
$$

Therefore, the assertion is proved both for problems (7)–(9) and for problems (33)–(35). □

Now we formulate solvability conditions for a boundary value problem with arbitrary operators $\mathcal{A}^{+/-}, \mathcal{C}^{+/-}, \mathcal{D}^{+/-}, \mathcal{B}^{+/-}$ satisfying the following equalities for given non-negative constants $\mathfrak{A}^{+/-}, \mathfrak{C}^{+/-}, \mathfrak{D}^{+/-}, \mathfrak{B}^{+/-}$:

$$(\mathcal{A}^+ \mathbf{1})(t) = \mathfrak{A}^+, \ t \in [0, T]; \quad (\mathcal{A}^- \mathbf{1})(t) = \mathfrak{A}^-, \ t \in [0, T];$$

$$(\mathcal{B}^+ \mathbb{1})_i = \sum_{j=0}^n b_{ij}^+ = \mathfrak{B}^+, \ i = 1, \ldots, n; \quad (\mathcal{B}^- \mathbb{1})_i = \sum_{j=0}^n b_{ij}^- = \mathfrak{B}^-, \ i = 1, \ldots, n;$$

$$(\mathcal{C}^+ \mathbb{1}) = \sum_{i=1}^n (c_i(t))^+ = \mathfrak{C}^+, \ t \in [0, T]; \quad (\mathcal{C}^- \mathbb{1}) = \sum_{i=1}^n (c_i(t))^- = \mathfrak{C}^-, \ t \in [0, T]; \tag{40}$$

$$\mathcal{D}_i^+ \mathbf{1} = \mathfrak{D}^+, \ i = 1, \ldots, n; \quad \mathcal{D}_i^- \mathbf{1} = \mathfrak{D}^-, \ i = 1, \ldots, n.$$

Denote

$$\widehat{\mathfrak{A}} \equiv T(\mathfrak{A}^+ + \mathfrak{A}^-), \quad \widehat{\mathfrak{B}} \equiv T(\mathfrak{B}^+ + \mathfrak{B}^-), \quad \mathfrak{A} \equiv \mathfrak{A}^+ - \mathfrak{A}^-, \quad \mathfrak{B} \equiv \mathfrak{B}^+ - \mathfrak{B}^-,$$

$$\widehat{\mathfrak{C}} \equiv T/n(\mathfrak{C}^+ + \mathfrak{C}^-), \quad \widehat{\mathfrak{D}} \equiv T/n(\mathfrak{D}^+ + \mathfrak{D}^-), \quad \mathfrak{C} \equiv \mathfrak{C}^+ - \mathfrak{C}^-, \quad \mathfrak{D} \equiv \mathfrak{D}^+ - \mathfrak{D}^-. \tag{41}$$

Theorem 8. *Let conditions* (40) *be fulfilled. If*

$$\widehat{\mathfrak{A}} < 4, \quad \widehat{\mathfrak{B}} < 4, \quad \mathfrak{C}\mathfrak{D} \neq \mathfrak{A}\mathfrak{B}, \quad \widehat{\mathfrak{C}}\widehat{\mathfrak{D}} < (4 - \widehat{\mathfrak{A}})(4 - \widehat{\mathfrak{B}}), \tag{42}$$

then periodic problems (10)–(12) *is uniquely solvable.*

The scheme of the proof is the same as in Theorem 7, where the factors p, q are chosen constant. We omit the complete proof due to its cumbersomeness caused by the need to consider both the positive and negative parts of the operators. It is important that the form of necessary and sufficient conditions turns out to be similar to conditions for the case of positive operators.

3.3. A Nonlinear Case

It Theorems 4–6, we obtained conditions for uniqueness and existence of periodic solutions in the linear case. Here we find non-existence conditions for the nonlinear system (see other results on similar nonlinear problems in References [23,24])

$$\dot{x}(t) = \varepsilon_{11}(\mathcal{F}_{11}x)(t) + \varepsilon_{12}(\mathcal{F}_{12}z)(t), \quad t \in [0, T], \tag{43}$$

$$z_i - z_{i-1} = \varepsilon_{21}(\mathcal{F}_{21}x)_i + \varepsilon_{22}(\mathcal{F}_{22}z)_i, \quad i = 1, \ldots, n, \tag{44}$$

$$x(0) = x(T), \quad z_0 = z_n, \tag{45}$$

where $\varepsilon_{ij} \in \{-1, 1\}, i, j = 1, 2$.

We impose on the operators $\mathcal{F}_{ij}$ some kind of the Lipschitz conditions:

- there exists a constant $\mathfrak{A}^+$ such that for every $x \in C$ the following inequality holds

$$\operatorname*{ess\,sup}_{t \in [0,T]}(\mathcal{F}_{11}x)(t) - \operatorname*{ess\,inf}_{t \in [0,T]}(\mathcal{F}_{11}x)(t) \leq \mathfrak{A}^+ \left(\max_{t \in [0,T]} x(t) - \min_{t \in [0,T]} x(t) \right); \tag{46}$$

- there exists a constant $\mathfrak{D}^+$ such that for every $x \in C$ the following inequality holds

$$\max_{i=1,\dots,n} (\mathcal{F}_{21}x)_i - \min_{i=1,\dots,n} (\mathcal{F}_{21}x)_i \leq \mathfrak{D}^+ \left(\max_{t \in [0,T]} x(t) - \min_{t \in [0,T]} x(t) \right); \tag{47}$$

- there exists a constant $\mathfrak{C}^+$ such that for every $z \in R^{n+1}$ the following inequality holds

$$\operatorname*{ess\,sup}_{t \in [0,T]} (\mathcal{F}_{12}z)(t) - \operatorname*{ess\,inf}_{t \in [0,T]} (\mathcal{F}_{12}z)(t) \leq \mathfrak{C}^+ \left(\max_{j=0,\dots,n} z_i - \min_{j=0,\dots,n} z_i \right); \tag{48}$$

- there exists a constant $\mathfrak{B}^+$ such that for every $z \in R^{n+1}$ the following inequality holds

$$\max_{i=1,\dots,n} (\mathcal{F}_{22}z)_i - \min_{i=1,\dots,n} (\mathcal{F}_{22}z)_i \leq \mathfrak{B}^+ \left(\max_{j=0,\dots,n} z_i - \min_{j=0,\dots,n} z_i \right). \tag{49}$$

It should be noted that under some natural conditions the product of the Nemytskii operator [7] (p. viii) and the inner superposition operator [7] (p. 93) satisfies these conditions.

The following theorem describes conditions under which non-constant business cycles become impossible whenever the nonlinear functional operators of system have point-wise restrictions.

Denote $\widehat{\mathfrak{A}}^+ = T\mathfrak{A}^+$, $\widehat{\mathfrak{B}} = T\mathfrak{B}^+$, $\widehat{\mathfrak{C}}^+ = T/n\,\mathfrak{C}^+$, $\widehat{\mathfrak{D}} = T/n\,\mathfrak{D}^+$.

Theorem 9. *If the inequalities*

$$\widehat{\mathfrak{A}}^+ < 4, \quad \widehat{\mathfrak{B}}^+ < 4, \quad \mathfrak{A}^+\mathfrak{B}^+ \neq \widehat{\mathfrak{C}}^+\widehat{\mathfrak{D}}^+ < \left(4 - \widehat{\mathfrak{A}}^+ \right)\left(4 - \widehat{\mathfrak{B}}^+ \right) \tag{50}$$

are fulfilled, then problems (43)–(45) under conditions (46)–(49) has no non-constant solutions.

Proof. From conditions (46)–(49), it is easy to see that if a pair (x, z) is a solution of periodic problems (43)–(45), then this pair is a solution of some linear non-homogeneous problem

$$\dot{x}(t) = \varepsilon_{11}(\mathcal{A}^+x)(t) + \varepsilon_{12}(\mathcal{C}^+z)(t) + c, \quad t \in [0, T],$$

$$z_i - z_{i-1} = \varepsilon_{21}(\mathcal{D}^+x)_i + \varepsilon_{22}(\mathcal{B}^+z)_i + d, \quad i = 1, \dots, n,$$

$$x(0) = x(T), \quad z_0 = z_n,$$

for some constants $c, d \in R$, where positive linear operators $\mathcal{A}^+$, $\mathcal{C}^+$, $\mathcal{D}^+$, $\mathcal{B}^+$ satisfy conditions (40). From (50) and Theorem 8, it follows that this problem has a unique solution. It is clear in this case, that both components of the solution x and z are constants. $\square$

4. Control Problems

Let us recall the general formulation of the control problem as applied to the hybrid system under control

$$\mathcal{L}y = Fu + q \tag{51}$$

with an operator $\mathcal{L} : AC^n \times FD^\nu \to L^n \times FD_1^\nu$ defined by the equality (5) and control $u : [0, T] \to R^r$ with components from the Hilbert space $L_2 = L_2[0, T]$ of square summable functions $v : [0, T] \to R^1$ equipped with the inner product $\langle v_1, v_2 \rangle = \int_0^T v_1(s)v_2(s)\,ds$ (as usually, $L_2^r = \underbrace{L_2 \times L_2 \times \cdots \times L_2}_{r \text{ times}}$); $F :$

$L_2^r \to L^n \times FD_1^\nu$ is a linear bounded Volterra [7], ([9] p. 106) operator, $q = \text{col}(f, g)$. The initial state of the system (51) is assumed to be fixed:

$$y(0) = \text{col}(x(0), z(0)) = \text{col}(\alpha, \delta). \tag{52}$$

The aim of control is prescribed by the equality

$$\ell y = \ell(x, z) = \beta \in R^N, \tag{53}$$

where $\ell : AC^n \times FD^\nu \to R^N$ is a given linear bounded vector-functional.

We say that the control problems (51)–(53) is solvable by a control u if the trajectory $y_u = \text{col}(x_u, z_u)$, generated by u, brings the prescribed value β to the on-target vector-functional ℓ.

4.1. General Results

Within this subsection, we consider the hybrid system (51) with operators $\mathcal{T}_{ij}$ defined as follows.

$(\mathcal{T}_{11}) \qquad\qquad \mathcal{T}_{11} : AC^n \to L^n, \quad (\mathcal{T}_{11}x)(t) = \int_0^t K^1(t, s)\dot{x}(s)\,ds + A^1(t)x(0), \ t \in [0, T].$

Here the elements $k_{ij}^1(t, s)$ of the kernel $K(t, s)$ are measurable on the set $0 \le s \le t \le T$ and such that $|k_{ij}^1(t, s)| \le \kappa(t)$, $i, j = 1, \dots, n$, $\kappa(\cdot)$ is summable on $[0, T]$, the $(n \times n)$-matrix A^1 has elements summable on $[0, T]$.

$(\mathcal{T}_{12}) \qquad\qquad \mathcal{T}_{12} : FD^\nu \to L^n, \quad (\mathcal{T}_{12}z)(t) = \sum_{\{j : t_j < t\}} B_j^1(t)z(t_j), \ t \in [0, T],$

where elements of matrices B_j^1, $j = 0, \dots, \mu$, are summable on $[0, T]$. As it usually is, here and in the sequel $\sum_{i=k}^l F_i = 0$ for any F_i if $l < k$.

$$\mathcal{T}_{21} : AC^n \to FD^\nu, \quad (\mathcal{T}_{21}x)(t_i) = \int_0^{t_{i-1}} K_i^2(s)\dot{x}(s)ds + A_i^2 x(0), \quad i = 1, \dots, \mu,$$

with measurable and essentially bounded on $[0, T]$ elements of matrices K_i^2 and constant $(\nu \times n)$-matrices $A_i^2, i = 1, \dots, \mu$.

$$\mathcal{T}_{22} : FD^\nu \to FD^\nu; \quad (\mathcal{T}_{22}z)(t_i) = \sum_{j=0}^{i-1} B_{ij}^2 z(t_j), \quad i = 1, \dots, \mu,$$

with constant $(\nu \times \nu)$-matrices B_{ij}^2.

This system is a special case of the general continuous-discrete system considered in detail in Reference [25]. Theorem 1 [25] gives the representation of the general solution in the form

$$\begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{Y} \begin{pmatrix} x(0) \\ z(0) \end{pmatrix} + \mathcal{C} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{54}$$

where $z = \text{col}(z(t_1), \dots, z(t_\mu))$, $g = \text{col}(g(t_1), \dots, g(t_\mu))$,

$$\mathcal{Y} = \begin{pmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{21} & \mathcal{Y}_{22} \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix}$$

are the fundamental matrix and the Cauchy operator respectively. The block-components $\mathcal{Y}_{ij}, \mathcal{C}_{ij}, i, j = 1, 2,$ are operators acting as follows:

$$\mathcal{Y}_{11} : R^n \to AC^n; \quad \mathcal{Y}_{12} : R^\nu \to AC^n; \quad \mathcal{Y}_{21} : R^n \to R^{\nu\mu}; \quad \mathcal{Y}_{22} : R^\nu \to R^{\nu\mu};$$

$$\mathcal{C}_{11} : L^n \to AC^n; \quad \mathcal{C}_{12} : R^{\nu\mu} \to AC^n; \quad \mathcal{C}_{21} : L^n \to R^{\nu\mu}; \quad \mathcal{C}_{22} : R^{\nu\mu} \to R^{\nu\mu}.$$

It should be noted that, with respect to the continuous time component $x(t)$, we restrict ourselves to the case $x \in AC^n$ and so ignore the impulsive component of the solution [25].

The general results on the solvability of (51)–(53) from Reference [8] are based on the following approach. The representation of the linear bounded vector-functional $\ell : AC^n \times FD^\nu \to R^N$ has the form

$$\ell(x, z) = \Psi x(0) + \int_0^T \Phi(s)\dot{x}(s)\, ds + \sum_{i=0}^\mu \Gamma_i z(t_i), \tag{55}$$

where $(N \times n)$-matrix Ψ and $(N \times \nu)$-matrices $\Gamma_i, i = 0, \ldots, \mu$ are constant, and $(N \times n)$-matrix $\Phi(s)$ have measurable and essentially bounded elements. The general form of ℓ covers many various special cases used while studying applied problems (see, for instance, Reference [26]). Here we restrict ourselves to the following example of (53):

$$\int_0^T \exp(-\lambda_1 t)\, F_1\, x(t)\, dt + \sum_{i=0}^\mu \exp(-\lambda_2 t_i)\, F_2 z(t_i) = \beta.$$

For this case we have

$$\Psi = \frac{1}{\lambda_1}(1 - \exp(-\lambda_1 T))F_1; \; \Phi(s) = \frac{1}{\lambda_1}(\exp(-\lambda_1 s) - \exp(-\lambda_1 T))F_1; \; \Gamma_i = \exp(-\lambda_2 t_i)F_2, \; i = 0, \ldots, \mu.$$

Using the representation (55) and the description of the set of all trajectories to the system under control

$$y = \begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{Y} \begin{pmatrix} x(0) \\ z(0) \end{pmatrix} + \mathcal{C} \begin{pmatrix} f \\ g \end{pmatrix} + \mathcal{C}Fu, \tag{56}$$

we can express the on-target conditions (53) in the form that is explicit with respect to the control $u \in L_2^r$:

$$\ell(x, z) = \ell(\mathcal{Y}\mathrm{col}(\alpha, \delta) + \mathcal{C}q) + \ell\mathcal{C}Fu = \beta, \tag{57}$$

or, denoting

$$\beta_0 = \ell(\mathcal{Y}\mathrm{col}(\alpha, \delta) + \mathcal{C}q),$$

as the system

$$\ell\mathcal{C}Fu = \beta_1 = \beta - \beta_0 \tag{58}$$

with respect to u. The solvability of (58) means the controlability of the hybrid system (51) with respect to the on-target vector-functional ℓ. In the case we consider, (58) may be reduced to the form

$$\int_0^T M(s)u(s)\, ds = \beta_1, \tag{59}$$

where $(N \times r)$-matrix $M(\cdot)$ is called the moment matrix to the problems (51)–(53). The justification of all transformations from (58) to (59) with an explicit form of $M(\cdot)$ requires the deep study of the Cauchy

operator properties which is done in Reference [25]. The system (59) opens a way to construct controls that provide the solutions to (51)–(53). In particular, a control can be found in the form

$$u(t) = M'(t)d$$

with a constant N-vector d ($(\cdot)'$ stands for transposition). In such a case, we obtain the linear algebraic system with respect to d:

$$Wd = \beta_1,$$

where

$$W = \int_0^T M(s)M'(s)\,ds.$$

Thus we obtain the condition

$$\det W \neq 0$$

for the solvability of (51)–(53). This condition is well known for many special classes of dynamic models with continuous and discrete times as applied to the very specific kind of the on-target vector-functional ℓ when $\ell(x,z) = \mathrm{col}(x(T), z(T))$. General results on the controlability of hybrid models with sufficient conditions and description of the programming control bringing the prescribed values β are presented in Reference [8]. It should be noted that possible constraints with respect to control, which are met in real world economic dynamics problems, are ignored in the formulation (51)–(53). Thus the results mentioned relate to the problems with unconstrained control. For a class of hybrid models, the case of constrained control where constraints are defined by a finite system of inequalities is considered in the next subsection.

4.2. Constrained Control of Hybrid Systems with Discrete Memory

In applied control problems with prescribed on-target indexes, a central place is occupied by constraints with respect to control. The rigidness of constraints impacts essentially onto the solvability of the problem that is the existence a control such that the trajectory generated by it brings the prescribed targeted values. Description of all values attainable under the action of controls with fixed constraints is one of the key problems in the frames of various control theory divisions (see, for instance, References [27–31]), including questions of the attainability set structure under various classes of constraints [32,33], their asymptotic [34] and statistical characteristics [35,36]. Therewith, as a rule, the attainability is understood with respect to the state of the system at a given instant. In control problems for economic systems, a more general point of view takes place when on-target indexes are given as integral characteristics and multipoint ones. Here we set the on-target indexes with the use of linear functionals having the general form that covers above mentioned cases. The system dynamics is described by the union of differential equations with delay and difference equations with discrete time. Such a description turns out to be urgent as applied to economic dynamics processes with interactions of variables having disparate kinds of changes, namely, continuous (say for production process) and discrete (say investment or financing). We pay the most attention to the external estimates of attainability sets with corresponding relationships and algorithms. The results are based on the general theory of functional differential equations [7] in terms of the solvability conditions and the representation of solutions, as well as on some of our previous results [25,26,37,38]. Here we recall some results on estimating attainability sets and propose estimates of switch points number to programming control for systems with discrete memory.

We consider the economic-mathematical model of interaction of the production subsystem

$$\dot{x}(t) = \sum_{i:t_i < t} A_i(t)x(t_i) + \sum_{i:t_i < t} B_i(t)z(t_i) + \int_0^t F(t,s)u(s)\,ds + f(t),\ t \in [0,T], \tag{60}$$

and the financial subsystem

$$z(t_i) = \sum_{j<i} D_j x(t_j) + \sum_{j<i} H_j z(t_j) + \int_0^{t_i} G_i(s)v(s)\, ds + g(t_i), \; i = 1, ..., \mu \tag{61}$$

with summable $(n \times n)$-matrices $A_i(t)$, $(n \times \nu)$-matrices $B_i(t)$ and constant $(\nu \times n)$-matrices D_j, $(\nu \times \nu)$-matrices H_j.

For definiteness, we consider $x = \text{col}(x_1, \ldots, x_n)$ in (60) as indexes (state variables) of functioning of a multiproduct productive system that change in continuous time $t \in [0, T]$. The rate of change depends on the productive accumulation at fixed points in time, $t_i, i = 0, \ldots, \mu$, $t_i > t_{i-1}, t_0 = 0, t_\mu = T$, with a given effectiveness (by elements of $A_i(t)$). In addition, there are used investments $z(t_i)$, $z = \text{col}(z_1, \ldots, z_\nu)$ governed by (61). The integral term in (60) simulates the direct control $u(t), u : [0, T] \to R^{r_1}$ of the indexes dynamics of $x(t)$ with an effectiveness defined by the kernel $F(t, s)$. The right-hand side of (61) includes at each t_i the previous investment values $z(t_j), j < i$, the productive accumulation, as a part of $x(t_j), j < i$, and the control $v(t), v : [0, T] \to R^{r_2}$ as a financial flow density. Therewith the integral term takes into account financial resources accumulated at t_i. The effectiveness of the use to all above factors is given by the matrix coefficients H_j, D_j, G_j. The functions $f(t)$ and $g(t_i)$ can be interpreted as external actions or disturbances, say, unforeseen lost or modeling errors. It should be noted that the specific kind of delay in (60) is of considerable current use as applied to dynamic economic models [39]. As in the general case, the initial state of (60) and (61) is assumed to be fixed:

$$x(0) = \alpha, \; z(0) = \delta. \tag{62}$$

Using (55), we define the aim of control by the equality

$$\ell(x, z) = \beta, \quad \beta \in R^N, \tag{63}$$

where β is a given vector of on-target values.

The problem of getting the values prescribed by (63) is considered with the constraints concerning control:

$$\Lambda \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \le \gamma \in R^{N_1}, \; t \in [0, T], \tag{64}$$

where Λ is a given constant $(N_1 \times (r_1 + r_2))$-matrix. The set of solutions to the linear inequalities system $\Lambda \begin{pmatrix} u \\ v \end{pmatrix} \le \gamma$ is denoted by $\mathcal{V}$ and assumed to be nonempty and bounded.

As is noted in the previous subsection, we reduce the problems (61)–(64) to the moment problem

$$\int_0^T M(s)w(s)\, ds = \beta_1, \tag{65}$$

$$\Lambda w \le \gamma, \tag{66}$$

where $w(s) = \text{col}(u(s), v(s))$, with the polyhedral constraints (66).

Having in mind that the Cauchy operator $\mathcal{C}$ to (61) and (62) can be found in the explicit form [38], we give here the representation of the moment matrix $M(s)$ in the terms of components

$\mathcal{C}_{ij}$, $i, j = 1, 2$. To do this, we shall use the representation of the solution to (61) and (62) in the case $x(0) = 0$, $z(0) = 0$, $f = 0$, $g = 0$:

$$x(t) = \int_0^t \mathcal{C}_{11}(t,s) \int_0^s F(s,\tau)u(\tau)\,d\tau\,ds + \sum_{i:i\geq 1;t_i\leq t} \mathcal{C}_{12}^i(t) \int_0^{t_i} G_i(s)v(s)\,ds, \ t \in [0,T],$$

$$z(t_i) = \sum_{j\leq i} \mathcal{C}_{21}^i \int_0^{t_j} F(t_i,s)u(s)\,ds + \sum_{j\leq i} \mathcal{C}_{22}^i \int_0^{t_j} G_j(s)v(s)\,ds, \ i = 1,\ldots,\mu. \tag{67}$$

Next we obtain the expression for $\dot{x}$, taking into account the properties of the kernel $\mathcal{C}_{11}(t,s)$:

$$\dot{x}(t) = \int_0^t \frac{\partial}{\partial t}\mathcal{C}_{11}(t,s) \int_0^s F(s,\tau)u(\tau)\,d\tau\,ds + \int_0^t F(t,s)u(s)\,ds$$

$$+ \sum_{i:i\geq 1;t_i\leq t} \frac{d}{dt}\mathcal{C}_{12}^i(t) \int_0^{t_i} G_i(s)v(s)\,ds, \ t \in [0,T], \tag{68}$$

or, after some evident transformations,

$$\dot{x}(t) = \int_0^t \theta_1(t,s)\,u(s)\,ds + \int_0^t \theta_2(t,s)\,v(s)\,ds, \tag{69}$$

where

$$\theta_1(t,s) = F(t,s) + \int_s^t \frac{\partial}{\partial t}\mathcal{C}_{11}(t,\tau)F(\tau,s)\,d\tau,$$

$$\theta_2(t,s) = \sum_{i:i\geq 1;t_i\leq t} \frac{d}{dt}\mathcal{C}_{12}^i(t)\,\chi_{[0,t_i]}(s)\,G_i(s).$$

Finally, after substitution of the right-hand sides of (67) and (69) into the representation of $\ell(x,z)$ we arrived at the representation of the matrices $M_1(s)$, $M_2(s)$, that form $M(s) = (M_1(s), M_2(s),)$:

$$M_1(s) = \int_s^T \Phi(\tau)\theta_1(\tau,s)\,d\tau + \sum_{i=1}^{\mu} \sum_{j\leq i} \Gamma_i \mathcal{C}_{21}^j \chi_{[0,t_j]}(s)F(t_j,s),$$

$$M_2(s) = \int_s^T \Phi(\tau)\theta_2(\tau,s)\,d\tau + \sum_{i=1}^{\mu} \sum_{j\leq i} \Gamma_i \mathcal{C}_{22}^j \chi_{[0,t_j]}(s)G_j(s).$$

Let us recall the results on estimates of the ℓ-attainability sets from Reference [37].

For any $\lambda \in R^N$, define $w(t,\lambda)$ by the equality

$$w(t,\lambda) = \operatorname{argmax}(\lambda' \cdot M(t) \cdot w : w \in \mathcal{W}). \tag{70}$$

In the case that the extremal value in (70) is reached at a number m of angle points $w_i(t,\lambda)$ of the polyhedron $\mathcal{W}$, we understand $w(t,\lambda)$ as $\frac{1}{m}\sum_{i=1}^{m} w_i(t,\lambda)$. By Theorem 1 [37], the set of all attainable β_1 in (65) and (66) is the set of all $\rho \in R^N$ such that the inequality

$$\lambda' \cdot \rho \leq \int_0^T \lambda' \cdot M(t) \cdot w(t,\lambda)\,dt \tag{71}$$

holds for any $\lambda \in R^N$.

Let us describe a way to obtain an internal estimate of the set of all attainable β_1 in (65) and (66). Fix a collection $\{\lambda_k \in R^N, \ k = 1, 2, ..., K\}$ and a collection $\{t_j^* \in [0, T], \ j = 1, 2, ..., J\}, t_j^* < t_{j+1}^*$. Define the control $w_k(t)$ by the equality

$$w_k(t) = \sum_{j=1}^{J+1} w(t_j^*, \lambda_k)\chi_{[t_{j-1}^*, t_j^*)}(t), \ t \in [0, T], \tag{72}$$

where $t_0^* = 0, t_{J+1}^* = T$. There takes place the following assertion.

Theorem 10. *Let the elements of the moment matrix $M(t)$ be piecewise continuous. Put*

$$\rho_k = \int_0^T M(t)w_k(t)\, dt, \ k = 1, 2, \dots, K.$$

Then the convex hull of $\{\rho_1, \dots, \rho_K\}$ is a subset of the set of all attainable β_1 in (65) and (66).

Proof. By construction of the controls $w_k(t), \ k = 1, 2, \dots, K$ those are admissible, and their convex hull is a subset to the set of all admissible controls. Any ρ_k is attainable by the control $w_k : [0, T] \to R^{r_1 + r_2}$. Let $\rho^* = \sum_{k=1}^K \omega_k \rho_k$, where $\omega_k \geq 0, \ \sum_{k=1}^K \omega_k = 1$. Then we have

$$\rho^* = \sum_{k=1}^K \omega_k \int_0^T M(t)w_k(t)\, dt = \int_0^T M(t) \sum_{k=1}^K \omega_k w_k(t)\, dt, \tag{73}$$

and, hence, ρ^* is attainable by the control

$$w^*(t) = \sum_{k=1}^K \omega_k w_k(t). \tag{74}$$

This completes the proof. $\square$

In applied problems, the question of a switch points number to the control bringing the prescribed on-target indexes is of special interest. A true number of switch points is defined by (70) and (71) and does depend on $\mathcal{W}$ and $M(t)$. An illustration is given in the next subsection.

4.3. An Example

Consider the system under control

$$\dot{x}(t) = 0.2x(0) + 0.1t\chi_{(1,4]}(t)x(1) + 0.1\cos(t)\chi_{(2,4]}(t)x(2) + 0.1t^2\chi_{(3,4]}(t)x(3)$$

$$-0.15z(0) - 0.1t\chi_{(1,4]}(t)z(1) + 0.1t^2\chi_{(2,4]}(t)z(2) - 0.15\chi_{(3,4]}(t)z(3) + 0.7\int_0^t u(s)\, ds, \ t \in [0, 4],$$

$$z(i) = 0.3x(0) + 0.2\chi_{(1,4]}(i)x(1) + 0.3\chi_{(2,4]}(i)x(2) + 0.2\chi_{(3,4]}(i)x(3)$$

$$+0.2z(0) + 0.2\chi_{(1,4]}(i)z(1) + 0.1\chi_{(2,4]}(i)z(i) + 0.4\chi_{(3,4]}(i)z(3) + \frac{1}{1+i}\int_0^i v(s)\, ds, \ i = 1, \dots, 4,$$

with the initial conditions

$$x(0) = 0, \ z(0) = 0,$$

the constraints with respect to control

$$0 \le u(t) \le 1; \ 0 \le v(t) \le 1; \ v(t) \le 2u(t); \ v(t) \le 2 - 2u(t); t \in [0,4],$$

and the on-target conditions

$$\ell_1(x,z) \equiv 0.2 \int_0^4 x(t) \, dt + 0.1z(3) = 0.6, \ \ell_2(x,z) \equiv x(4) + z(4) = 8.0.$$

With the use of the results [25] on the Cauchy operator we construct the moment matrix $M(t) = (m_{ij}(t))_{i,j=1,2}$ that corresponds to the on-target functional:

$$m_{11}(t) = (0.483301 - 0.321150t)\chi_{[0,1]}(t) + (0.246135 - 0.083984t)\chi_{(1,2]}(t)$$

$$+(0.234500 - 0.078167t)\chi_{(2,3]}(t);$$

$$m_{12}(t) = 0.263258\chi_{[0,1]}(t) + 0.287194\chi_{(1,2]}(t) + 0.123083\chi_{(2,3]}(t);$$

$$m_{21}(t) = (9.110666 - 5.600451t)\chi_{[0,1]}(t) + (5.487099 - 1.976883t)\chi_{(1,2]}(t)$$

$$+(4.60000 - 1.533333t)\chi_{(2,3]}(t);$$

$$m_{22}(t) = 1.867111\chi_{[0,1]}(t)\chi_{(1,2]}(t) + 2.134306\chi_{(1,2]}(t) + 1.040417\chi_{(2,3]}(t)$$

$$+0.200000\chi_{(3,4]}(t).$$

Here the numerical coefficients are displayed to six places of decimals.

In this case we find the following three points:

$$P_1 = (0,0), P_2 = (0.812625, 11,329384), P_3 = (0.889121, 9.617857)$$

forming the triangle that is a subset of the attainability set and contains the given point $P_0 = (0.6, 8.)$ from the on-target conditions. That point is the convex linear combination of the above three points with the coefficients $0.594579, 0.274021, 0.131399$ respectively. Having in mind that the controls bringing P2 and P3 has two switch points $t = 0.8$ and $t = 2.8$, and P_1 is attainable by $u(t) = 0, v(t) = 0$, we conclude that the problem under consideration may be solved by the control with two switch points.

5. Stability and Asymptotic Behavior of Solutions

The currently constructed theory [7] permits the clear description of main properties of FDEs, among them the stability properties of solutions [40], whereas many relevant classes of HFDEs are not encompassed by the constructed theory and are beyond the attention of experts employing systems with aftereffect while modelling real-world processes. Hybrid analogs of the main statements of the FDEs theory concerning stability questions are proposed below.

5.1. The Scheme of N. V. Azbelev's W-Method as Applied to Hybrid Models

Let us first consider the case when one equation is a linear functional differential equation with aftereffect on the semi-axis, and the other is a linear difference equation, and describe for this case the scheme of N. V. Azbelev's W-method.

Introduce the space L as the space of locally summable functions $f : [0, \infty) \to R^n$ with semi-norms $||f||_{L[0,T]} = \int_0^T |f(t)|_n \, dt$ for all and the space locally absolutely continuous functions $x : [0, \infty) \to R^n$ with semi-norms $||x||_{AC[0,T]} = ||\dot{x}||_{L[0,T]} + |x(0)|_n$ for all $T > 0$. Denote by

$$z = \{z(-1), z(0), z(1), \ldots, z(N), \ldots\}$$

the infinite matrix with the ν-columns z(-1), z(0), z(1), ..., z(N), ... and, analogously,

$$g = \{g(0), g(1), \ldots, g(N), \ldots\}$$

will stand for the infinite matrix with the ν-columns $g(0)$, $g(1), \ldots, g(N), \ldots$. By any z we define the function $z(t)$:

$$z(t) = z(-1)\chi_{[-1,0)}(t) + z(0)\chi_{[0,1)}(t) + z(1)\chi_{[1,2)}(t) + \cdots + z(N)\chi_{[N,N+1)}(t) + \ldots, \ t \in [-1, \infty),$$

and by any g define the function $g(t)$:

$$g(t) = g(0)\chi_{[0,1)}(t) + g(1)\chi_{[1,2)}(t) + \cdots + g(N)\chi_{[N,N+1)}(t) + \ldots, \ t \in [0, \infty).$$

In the sequel we denote $z[t] = z([t])$ and $g[t] = g([t])$ where $[t]$ is the integer part of t. Let ℓ_0 be the space of functions $z[\cdot]$ with semi-norms $||z||_{\ell_{0T}} = \sum_{i=-1}^T |z(i)|_\nu$ for all integer $T \geq -1$, and ℓ be the space of functions $g[\cdot]$ with semi-norms $||g||_{\ell_T} = \sum_{i=0}^T |g(i)|_\nu$ for all integer $T \geq 0$.

Further, define $(\Delta z)(t) = z[t] - z[t-1], t \in [1, \infty)$, $(\Delta z)(t) = z[t] = z(0), t \in [0,1)$.

Consider the hybrid functional differential system

$$\begin{aligned} \dot{x} &= \mathcal{T}_{11}x + \mathcal{T}_{12}z + f, \\ \Delta z &= \mathcal{T}_{21}x + \mathcal{T}_{22}z + g. \end{aligned} \tag{75}$$

For convenience, let us introduce operators $\mathcal{L}_{ij}, i, j = 1, 2$:

$$\mathcal{L}_{11}x = \dot{x} - \mathcal{T}_{11}x, \ \mathcal{L}_{12}z = -\mathcal{T}_{12}z, \ \mathcal{L}_{21}x = \mathcal{T}_{21}x, \ \mathcal{L}_{22}z = \Delta z - \mathcal{T}_{22}z.$$

The operators $\mathcal{L}_{11}, \mathcal{T}_{11} : AC \to L$, $\mathcal{L}_{12}, \mathcal{T}_{12} : \ell_0 \to L$, $\mathcal{L}_{21}, \mathcal{T}_{21} : AC \to \ell$, $\mathcal{L}_{22}, \mathcal{T}_{22} : \ell_0 \to \ell$ are assumed to be linear continuous and Volterra. Denote $\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}$. Then (75) takes the form $\mathcal{L}(x, z) = \mathrm{col}(f, g)$.

Let

$$\dot{x} = \mathcal{T}_{11}^0 x + \mathcal{T}_{12}^0 z + u, \ \Delta z = \mathcal{T}_{21}^0 x + \mathcal{T}_{22}^0 z + v$$

be a model system such that, for any $x(0) \in R^n$ and $z(-1) \in R^\nu$, the Cauchy problem be uniquely solvable (as for operators $\mathcal{T}_{ij}^0$, we save the same assumptions as to $\mathcal{T}_{ij}$). The model system may be written in the form $\mathcal{L}_0(x, z) = \mathrm{col}(u, v)$. Its solution has the representation

$$\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x(0) \\ z(-1) \end{pmatrix} + \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where $\mathcal{W} : L \times \ell \to AC \times \ell_0$ is the Cauchy operator, $\mathcal{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$; $U : R^n \times R^\nu \to AC \times \ell_0$ is

the fundamental matrix, $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$.

If elements $\mathrm{col}(x,z) : [0,\infty) \times [-1.\infty) \to R^n \times R^\nu$ constitute a Banach space $\mathbf{D} \times \mathbf{M}_0 \cong (\mathbf{B} \times R^n) \times ((\mathbf{B} \times R^\nu)$ $(\mathbf{D} \subset AC, \mathbf{M}_0 \cong (\mathbf{M} \oplus R^\nu \subset \ell_0), \mathbf{B} \subset L, \mathbf{M} \subset \ell, \mathbf{B}, \mathbf{M}$ are Banach spaces) and possess certain specific properties, say, $\sup_{t \geq 0} |x(t)|_n + \sup_{k=-1,0,1,\dots} |z(k)|_\nu < \infty$ and the Cauchy problem for an equation $\mathcal{L}(x,z) = \mathrm{col}(f,g)$ with a linear bounded operator $\mathcal{L} : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$ is uniquely solvable, then its solutions have the same asymptotic properties. This follows from the following theorem [41].

Theorem 11. *Let* $\mathcal{W} : \mathbf{B} \times \mathbf{M} \to \mathbf{D} \times \mathbf{M}_0$ *be the bounded Cauchy operator to a model problem* $\mathcal{L}_0(x,z) = \mathrm{col}(f,g)$ *with* $\mathcal{L}_0 : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$, *and* $U : R^n \times R^\nu \to AC \times \ell_0$ *is the fundamental matrix to* $\mathcal{L}_0(x,z) = \mathrm{col}(0,0)$. *Let, further, a linear operator* $\mathcal{L} : \mathbf{D} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$ *be bounded, C is the Cauchy operator to* $\mathcal{L}(x,z) = \mathrm{col}(f,g)$, *and X is its fundamental matrix. Then the equality*

$$\mathcal{W}\{\mathbf{B}, \mathbf{M}\} \oplus U\{R^n, R^\nu\} = C\{\mathbf{B}, \mathbf{M}\} \oplus X\{R^n, R^\nu\} \tag{76}$$

holds if and only if the operator $\mathcal{L}\mathcal{W}$ *has the bounded inverse* $(\mathcal{L}\mathcal{W})^{-1} : \mathbf{B} \times \mathbf{M} \to \mathbf{B} \times \mathbf{M}$.

Corollary 2 ([41]). *If an operator* $\mathcal{L} : \mathbf{B} \times \mathbf{M}_0 \to \mathbf{B} \times \mathbf{M}$ *is bounded and the inequality* $||(\mathcal{L} - \mathcal{L}_0)\mathcal{W}||_{\mathbf{B} \times \mathbf{M} \to \mathbf{B} \times \mathbf{M}} < 1$ *holds, then the equality* (76) *takes place.*

In the case of equality (76) (the coincidence of the solution space of the model and the studied equations), we say that the equation $\mathcal{L}(x,z) = \mathrm{col}(f,g)$ has the property $\mathbf{D} \times \mathbf{M}_0$ or, for short: the equation is $\mathbf{D} \times \mathbf{M}_0$-stable. We note the connection of the concept of $\mathbf{D} \times \mathbf{M}_0$-stability with the monograph by J. L. Massera and J. J. Schaffer on the admissibility of pairs of spaces [42], and with the monograph by E. A. Barbashin on preserving the properties of solutions when perturbations accumulate [43]. Let the model equation [7,44–46] $\mathcal{L}_{11} = u$ and the Banach space $\mathbf{B}$ with elements from the space L (recall that $\mathbf{B} \subset L$ and this embedding is continuous) be chosen so that the solutions of this equation have the asymptotic properties that are of interest to us.

For example, let $\sup_{t \geq 0} |x(t)|_n < \infty$ be hold. Then, putting $\mathcal{L}_{11}x \equiv \dot{x} + x = u$, we take as $\mathbf{B}$ the Banach space L_∞ of measurable and essentially bounded functions $u : [0,\infty \to R^n)$ with the norm $||u||_{L_\infty} = \mathrm{vrai} \sup_{t \geq 0} |u(t)|_n < \infty$. The space $AC(\mathcal{L}_{11}, L_\infty)$ generated by the model equation consists of the solutions

$$x(t) = (\mathcal{W}_{11}u)(t) + (\mathcal{U}_{11}\alpha)(t) = \int_0^t e^{-(t-s)} u(s)\, ds + \alpha e^{-t} \ (\alpha \in R^n, z \in L_\infty)$$

that are bounded ($\sup_{t \geq 0} |x(t)|_n < \infty$) and have the derivative $\dot{x} = -x + u$ from L_∞.

All these solutions form the Banach space equipped by the norm

$$||x||_{AC(\mathcal{L}_{11}, L_\infty)} = \mathrm{vrai} \sup_{t \geq 0} |\dot{x}(t) + x(t)|_n + |x(0)|_n < \infty.$$

This space is isomorphic to the Sobolev space $W_\infty^{(1)}[0, \infty)$ with the norm

$$||x||_{W_\infty^{(1)}[0,\infty)} = \sup_{t \geq 0} |x(t)|_n + \text{vrai} \sup_{t \geq 0} |\dot{x}(t)|_n.$$

In the sequel we denote this space by W_{L_∞}, note that $W_{L_\infty} \subset AC$ and the embedding is continuous. Similarly, introduce the space $AC(\mathcal{L}_{11}, \mathbf{B})$, $\mathbf{B} \subset L$, with the norm

$$||x||_{AC(\mathcal{L}_{11}, \mathbf{B})} = ||\dot{x} + x||_{\mathbf{B}} + |x(0)|_n.$$

Under natural conditions [40,44–46], the space $AC(\mathcal{L}_{11}, \mathbf{B})$ is isomorphic to the Sobolev space $W_{\mathbf{B}}^{(1)}[0, \infty)$ with the norm

$$||x||_{W_{\mathbf{B}}^{(1)}}[0, \infty) = ||\dot{x}||_{\mathbf{B}} + ||x||_{\mathbf{B}},$$

which we denote by $\mathbf{W_B}$.

The equation $\mathcal{L}_{11}x = u$ with an operator $\mathcal{L}_{11} : \mathbf{W_B} \to \mathbf{B}$ is $AC(\mathcal{L}_{11}, \mathbf{B})$-stable if and only if it is strictly $\mathbf{B}$-stable, namely, for any $u \in \mathbf{B}$, every solution x to this equation possesses the property: $x \in \mathbf{B}$ and $\dot{x} \in \mathbf{B}$ [40,45].

5.2. The Case of Two Scalar Equations

Let us apply the scheme of Section 5.1 to the case of two scalar Equation (75). The operators $\mathcal{L}_{11} : AC \to L$, $\mathcal{L}_{12} : \ell_0 \to L$, $\mathcal{L}_{21} : AC \to \ell$, $\mathcal{L}_{22} : \ell_0 \to \ell$ are considered as operators of reducing to the pairs $(\mathbf{W_B}, \mathbf{B})$, $(\mathbf{M}_0, \mathbf{B})$, $(\mathbf{W_B}, \mathbf{M})$, and $(\mathbf{M}_0, \mathbf{M})$, under the assumption that the operators are linear bounded and Volterra and the general solution of the equation $\mathcal{L}_{22}z = g$, where , $g \in \mathbf{M}$ belongs to the space $\mathbf{M}_0$ and has the representation

$$z[t] = Z_{22}[t] z(-1) + \sum_{s=0}^{[t]} C_{22}[t, s] g[s],$$

where $C_{22}[t, s] = C_{22}([t], [s])$.

Denote

$$(C_{22}g)[t] = \sum_{s=0}^{[t]} C_{22}[t, s]g[s], \quad (Z_{22}z(-1))[t] = Z_{22}[t]z(-1).$$

Then every solution z to the second equation in (75) may be written in the form

$$z = -C_{22}\mathcal{L}_{21}x + Z_{22}z(-1) + C_{22}g.$$

After substitution this expression into the first equation of (75) we obtain

$$\mathcal{L}_{11}x + \mathcal{L}_{12}z = \mathcal{L}_{11}x - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}x + \mathcal{L}_{12}Z_{22}z(-1) + \mathcal{L}_{12}C_{22}g = f,$$

or

$$\mathcal{L}_{11}x - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}x = f_1 = f - \mathcal{L}_{12}Z_{22}z(-1) - \mathcal{L}_{12}C_{22}g.$$

Now, putting $\mathcal{L} = \mathcal{L}_{11} - \mathcal{L}_{12}C_{22}\mathcal{L}_{21}$, rewrite this equation in the form $\mathcal{L}x = f_1$. In the case when the Volterra operator $\mathcal{L} : (\mathbf{W_B})^0 \to \mathbf{B}$ is invertible and the inverse is Volterra too we have that, for any $\{f, g\} \in \mathbf{B} \times \mathbf{M}$, the corresponding solution $\{x, z\}$ of (75) is from the space $\mathbf{W_B}$.

Example 1. *Consider the equations*

$$\dot{x}(t) + a(1 - 0.5\sin(t))x(t) + bz[t] = f(t), \ t \in [0, \infty),$$
$$z[t] - dz[t-1] + cx[t] = g[t], \ t \in [0, \infty),$$

(77)

assuming that

$$z(0) - dz(-1) + cx(0) = g(0).$$

Introduce the spaces

$$\ell_{\infty 0} = \{z \in \ell_0 : ||z||_{\ell_0} = \sup_{k=-1,0,1,\dots} |z(k)|_\nu < +\infty\},$$

$$\ell_\infty = \{g \in \ell_0 : ||g||_{\ell_\infty} = \sup_{k=0,1,\dots} |g(k)|_\nu < +\infty\}.$$

Define the operator S by the equalities $(Sz)(t) = dz(t-1)$, $t \geq 1$; $(Sz)(t) = 0$, $t \in [0,1)$, *then the second equation takes the form*

$$z(t) - (Sz)(t) + cx(t) = g_1(t) = g(t) + dz[t-1], \ t \in [0,1),$$

$$z(t) - (Sz)(t) + cx(t) = g(t), \ t \in [1, \infty).$$

As is known, the operator $(I - S) : \ell_\infty \to \ell_\infty$ *has the bounded inverse if and only if the spectral radius* $\rho_{\ell_\infty}(S)$ *is less than one, or, which is the same, the inequality* $|d| < 1$ *holds. Let*

$$(\mathcal{L}_{11}x)(t) = \dot{x}(t) + a(1 - 0.5\sin(t))x(t), \quad t \geq 0,$$

$$(\mathcal{L}_{12}z)(t) = bz[t], \quad t \geq 0,$$

$$(\mathcal{L}_{21}x)[t] = cx[t], \quad t \geq 0,$$

$$(\mathcal{L}_{22}z)[t] = z[t] - (Sz)[t], \quad t \geq 0.$$

Construct the Cauchy function C_{22} *and the fundamental solution* Z_{22} *to the equation* $z[t] - dz[t-1] = g[t]$: $d^{[t]+1}z(-1) + \sum_{s=0}^{[t]} g[s]d^{[t]-s} = Z_{22}[t]z(-1) + (C_{22}g)[t]$. *Now express* $z[t]$ *from the second equation of* (77):

$$z[t] = Z_{22}z(-1) + (C_{22}(g - cx))[t].$$

After substitution this into the first formula of (75) *we obtain*

$$(\mathcal{L}_{11}x)(t) + (\mathcal{L}_{12}z)(t) = \dot{x}(t) + a(1 - 0.5\sin(t))x(t) + bd^{[t]+1}z(-1) + b\sum_{s=0}^{[t]} (g[s] - cx[s])d^{[t]-s} = f(t).$$

Further we have

$$(\mathcal{L}x)(t) = ((\mathcal{L}_{11} - \mathcal{L}_{12}C_{22}\mathcal{L}_{21})x)(t) = \dot{x}(t) + a(1 - 0.5\sin(t))x(t) - bc\sum_{s=0}^{[t]} x[s])d^{[t]-s} = f_1(t)$$

$$= f(t) - bd^{[t]+1}z(-1) - b\sum_{s=0}^{[t]} g[s])d^{[t]-s}.$$

It is clear that $f_1 \in L_\infty$ if $|d| < 1$. Write the Cauchy formula as applied to the equation $(\mathcal{L}_{11}x)(t) = bc \sum_{s=0}^{[t]} x[s])d^{[t]-s} + f_1(t)$:

$$x(t) = X_{11}x(0) + \int_0^t C_{11}(t,s)\,(bc\sum_{i=0}^{[s]} x[i])d^{[s]-i} + f_1(s))\,ds.$$

Here we have $X_{11}(t) = e^{-a(t+0.5\cos(t))}$, $C_{11}(t,s) = e^{-a(t+0.5\cos(t))} \cdot e^{a(s+0.5\cos(s))}$. For the case of a positive a, the estimates

$$\sup_{t\geq 0} |\int_0^t C_{11}bc\sum_{i=0}^{[s]} x[i]d^{[s]-i}\,ds| \;\leq\; |bc|\cdot\frac{1}{1-|d|}\cdot\sup_{t\geq 0}(e^{-a(t+0.5\cos(t))}\int_0^t e^{a(s+0.5\cos(s))}\,ds)\cdot||x||_{L_\infty}$$

$$\leq \frac{2}{a}\cdot|bc|\cdot\frac{1}{1-|d|}\cdot||x||_{L_\infty}$$

hold. Therefore, the norm of $bC_{11}cC_{22}$ is less than 1 if $|bc| < \frac{a}{2}(1-|d|)$, and under this condition we have that, for any $f_1 \in L_\infty$, the solution x of $\mathcal{L}x = f_1$ belongs to the space L_∞ together with its derivative $\dot{x}$. Thus it is established that, for any $f_1 \in L_\infty$, solution x to $\mathcal{L}x = f_1$ belongs to the space W_{L_∞}. Therefore, for Equation (77), we obtain the conditions

$$|d| < 1, \ |bc| < \frac{a}{2}(1-|d|), \tag{78}$$

under which, for any $\{f,g\} \in L_\infty \times \ell_\infty$, solution of it belongs to the space $\mathbf{D} \cong W_{L_\infty} \times \ell_{\infty 0}$.

5.3. A Case of Periodic Parameters: Direct Study of Asymptotic Behavior

The case of economic dynamic models with periodic parameters is of special interest [47].

Here we consider a hybrid model with short discrete memory and demonstrate a way to study asymptotic behavior of solutions, namely, the existence of a finite limit state of the system as $t \to \infty$.

Consider the system

$$\begin{aligned}\dot{x}(t) &= A(t)x[t-1] + B(t)z[t-1] + f(t), \ t \in [1,\infty),\\ z[t] &= Dx[t-1] + Hz[t-1] + g[t], \ t \in [1,\infty),\end{aligned} \tag{79}$$

with 1-periodic matrices $A(t)$ and $B(t)$ of the dimensions $n \times n$ and $n \times \nu$ respectively, constant matrices D and H of the dimensions $\nu \times n$ and $\nu \times \nu$, respectively, 1-periodic functions f, g and a given initial state:

$$x(0) = \alpha, \ z(0) = \beta. \tag{80}$$

Define the constant matrices P, Q by the equalities

$$P = \int_0^1 A(s)\,ds, \ Q = \int_0^1 B(s)\,ds, \ \phi = \int_0^1 f(s)\,ds,$$

and the $(n+\nu) \times (n+\nu)$-matrix $\mathcal{F}$ by the equality

$$\mathcal{F} = \begin{pmatrix} E_n + P & Q \\ D & H \end{pmatrix},$$

where E_n is the identity $(n \times n)$-matrix.

Theorem 12. *Let the spectral radius of $\mathcal{F}$ be less than 1. Then there exists a vector $y = \text{col}(\gamma, \delta)$ with finite components such that the solution $y = \text{col}(x(t), z[t])$ of the problems* (79) *and* (80) *possesses the property*

$$X(N) \to \gamma, \ z(N) \to \delta \text{ as integer } N \to +\infty.$$

Proof. Let us integrate the first equation of (13) from $[t-1]$ to $[t]$, $t \in [1, \infty)$:

$$x([t]) = x([t-1]) + \int_{[t-1]}^{[t]} A(s) \, ds \, x([t-1]) + \int_{[t-1]}^{[t]} B(s) \, ds \, z[t-1] + \int_{[t-1]}^{[t]} f(s) \, ds.$$

Put $y_i = \text{col}(x(i), z(i))$, then we have

$$y_i = \mathcal{F} y_{i-1} + \psi, \ i = 1, 2, \ldots, \tag{81}$$

where $\psi = \text{col}(\phi, g(1))$. By the assumptions, the successive approximations (81) converge to y being the solution of the system $y_i = \mathcal{F} y + \psi$, $: y = (E_{n+\nu} - \mathcal{F})^{-1} \psi$. $\square$

Example 2. *Consider the system*

$$\dot{x} = 0.05(1 + \sin(2\pi t))x[t-1] - 0.5(1 - \cos(2\pi t))z[t-1] + f(t), \ t \in [1, \infty),$$

$$z[t] = 0.3x[t-1] - 0.5z[t-1] + g[t], \ t \in [1, \infty),$$

where $f(t) = 0.2t$, $t \in [0, 1)$, $g(i) = 0.2$, $I = 1, 2, \ldots$, with the initial state defined by the equalities $x(0) = 2$, $z(0) = 1$.

For the case, we have $\mathcal{F} = \begin{pmatrix} 1 + 0.05 & -0.5 \\ 0.3 & -0.5 \end{pmatrix}$, $\rho(\mathcal{F}) < 0.94628608$. As for the limit state y,

its approximate value after 50 iterations is $\text{col}(0.740251217, 0.281930151)$ (the components are displayed to nine decimals).

6. Discussion

In this paper we have presented the results concerning three types of problem with respect to trajectories of continuous-discrete dynamic models used in Mathematical Economics.

For boundary value problems with periodic boundary conditions, the conditions of the solvability and conditions of the insolvability are obtained which can be applied to the problem of existence/lack of business-cycles. A feature of these results is the possibility for taking into account both current and future states of economic system. The results improve known conditions even in the case of continuous problems. Note, the obtained solvability conditions are results of applying original approach developed in Reference [20] and not connected with fixed points principles at all. As for a direction to future work, we plan to extend our method [20–22] to other boundary value problems for hybrid objects including the case of multidimensional (in the continuous part) models.

The results on the control problems with constrained control are based on the construction of the Cauchy operator to hybrid dynamic models and allow one to give a constructive description of attainability sets for a wide class of on-target conditions. The next steps in this direction can be thought of as computer-aided implementation of constructions and algorithms with the use of reliable computing [48].

The conditions for the stability of solutions to the models under consideration are established on the base of the elementary models technique that demonstrates the efficiency of the approach proposed

and provides the researcher with a way to encompass more and more wide classes of models. For entire collection of elementary models we refer to References [49–51].

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