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Double Cyclic Codes over $\mathbb{F}_q + v\mathbb{F}_q$

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Abstract: In this paper, an algebraic structure of a type of double cyclic codes is investigated, which covers some existing codes as special cases. The paper presents generic results about the generating polynomials, minimal generating sets, matrices and dual codes of the proposed codes.

Keywords: double cyclic codes; generating matrices; generating polynomials; minimal generating sets; non-chain rings

1. Introduction

Due to the nice algebraic structures of finite rings, the coding theory over finite rings has attracted the attention of many scholars since the early 1970s, which can be regarded as the generalization of the classical coding theory over the finite fields.

For the ring $R = \mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = v$, it can be seen that it is a commutative finite ring which can be viewed as a 2-dimensional vector space over $\mathbb{F}_q$. In fact, it is also considered to be isomorphic to $\mathbb{F}_q \times \mathbb{F}_q$. Therefore, the ring $R = \mathbb{F}_q + v\mathbb{F}_q$ is a generalization of the classical finite fields.

In 1973, Delsarte [1] introduced the concept of additive codes and gave the explicit construction of such codes. Since then, many scholars have began to focus on these codes. In 2010, Borges et al. proposed a new concept called $\mathbb{Z}_2\mathbb{Z}_4$-linear codes in [2]. It is clear that $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are $\mathbb{Z}_4$ additive subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ where α and β are index positive integers. For two index positive integers, $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are $\mathbb{Z}_4$ additive subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ as can be seen in their structure. Note that $\mathbb{Z}_2\mathbb{Z}_4$-additive code is a quaternary linear code for $\alpha = 0$, while it is a binary linear code for $\beta = 0$. Hence, $\mathbb{Z}_2\mathbb{Z}_4$-additive codes generalize both the binary linear codes and the quaternary linear codes. Some good results related to $\mathbb{Z}_2\mathbb{Z}_4$-codes can be found in [3–5]. However, there are two important problems about these codes that deserve further investigation: the one is to broaden the alphabet and the other is to improve the structure of the codes further. For the first problem, the structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive cyclic codes and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes in [6,7], respectively. Here, it is worth emphasizing that the second problem is more closely related to this paper. Some good results about this problem have been obtained in some references. For instance, in 2014, Abusltun et al. [8] studied the algebraic structure of $\mathbb{Z}_2\mathbb{Z}_4$-cyclic codes systematically. In [9], Borges et al. replaced $\mathbb{Z}_4$ with $\mathbb{Z}_2$ in the last part of the alphabet and then explored the related properties about $\mathbb{Z}_2$-double cyclic codes. It should be pointed out that the initial concept of double cyclic codes is also proposed in [9]. Short after, Gao et al. [10] replaced $\mathbb{Z}_2$ with $\mathbb{Z}_4$ in the first part of the alphabet and gave some good results about double cyclic codes over $\mathbb{Z}_4$. Considering the relationship between $\mathbb{Z}_2$ and $\mathbb{Z}_4$, Wang and Gao [11] investigated the double λ-constacyclic codes over finite chain rings, which generalizes the previous works. For a non-chain ring, the double cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = v$ will be investigated in this paper. The contributions of various authors in the history of double cyclic codes are listed as follows in Table 1.

Table 1. The table of authors' contributions to the history of double cyclic codes.

References	Titles	Contributions
[2]	$\mathbb{Z}_2\mathbb{Z}_4$-linear codes:generator matrices and duality	$\mathbb{Z}_2\mathbb{Z}_4$-linear codes
[6,7]	The structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive cyclic codes, On $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes	Various extensions and deformations of $\mathbb{Z}_2\mathbb{Z}_4$.
[9]	$\mathbb{Z}_2$-double cyclic codes	The original definition of double cyclic codes
[10]	On double cyclic codes over $\mathbb{Z}_4$	Doule cyclic codes over another one finite chain ring
[11]	Double λ-constacyclic codes over finite chain rings	Further extension of double cyclic codes over generally finite chain rings

The notion of double cyclic codes derives from the research of $\mathbb{Z}_2\mathbb{Z}_4$-codes. Therefore, a natural idea is to consider double cyclic codes over $\mathbb{Z}_2$ and $\mathbb{Z}_4$. For more details, readers can refer to [9,10]. Since then, coding researchers have tried to study the structure of double cyclic codes over the other finite rings. Generally speaking, this article is the concrete form of the double cyclic codes under another kind of ring. The following three remarks about this paper should be pointed out.

- Similar to the literature [10,11], this article shows the results over another type of ring. Compared with literature [9,12], the theoretical results provided in this paper are more general. Therefore, this paper will improve and generalize the concrete forms of those codes shown in [9,12] for further research of the double cyclic codes over finite fields.
- Since the ring $\mathbb{F}_q + v\mathbb{F}_q$ is a finite field locally, the results shown in this paper reflect some properties of double cyclic codes over finite fields locally as a result. Through some special details of this ring, this double cyclic code can be viewed as a code over finite fields directly, rather than having been implemented by the Gray map as some codes over traditional finite rings. This point will be reflected by the examples of this paper.
- Throughout this article, double cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$ are found to be a linear combination of two $\mathbb{F}_q$-double cyclic codes with the same length, which also provides a new technical method for us to obtain some codes with new parameters. Finally, some examples that are the linear combination of two double cyclic codes over finite fields are presented, which is helpful for acquiring some codes with new parameters over finite fields.

This paper is organized as follows. In Section 2, some preliminaries about this paper are reviewed. Section 2.1 explains the structure of this ring, Section 2.2 just lists some reference results and Section 2.3 mainly leads to various marks and mathematical objects of this paper. Sections 3 and 4 are the kernel of this paper. It utilizes the polynomial theory to give some results about double cyclic codes and their dual codes. Section 3 illustrates the basic forms of the double cyclic codes, which include the forms of generating polynomials, minimal generating sets and generating matrices. Section 4 explains the generating polynomial relationship between the dual codes and their proposed codes. Some examples of double cyclic codes over $\mathbb{F}_2 + v\mathbb{F}_2$ are also presented as the theoretical application of this article. Section 5 summarizes this paper.

2. Preliminaries

Let $\mathbb{F}_q$ be the finite field with q elements, where $q = p^s$ is a prime power for some positive integer s. Along this paper, let R denote the commutative finite ring $\mathbb{F}_q + v\mathbb{F}_q = \{a + bv \mid a, b \in \mathbb{F}_q\}$ with $v^2 = v$. This section mainly introduces some properties of R and illustrates some necessary knowledge of this article.

It is well known that $R = \mathbb{F}_q + v\mathbb{F}_q$ is a principal ideal ring and it has only two non-trivial ideals, namely $\langle v \rangle = \{av \mid a \in \mathbb{F}_q\}$ and $\langle 1 - v \rangle = \{b(1 - v) \mid b \in \mathbb{F}_q\}$. It can be easily checked that $\langle v \rangle$ and

$\langle 1-v\rangle$ are all maximal ideals in R. Hence, R is not a chain ring. R is a Frobenius ring from the results of paper [13].

The definition of linear codes, cyclic codes over general rings obey the usual forms. Readers can refer to [14].

2.1. The Basic Consequence about Polynomial Theory over R

For $r \in R = \mathbb{F}_q + v\mathbb{F}_q$, let $r = a + bv$ with $a, b \in \mathbb{F}_q$. Obviously, we know that

$$a + bv = (a + b)v + a(1 - v).$$

For the sake of convenience, set $1 - v = w$. Notice that $v^2 = v, w^2 = w$ and $vw = wv = 0$. Let $r, s \in R$ with $\begin{cases} r = r_v v + r_w w, \\ s = s_v v + s_w w, \end{cases}$ then $\begin{cases} r + s = (r_v + s_v)v + (r_w + s_w)w, \\ rs = (r_v s_v)v + (r_w s_w)w. \end{cases}$

It is indicated that this special non-chain ring can be considered as a 2-dimensional algebra over $\mathbb{F}_q$ with $\{v, w\}$ being its basis. The significance of this basis is that v, w are idempotent and orthogonal.

Define two canonical projective maps $\begin{cases} P_v : R \longrightarrow \mathbb{F}_q & r \longmapsto r_v, \\ P_w : R \longrightarrow \mathbb{F}_q & r \longmapsto r_w. \end{cases}$ The above facts illustrate that P_v and P_w are $\mathbb{F}_q$-algebra homomorphism. Similarly, for all $n \in \mathbb{N}$, it can be seen that $\begin{cases} P_v : R^n \longrightarrow \mathbb{F}_q^n & (r_1, \ldots, r_n) \longmapsto ((r_1)_v, \ldots, (r_n)_v), \\ P_w : R^n \longrightarrow \mathbb{F}_q^n & (r_1, \ldots, r_n) \longmapsto ((r_1)_w, \ldots, (r_n)_w), \end{cases}$ are $\mathbb{F}_q$-algebra homomorphism.

As the application to polynomial theory, extend this thought from R to polynomial ring $R[x]$ naturally. Let $\sum_i r_i x^i = r(x) \in R[x]$ with $r_i \in R$. Set $r_i = P_v(r_i)v + P_w(r_i)w$. Then,

$$r(x) = \sum_i (P_v(r_i)v + P_w(r_i)w)x^i = (\sum_i P_v(r_i)x^i)v + (\sum_i P_w(r_i)x^i)w = r_v(x)v + r_w(x)w,$$

where $r_v(x), r_w(x) \in F_q[x]$.

For $r(x), s(x) \in R[x]$, let $r(x) = r_v(x)v + r_w(x)w$ and $s(x) = s_v(x)v + s_w(x)w$, it is clear that

$$\begin{cases} r(x) + s(x) & = (r_v(x) + s_v(x))v + (r_w(x) + s_w(x))w, \\ r(x)s(x) & = (r_v(x)s_v(x))v + (r_w(x)s_w(x))w. \end{cases}$$

Consequently, define $\begin{cases} P_v : R[x] \longrightarrow \mathbb{F}_q[x] & r(x) \longmapsto r_v(x), \\ P_w : R[x] \longrightarrow \mathbb{F}_q[x] & r(x) \longmapsto r_w(x), \end{cases}$ the two maps still are $\mathbb{F}_q[x]$-algebra homomorphism. For simplicity, if one mathematical object appears to subscript v or w, it uses the projective maps P_v or P_w by default.

It follows from the above illustration that $R[x] = \mathbb{F}_q[x]v \oplus \mathbb{F}_q[x]w$. For $f(x) \in \mathbb{F}_q[x]$, write $f(x) = f(x)v + f(x)w$. This guarantees that the element of $\mathbb{F}_q[x]$ can be viewed as the element of $R[x]$.

Next, let us consider the divisibility between any two elements in $R[x]$. Then, the following proposition holds.

Proposition 1. *For $r(x), s(x) \in R[x]$, let $r(x) = r_v(x)v + r_w(x)w, s(x) = s_v(x)v + s_w(x)w$ with $r_v(x), r_w(x), s_v(x), s_w(x) \in \mathbb{F}_q[x]$. Then, $s(x)|r(x)$ in $R[x]$ if and only if $s_v(x)|r_v(x), s_w(x)|r_w(x)$ in $\mathbb{F}_q[x]$.*

Proof. For $s(x)|r(x)$, let $r(x) = \varepsilon(x)s(x)$. So $r_v(x)v + r_w(x)w = (\varepsilon_v(x)v + \varepsilon_w(x)w)(s_v(x)v + s_w(x)w)$. Then, $r_v(x)v + r_w(x)w = (\varepsilon_v(x)s_v(x))v + (\varepsilon_w(x)s_w(x))w$. Due to the expression by the basis $\{v, w\}$, it follows that $r_v(x) = \varepsilon_v(x)s_v(x)$ and $r_w(x) = \varepsilon_w(x)s_w(x)$, which means that $s_v(x)|r_v(x), s_w(x)|r_w(x)$.

Conversely, for $s_v(x)|r_v(x), s_w(x)|r_w(x)$, let $r_v(x) = \varepsilon_v(x)s_v(x)$, $r_w(x) = \varepsilon_w(x)s_w(x)$. Then,

$$r_v(x)v + r_w(x)w = (\varepsilon_v(x)s_v(x))v + (\varepsilon_w(x)s_w(x))w = (\varepsilon_v(x)v + \varepsilon_w(x)w)(s_v(x)v + s_w(x)w)$$

Let $\varepsilon(x) = \varepsilon_v(x)v + \varepsilon_w(x)w$, then $\varepsilon(x) \in R[x]$ and $s(x)|r(x)$ in $R[x]$. □

Remark 1. *Proposition 1 implies that* $(\frac{r(x)}{s(x)})_v = \frac{r_v(x)}{s_v(x)}$ *and* $(\frac{r(x)}{s(x)})_w = \frac{r_w(x)}{s_w(x)}$ *for any* $r(x), s(x) \in R[x]$ *with* $s(x)|r(x)$.

Although $\mathbb{F}_q + v\mathbb{F}_q$ is not a field, it inherits the nice properties related to $\mathbb{F}_q$, particularly with regard to factorization. Therefore, let us consider the greatest common divisor between any two elements in $R[x]$. Similarly, the following proposition is obtained.

Proposition 2. *For* $r(x), s(x) \in R[x]$, *let* $r(x) = r_v(x)v + r_w(x)w, s(x) = s_v(x)v + s_w(x)w$ *with* $r_v(x), r_w(x), s_v(x), s_w(x) \in \mathbb{F}_q[x]$. *Then, in* $R[x]$,

$$\gcd(r(x), s(x)) = \gcd(r_v(x), s_v(x))v + \gcd(r_w(x), s_w(x))w,$$

where the symbol $\gcd(-,-)$ *on the right hand side of the equation denotes the greatest common divisor in* $\mathbb{F}_q[x]$.

Proof. Let $\gcd(r_v(x), s_v(x))v + \gcd(r_w(x), s_w(x))w = \Theta(x)$. Clearly, $\Theta(x) \in R[x]$. From Proposition 1, $\Theta(x)|r(x)$ holds. For the same reason, $\Theta(x)|s(x)$ also holds. These indicate that $\Theta(x)$ is a common divisor of $r(x)$ and $s(x)$ in $R[x]$.

For every $\delta(x) \in R[x]$ with $\delta(x)|r(x), \delta(x)|s(x)$, in terms of Proposition 1, write $\delta(x) = \delta_v(x)v + \delta_w(x)w$, where $\delta_v(x)|r_v(x), \delta_v(x)|s_v(x)$, $\delta_w(x)|r_w(x)$ and $\delta_w(x)|s_w(x)$. As a result of the conventional polynomial theory over finite fields, $\delta_v(x)|\gcd(r_v(x), s_v(x))$ and $\delta_w(x)|\gcd(r_w(x), s_w(x))$ are obtained. Applying Proposition 1 again obtains $\delta(x)|\Theta(x)$.

Hence, $\gcd(r_v(x), s_v(x))v + \gcd(r_w(x), s_w(x))w = \Theta(x) = \gcd(r(x), s(x))$ holds. □

Remark 2. *According to the above proof,* $(\gcd(r(x), s(x)))_v = \gcd_v(r(x), s(x)) = \gcd(r_v(x), s_v(x))$ *and* $(\gcd(r(x), s(x)))_w = \gcd_w(r(x), s(x)) = \gcd(r_w(x), s_w(x))$ *also holds*

The above analysis indicated that for $r(x) \in R[x]$, one has $r(x) = r_v(x)v + r_w(x)w$ with $r_v(x), r_w(x) \in \mathbb{F}_q[x]$, which derives that

$$R[x]/\langle r[x]\rangle = (\mathbb{F}_q[x]v \oplus \mathbb{F}_q[x]w)/\langle r_v(x)v + r_w(x)w\rangle = (\mathbb{F}_q[x]/\langle r_v(x)\rangle)v \oplus (\mathbb{F}_q[x]/\langle r_w(x)\rangle)w.$$

2.2. Some Results about Cyclic Codes over R

This section lists some important results about cyclic codes over R, which will be used to obtain the main results of this paper. For more details, please refer to [15].

Lemma 1. *([15]) Let* $C = vC_1 \bigoplus (1-v)C_2$ *be a linear code of length n over R. Then C is a cyclic code of length n over R if and only if* C_1 *and* C_1 *are cyclic codes of length n over* $\mathbb{F}_q$.

Lemma 2. *([15]) Let* $C = vC_1 \bigoplus (1-v)C_2$ *be a cyclic code of length n over R. Then there exists a unique polynomial* $f(x)$ *such that* $C = \langle f(x)\rangle$, *where* $f(x) = vf_1(x) + (1-v)f_2(x)$.

Lemma 3. *([15]) Let* $C = vC_1 \bigoplus (1-v)C_2$ *be a cyclic code of length n over R and* $f_1(x)$, $f_2(x)$ *are the generator polynomials of* C_1 *and* C_2, *respectively. Then,* $|C| = |C_1||C_2| = q^{2n-deg(f_1(x))-deg(f_2(x))}$.

Remark 3. *Denote by* $\mathcal{C}_n(R)$ *the set of all single cyclic codes of length n over R for* $n \in \mathbb{N}$.

2.3. Further Results about Polynomial Theory over R

Definition 1. *Let C be an R-submodule of R^{m+n}, C is called a double cyclic code of length (m,n) over R if $(c'_0,\ldots,c'_{m-2},c'_{m-1}|c''_0,\ldots,c''_{n-2},c''_{n-1}) \in C$ implies that*

$$\tau((c'_0,\ldots,c'_{m-2},c'_{m-1}|c''_0,\ldots,c''_{n-2},c''_{n-1})) = (c'_{m-1},c'_0,\ldots,c'_{m-2}|c''_{n-1},c''_0,\ldots,c''_{n-2}) \in C.$$

Remark 4. *Denote by $\mathcal{C}_{m,n}(R)$ the set of all double cyclic codes of length (m,n) over R.*

For $C \in \mathcal{C}_{m,n}(R)$, let C_m be the coordinate projection of C on the first m coordinates, and C_n be the coordinate projection of C on the second n coordinates. These mean that $\begin{cases}\Phi_m : C_{m,n} \to C_m & (c'_0,\ldots,c'_{m-1}|c''_0,\ldots,c''_{n-1}) \mapsto (c'_0,\ldots,c'_{m-1}),\\ \Phi_n : C_{m,n} \to C_n & (c'_0,\ldots,c'_{m-1}|c''_0,\ldots,c''_{n-1}) \mapsto (c''_0,\ldots,c''_{n-1}),\end{cases}$ are R-linear map and $C_m = \Phi_m(C)$, $C_n = \Phi_n(C)$ (For the convenience of writing, one mathematical object has the subscript m or n, which also means that it used by the coordinates projected to m or n). Hence, $C \in \mathcal{C}_{m,n}(R)$ if and only if $C_m \in \mathcal{C}_m(R)$ and $C_n \in \mathcal{C}_n(R)$.

Remark 5. *For simplicity, P_v, P_w are called canonical projective maps, and Φ_m, Φ_n are called coordinate projective maps.*

Definition 2. *A code $C \in \mathcal{C}_{m,n}(R)$ is separable if C is the direct product of C_m and C_n.*

Just like the situation of cyclic codes over finite fields, there exists a bijection between $R^m \times R^n$ and $(R[x]/\langle x^m-1\rangle) \times (R[x]/\langle x^n-1\rangle)$ given by

$$(c'_0,\ldots,c'_{m-1}|c''_0,\ldots,c''_{n-1}) \mapsto (c'_0 + c'_1x + \cdots + c'_{m-1}x^{m-1}|c''_0 + c''_1x + \cdots + c''_{n-1}x^{n-1}).$$

Let this bijective map expressed by π. Set $\begin{cases}R_m[x] = R[x]/\langle x^m-1\rangle,\\ R_{m,n}[x] = (R[x]/\langle x^m-1\rangle) \times (R[x]/\langle x^n-1\rangle),\\ R_n[x] = R[x]/\langle x^n-1\rangle.\end{cases}$

Then the rings $R_m[x]$, $R_{m,n}[x]$ and $R_n[x]$ with this action, which is induced by the action of $R[x]$ on $R_m[x]$, $R_{m,n}[x]$ and $R_n[x]$ from the multiplication of $R[x]$, become the $R[x]$-module. Simultaneously, define two maps

$$\begin{cases}\Phi_m : R_{m,n}[x] \to R_m[x] & (p(x)|q(x)) \mapsto p(x),\\ \Phi_n : R_{m,n}[x] \to R_n[x] & (p(x)|q(x)) \mapsto q(x).\end{cases}$$

Then, Φ_m and Φ_n are still $R[x]$-module homomorphism.

This subsection reveals the fact that $C \in \mathcal{C}_{m,n}(R)$ if and only if $\pi(C)$ is a $R[x]$-submodule to $R_{m,n}[x]$. Hence, the issue of $R[x]$-submodule of $R_{m,n}[x]$ needs to be of concern in this paper. Based on the bijection of π, double cyclic codes over R as the $R[x]$-submodule of $R_{m,n}[x]$ will be studied.

3. Double Cyclic Codes

3.1. Generating Polynomial Forms

From the above necessary preliminaries, the first important theorem of this paper is provided as follows.

Theorem 1. *Let C be a double cyclic code of length (m,n) over R. Then there exist $\xi(x), \ell(x) \in R_m[x]$, $o(x) \in R_n[x]$ with $\xi(x) = \xi_v(x)v + \xi_w(x)w, \ell(x) = \ell_v(x)v + \ell_w(x)w, o(x) = o_v(x)v + o_w(x)w$, such that*

$$C = \langle(\xi(x)|0), (\ell(x)|o(x))\rangle = \langle(\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w)\rangle,$$

where $\xi_v(x)$, $\xi_w(x)|x^m-1$, $o_v(x)$, *and* $o_w(x)|x^n-1$.

Proof. For $C \in \mathcal{C}_{m,n}(R)$, define $\widetilde{C} = \{(p(x)|q(x)) \in C \mid q(x) = 0\}$. It is obviously that $\widetilde{C} \cong \Phi_m(\widetilde{C})$ by using the map $(p(x)|0) \mapsto p(x)$. There are $\Phi_m(\widetilde{C}) \in \mathcal{C}_m(R)$ and $C_n = \Phi_n(C) \in \mathcal{C}_n(R)$ from the setting of the coordinate projections. Due to the results about cyclic codes over R (see Lemmas 1, 2, 3 in Section 2.2), write $\Phi_m(\widetilde{C}) = \langle \xi(x) \rangle$, in which $\xi(x) = \xi_v(x)v + \xi_w(x)w$ such that $\xi_v(x)|(x^m-1), \xi_w(x)|(x^m-1)$, and $\Phi_n(C) = C_n = \langle o(x) \rangle$, where $o(x) \in R[x]$ with $o(x) = o_v(x)v + o_w(x)w$ such that $o_v(x)|(x^n-1), o_w(x)|(x^n-1)$. Hence, $(\xi(x)|0)$ is a generator polynomial of $\widetilde{C}$ and there exists $\ell(x) \in R_m[x]$ such that $(\ell(x)|o(x)) \in C$.

It remains to prove that $C = \langle (\xi(x)|0), (\ell(x)|o(x)) \rangle$.

Obviously, $q(x) = \Phi_n((p(x)|q(x))) \in \Phi_n(C)$ holds for any $(p(x)|q(x)) \in C$. Hence, there exists $\nu(x) \in R_n[x]$ such that $q(x) = \nu(x)o(x)$. Then,

$$(p(x) \mid q(x)) - \nu(x)(\ell(x)|o(x)) = (p(x) - \nu(x)\ell(x)|0) \in \widetilde{C},$$

which implies that there exists $\mu(x) \in R_m[x]$ such that $(p(x) - \nu(x)\ell(x)|0) = \mu(x)(\xi(x)|0)$. Thus $(p(x)|q(x)) = \mu(x)(\xi(x)|0) + \nu(x)(\ell(x)|o(x))$. It is sufficient to show that C is finite generated by $\{(\xi(x)|0), (\ell(x)|o(x))\}$. □

Remark 6. *From the process of the above proof, it is easy to see that* C_m *is generated by the polynomial* $\gcd(\xi(x), \ell(x))$ *and* C_n *is generated by* $o(x)$.

There are three propositions about $\ell(x) = \ell_v(x)v + \ell_w(x)w$.

Proposition 3. *Let* $C = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle \in \mathcal{C}_{m,n}(R)$. *As the minimal forms of generating polynomials, it has*

$$\deg(\ell_v(x)) < \deg(\xi_v(x)) \text{ and } \deg(\ell_w(x)) < \deg(\xi_w(x)).$$

Proof. Otherwise, $\deg(\ell_v(x)) \geq \deg(\xi_v(x))$ or $\deg(\ell_w(x)) \geq \deg(\xi_w(x))$. Without loss of generality, let $\deg(\ell_v(x)) \geq \deg(\xi_v(x))$. Set $i = \deg(\ell_v(x)) - \deg(\xi_(x))$, $i \geq 0$ and let D be the code generated by $\{(\xi_v(x)v + \xi_w(x)w|0), ((\ell_v(x) - x^i\xi_v(x))v + \ell_w(x)w|o_v(x)v + o_w(x)w)\}$. It is obvious that $\deg((\ell_v(x) - x^i\xi_v(x)) < \deg(\ell_v(x))$. Since the generators of D belong to C, $D \subset C$. On the other hand,

$$\begin{aligned}&(\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) = \\ &((\ell_v(x) - x^i\xi_v(x))v + \ell_w(x)w|o_v(x)v + o_w(x)w) + vx^i(\xi_v(x)v + \xi_w(x)w|0).\end{aligned}$$

Then, $(\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \subset D$. This shows that $C \subset D$. Consequently, $D = C$. Repeating the above process, the desired results will be obtained. □

Proposition 4. *Let* C *be a double cyclic code of length* (m, n) *over* R, *and set*

$$C = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle .$$

Then, $\xi_v(x)|\frac{x^n-1}{o_v(x)}\ell_v(x)$ *and* $\xi_w(x)|\frac{x^n-1}{o_w(x)}\ell_w(x)$.

Proof. From the setting about coordinate projective homomorphism of $R[x]$-module defined by $\Phi_n|_C$: $C \to R_n[x] \mid (p(x)|q(x)) \mapsto q(x)$, it is easy to verify that $\mathrm{Ker}(\Phi_n|_C) = \langle (\xi(x)|0) \rangle$. Focus on the codewords of $\frac{x^n-1}{o(x)}(\ell(x)|o(x))$ now.

Since $\frac{x^n-1}{o(x)}(\ell(x)|o(x)) = (\frac{x^n-1}{o(x)}\ell(x)|0) \in \mathrm{Ker}(\Phi_n|_C)$, one has $\xi(x)|\frac{x^n-1}{o(x)}\ell(x)$. It follows from Proposition 1 that $\xi_v(x)|\frac{x^n-1}{o_v(x)}\ell_v(x)$ and $\xi_w(x)|\frac{x^n-1}{o_w(x)}\ell_w(x)$. □

Corollary 1. *Let C be a double cyclic code of length (m,n) over R and let*

$$C = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle .$$

Then, $\xi_v(x)|\frac{x^n-1}{o_v(x)}\gcd(\xi_v(x),\ell_v(x))$ and $\xi_w(x)|\frac{x^n-1}{o_w(x)}\gcd(\xi_w(x),\ell_w(x))$.

Proof. By Theorem 1, someone has $o_v(x)|x^n-1$ and $o_w(x)|x^n-1$, which shows $\xi_v(x)|\frac{x^n-1}{o_v(x)}\xi_v(x)$ and $\xi_w(x)|\frac{x^n-1}{o_w(x)}\xi_w(x)$. From Proposition 4, it follows that $\xi_v(x)|\frac{x^n-1}{o_v(x)}\ell_v(x)$ and $\xi_w(x)|\frac{x^n-1}{o_w(x)}\ell_w(x)$. Hence, $\xi_v(x)|\frac{x^n-1}{o_v(x)}\gcd(\xi_v(x),\ell_v(x))$ and $\xi_w(x)|\frac{x^n-1}{o_w(x)}\gcd(\xi_w(x),\ell_w(x))$. □

Proposition 5. *If $C = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle$ is a separable R-double cyclic code, then $\ell_v(x) = \ell_w(x) = 0$.*

Proof. Combining the definition of separable R-double cyclic codes with Theorem 1, one can derive the desired results. □

3.2. Generating Set Forms

Proposition 6. *Let C be a double cyclic code of length (m,n) over R with*

$$C = \langle ((\xi(x) \mid 0), (\ell(x)|o(x)) \rangle = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle .$$

Define the sets

$$\begin{aligned}
G_1^v &= \{v(\xi(x)|0), vx(\xi(x)|0), \ldots, vx^{m-\deg(\xi_v(x))-1}(\xi(x)|0)\} \\
&= \{(\xi_v(x)v|0), (x\xi_v(x)v|0), \ldots, (x^{m-\deg(\xi_v(x))-1}\xi_v(x)v|0)\}, \\
G_1^w &= \{w(\xi(x)|0), wx(\xi(x)|0), \ldots, wx^{m-\deg(\xi_w(x))-1}(\xi(x)|0)\} \\
&= \{(\xi_w(x)w|0), (x\xi_w(x)w|0), \ldots, (x^{m-\deg(\xi_w(x))-1}\xi_w(x)w|0)\}, \\
G_2^v &= \{v(\ell(x)|o(x)), vx(\ell(x)|o(x)), \ldots, vx^{n-\deg(o_v(x))-1}(\ell(x)|o(x))\} \\
&= \{(\ell_v(x)v|o_v(x)v), \ldots, (x^{n-\deg(o_v(x))-1}\ell_v(x)v|x^{n-\deg(o_v(x))-1}o_v(x)v)\}, \\
G_2^w &= \{w(\ell(x)|o(x)), wx(\ell(x)|o(x)), \ldots, wx^{n-\deg(o_w(x))-1}(\ell(x)|o(x))\} \\
&= \{(\ell_w(x)w|o_w(x)w), \ldots, (x^{n-\deg(o_w(x))-1}\ell_w(x)w|x^{n-\deg(o_w(x))-1}o_w(x)w)\}.
\end{aligned}$$

Then, $G_1^v \cup G_1^w \cup G_2^v \cup G_2^w$ forms a minimal generating set for C as a $\mathbb{F}_q$-vector space.

Proof. It is obvious that the codewords of $G_1^v \cup G_1^w \cup G_2^v \cup G_2^w$ are $\mathbb{F}_q$-linear independent. For $c(x) \in C$, let $c(x) = p(x)(\xi(x)|0) + q(x)(\ell(x)|o(x))$ with $p(x), q(x) \in R[x]$. Then, $c(x) \in \langle G_1^v \cup G_1^w \cup G_2^v \cup G_2^w \rangle_{\mathbb{F}_q}$ will be verified in the following discussion.

If $\deg(p_v(x)) \leq m - \deg(\xi_v(x)) - 1$ and $\deg(p_w(x)) \leq m - \deg(\xi_w(x)) - 1$, then

$$p(x)(\xi(x)|0) = (p_v(x)v + p_w(x)w)(\xi(x)|0) \in \langle G_1^v \cup G_1^w \rangle_{\mathbb{F}_q} .$$

Conversely, suppose that $\deg(p_v(x)) > m - \deg(\xi_v(x)) - 1$ or $\deg(p_w(x)) > m - \deg(\xi_w(x)) - 1$. Without loss of generality, let $\deg(p_v(x)) > m - \deg(\xi_v(x)) - 1$. Applying the Division Algorithm over $\mathbb{F}_q[x]$, consider $p_v(x) = \tilde{p}_v(x)\frac{x^m-1}{\xi_v(x)} + \tilde{\tilde{p}}_v(x)$ with $\deg(\tilde{\tilde{p}}_v(x)) \leq m - \deg(\xi_v(x)) - 1$. Then

$$\begin{aligned} p(x)(\xi(x)|0) &= (p_v(x)v + p_w(x)w)(\xi_v(x)v + \xi_w(x)w|0) \\ &= ((\tilde{p}_v(x)\frac{x^m-1}{\xi_v(x)} + \tilde{\tilde{p}}_v(x))v + p_w(x)w)(\xi_v(x)v + \xi_w(x)w|0) \\ &= (\tilde{\tilde{p}}_v(x)v + p_w(x)w)(\xi_v(x)v + \xi_w(x)w|0) \in \langle G_1^v \cup G_1^w \rangle_{\mathbb{F}_q} \end{aligned}$$

The statement $c(x) \in \langle G_1^v \cup G_1^w \cup G_2^v \cup G_2^w \rangle_{\mathbb{F}_q}$ will be proved once someone illustrates that $q(x)(\ell(x)|o(x)) \in \langle G_1^v \cup G_1^w \cup G_2^v \cup G_2^w \rangle_{\mathbb{F}_q}$.

If $\deg(q_v(x)) \leq n - \deg(o_v(x)) - 1$ and $\deg(q_w(x)) \leq n - \deg(o_w(x)) - 1$, then $q(x)(\ell(x)|o(x)) \in \langle G_1^v \cup G_1^w \cup G_2^v \cup G_2^w \rangle_{\mathbb{F}_q}$. Otherwise, assume that $\deg(q_v(x)) > n - \deg(o_v(x)) - 1$. Using the Division with Remainder similarly, let $q_v(x) = \tilde{q}_v(x)\frac{x^n-1}{o_v(x)} + \tilde{\tilde{q}}_v(x)$ with $\deg(\tilde{\tilde{q}}_v(x)) \leq n - \deg(o_v(x)) - 1$. Hence,

$$\begin{aligned} q(x)(\ell(x) \mid o(x)) &= (q_v(x)v + q_w(x)w)(\ell(x)|o(x)) \\ &= ((\tilde{q}_v(x)\frac{x^n-1}{o_v(x)} + \tilde{\tilde{q}}_v(x))v + q_w(x)w)(\ell(x)|o(x)) \\ &= \tilde{q}_v(x)\frac{x^n-1}{o_v(x)}v(\ell(x)|o(x)) + (\tilde{\tilde{q}}_v(x)v + q_w(x)w)(\ell(x)|o(x)). \end{aligned}$$

On the one hand, $(\tilde{\tilde{q}}_v(x)v + q_w(x)w)(\ell(x)|o(x)) \in \langle G_2^v \cup G_2^w \rangle_{\mathbb{F}_q}$. On the other hand,

$$\begin{aligned} \tilde{q}_v(x)\frac{x^n-1}{o_v(x)}v(\ell(x)|o(x)) &= \tilde{q}_v(x)\frac{x^n-1}{o_v(x)}v(\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \\ &= (\tilde{q}_v(x)\frac{x^n-1}{o_v(x)}\ell_v(x)v|\tilde{q}_v(x)\frac{x^n-1}{o_v(x)}o_v(x)v) \\ &= (\tilde{q}_v(x)\frac{x^n-1}{o_v(x)}\ell_v(x)v|0). \end{aligned}$$

From Proposition 4, it follows that $\xi_v(x)|\frac{x^n-1}{o_v(x)}\ell_v(x)$ which leads to $(\tilde{q}_v(x)\frac{x^n-1}{o_v(x)}\ell_v(x)v|0) \in \langle G_1^v \rangle_{\mathbb{F}_q}$. Therefore, the desired results follow. □

3.3. Generating Matrix Forms

Through the generating polynomials of C, it is not difficult to gain the generating matrix forms of C in the following theorem.

Theorem 2. *Let $C = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle \in \mathcal{C}_{m,n}(R)$. Then, C is permutation equivalent to an $\mathbb{F}_q$-linear code with generating matrix*

$$G = \left(\begin{array}{ccc|ccc} I_{m-\deg(\xi_v(x))}v & \dot{A}_v v & \ddot{A}_v v & 0 & 0 & 0 \\ 0 & \dot{B}_v v & \ddot{B}_v v & \dddot{B}_v v & I_{k_v}v & 0 \\ 0 & 0 & 0 & \dot{M}_v v & \ddot{M}_v v & I_{n-\deg(\xi_v(x))-k_v}v \\ I_{m-\deg(\xi_w(x))}w & \dot{A}_w w & \ddot{A}_w w & 0 & 0 & 0 \\ 0 & \dot{B}_w w & \ddot{B}_w w & \dddot{B}_w w & I_{k_w}w & 0 \\ 0 & 0 & 0 & \dot{M}_v v & \ddot{M}_w w & I_{n-\deg(\xi_w(x))-k_w}w \end{array} \right),$$

where $k_v = \deg(\xi_v(x)) - \deg(\gcd(\xi_v(x), \ell_v(x)))$ and $k_w = \deg(\xi_w(x)) - \deg(\gcd(\xi_w(x), \ell_w(x)))$ are two integers with the subscript.

Proof. Due to the fact in Proposition 6, C is generated by the matrix whose rows are the elements of the set $G_1^v \cup G_1^w \cup G_1^v \cup G_1^w$. Note that $m - \deg(\xi_v(x))$ and $n - \deg(o_v(x))$ are the dimensions of the

matrices generated by the shifts of $\xi_v(x)$ and $o_v(x)$, respectively. Then, the generating matrix of the code C is permutation equivalent to the following matrix

$$\left(\begin{array}{cc|cc} I_{m-\deg(\xi_v(x))}v & A_v v & 0 & 0 \\ 0 & B_v v & X_v v & I_{n-\deg(o_v(x))}v \\ I_{m-\deg(\xi_w(x))}w & A_w w & 0 & 0 \\ 0 & B_w w & X_w w & I_{n-\deg(o_w(x))}v \end{array}\right).$$

It is clear that $(C_m)_v$ is a conventional cyclic code generated by $\gcd(\xi_v(x), \ell_v(x))$. Then, the submatrix B_v has rank $k_v = \deg(\xi_v(x) - \deg(g(\xi_v(x), \ell_v(x))))$. Obviously, the same reason applies to the submatrix of B_w. Moreover, the generating matrix of C_m is permutation equivalent to the matrix

$$\begin{pmatrix} I_{m-\deg(\xi_v(x))}v & \dot{A}_v v & \ddot{A}_v v \\ 0 & \dot{B}_v v & \ddot{B}_v v \\ 0 & 0 & 0 \\ I_{m-\deg(\xi_w(x))}w & \dot{A}_w w & \ddot{A}_w w \\ 0 & \dot{B}_w w & \ddot{B}_w w \\ 0 & 0 & 0 \end{pmatrix},$$

where $\dot{B}_v$ is a full rank square matrix of size $k_v \times k_v$ and $\dot{B}_w$ is a full rank square matrix of size $k_w \times k_w$. Applying the convenient permutations and linear combinations, C is permutation equivalent to a $\mathbb{F}_q$-linear code with the above generating matrix in the end. □

From the generating matrix of the R-double cyclic code, it is easy to prove that

Corollary 2. *Let* $C = \langle(\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w)\rangle \in \mathcal{C}_{m,n}(R)$. *Then*, C *is a* $\mathbb{F}_q$*-linear code of dimension* $2m + 2n - \deg(\xi_v(x)) - \deg(o_v(x)) - \deg(\xi_w(x)) - \deg(o_w(x))$.

Consider some examples as the end of this section.

Example 1. *Let* $\mathbb{F}_q = \mathbb{F}_2$, $m = 7, n = 7$,

$$\begin{cases} \xi(x) &= vx^6 + vx^5 + x^4 + x^3 + x^2 + vx + 1 \\ &= (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)v + (x^4 + x^3 + x^2 + 1)w, \\ \ell(x) &= x^3 + vx^2 + (1 + v)x + 1 = (x^3 + x^2 + 1)v + (x^3 + x + 1)w, \\ o(x) &= vx^4 + (1 + v)x^3 + x^2 + vx + 1 = (x^4 + x^2 + x + 1)v + (x^3 + x^2 + 1)w. \end{cases}$$

Therefore,

$$C = \langle(\xi(x)|0), (\ell(x)|o(x))\rangle = \langle(\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w)\rangle$$

is an $\mathbb{F}_2 + v\mathbb{F}_2$*-double cyclic code. According to Proposition 6, the minimal generating set of* C *is* $G_1^v \cup G_1^w \cup G_2^v \cup G_2^w$, *where*

$$\begin{cases} G_1^v = \left\{v(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1|0)\right\}, \\ G_1^w = \left\{w(x^4 + x^3 + x^2 + 1|0), wx(x^4 + x^3 + x^2 + 1|0), wx^2(x^4 + x^3 + x^2 + 1|0)\right\}, \\ G_2^v = \Big\{v(x^3 + x^2 + 1|x^4 + x^2 + x + 1), vx(x^3 + x^2 + 1|x^4 + x^2 + x + 1), \\ \qquad vx^2(x^3 + x^2 + 1|x^4 + x^2 + x + 1)\Big\}, \\ G_2^w = \Big\{w(x^3 + x + 1|x^3 + x^2 + 1), wx(x^3 + x + 1|x^3 + x^2 + 1), \\ \qquad wx^2(x^3 + x + 1|x^3 + x^2 + 1), wx^3(x^3 + x + 1|x^3 + x^2 + 1)\Big\}. \end{cases}$$

Hence, the generating matrix of C is

$$\left(\begin{array}{ccccccc|ccccccc} v & v & v & v & v & v & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & v & v & 0 & 0 & 0 & v & v & v & 0 & v & 0 & 0 \\ 0 & v & 0 & v & v & 0 & 0 & 0 & v & v & v & 0 & v & 0 \\ 0 & 0 & v & 0 & v & v & 0 & 0 & 0 & v & v & v & 0 & v \\ w & 0 & w & w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & w & w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & w & w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & w & 0 & w & 0 & 0 & 0 & w & 0 & w & w & 0 & 0 & 0 \\ 0 & w & w & 0 & w & 0 & 0 & 0 & w & 0 & w & w & 0 & 0 \\ 0 & 0 & w & w & 0 & w & 0 & 0 & 0 & w & 0 & w & w & 0 \\ 0 & 0 & 0 & w & w & 0 & w & 0 & 0 & 0 & w & 0 & w & w \end{array}\right).$$

This means that the generating matrices of $P_v(C)$, $P_w(C)$ are

$$\left(\begin{array}{ccccccc|ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right), \left(\begin{array}{ccccccc|ccccccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right),$$

respectively. Then,

$$\begin{cases} \text{the length of } P_v(C) \text{ is } 14,\ \dim_{\mathbb{F}_2}(P_v(C)) = 4,\ d_{min}(P_v(C)) = 7; \\ \text{the length of } P_w(C) \text{ is } 14,\ \dim_{\mathbb{F}_2}(P_w(C)) = 7,\ d_{min}(P_w(C)) = 4. \end{cases}$$

Consequently, the codes $P_v(C)$ and $P_w(C)$ have parameters $[14, 4, 7]$ and $[14, 7, 4]$, respectively. Both of them are optimal from the table in [16] and the $\mathbb{F}_2 + v\mathbb{F}_2$-double cyclic code C has the paramter of $[28, 11, 4]$.

Example 2. *Let $\mathbb{F}_q = \mathbb{F}_2$, $m = 7, n = 14$ and*

$$\begin{cases} \xi(x) &= vx^7 + (1+v)x^6 + (1+v)x^5 + (1+v)x^4 + (1+v)x^3 + (1+v)x^2 + (1+v)x + 1 \\ &= (x^7+1)v + (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)w, \\ \ell(x) &= (1+v)x^4 + x^3 + vx + 1 \\ &= (x^3 + x + 1)v + (x^4 + x^3 + 1)w, \\ o(x) &= vx^9 + vx^8 + vx^6 + x^5 + vx^4 + vx^3 + (1+v)x^2 + (1+v)x + 1 \\ &= (x^9 + x^8 + x^6 + x^5 + x^4 + x^3 + 1)v + (x^5 + x^2 + x + 1)w. \end{cases}$$

As in Example 1, $P_v(C)$, $P_w(C)$, C have the parameters of $[21, 5, 10]$, $[21, 10, 7]$, $[42, 15, 7]$, respectively.

4. The Dual Codes

As the generalization of cyclic codes over R, one can obtain the following results about the dual codes of cyclic codes over R. For more information, someone can consult [15].

Lemma 4. *([15]) Let $C = vC_1 \oplus (1-v)C_2$ be a cyclic code of length n over R, then its dual code $C^\perp$ is also a cyclic and moreover we have $C^\perp = vC_1^\perp \oplus (1-v)C_2^\perp$.*

Lemma 5. *([15]) Let $C = \langle vf_1(x), (1-v)f_2(x)\rangle$ be a cyclic code of length n over R, with $f_1(x)$ and $f_2(x)$ as the generator polynomials of C_1 and C_2, respectively such that $x^n - 1 = f_1(x)h_1(x)$ and $x^n - 1 = f_2(x)h_2(x)$. Then*

(i) $C^\perp = \langle vh_1^*(x), (1-v)h_2^*(x)\rangle$ *and* $|C^\perp| = q^{\deg f_1(x) + \deg f_2(x)}$,
(ii) $C^\perp = \langle h(x)\rangle$ *where* $h(x) = vh_1^*(x) + (1-v)h_2^*(x)$.

Lemma 6. *([15]) Let C_1 and C_2 be two linear codes of length n over $\mathbb{F}_q$ and*

$$C = vC_1 \oplus (1-v)C_2 = \{vc_1 + (1-v)c_2 \mid c_1 \in C_1, c_2 \in C_2\}.$$

We have

$$C^\perp = vC_1^\perp \oplus (1-v)C_2^\perp = \left\{(vc_1 + (1-v)c_2), c_1 \in C_1^\perp, c_2 \in C_2^\perp\right\}.$$

4.1. Background Knowledge

Motivated by the idea in [10]: some new definitions will be introduced as follows.

Definition 3. *Let C be a double cyclic code of length (m, n) over R. Define*

$$\langle c, d\rangle = \sum_{i=0}^{m-1} c_i' d_i' + \sum_{j=0}^{n-1} c_j'' d_j'',$$

where $c = (c_0', \ldots, c_{m-1}' | c_0'', \ldots, c_{n-1}'')$, $d = (d_0', \ldots, d_{m-1}' | d_0'', \ldots, d_{n-1}'')$.

Similarly, give the following definition.

Definition 4. *For $C \in \mathcal{C}_{m,n}(R)$, define $C^\perp = \{d \in R^{m+n} \mid \langle d, c\rangle = 0\ \forall c \in C\}$ as its dual code.*

For $r(x) \in R[x]$, let $r(x) = r_v(x)v + r_w(x)w$ with $r_v(x), r_w(x) \in \mathbb{F}_q[x]$. Inspired by the results in [10],

Definition 5. *Define the monic reciprocal polynomial of $r(x)$ as*

$$r^*(x) = r_v^*(x)v + r_w^*(x)w = (r_v(0))^{-1}x^{\deg(r_v(x))}r_v(x^{-1})v + (r_w(0))^{-1}x^{\deg(r_w(x))}r_w(x^{-1})w.$$

Remark 7. *Obviously, $r^*(x) \in R[x]$ and Definition 5 generalizes the trivial case for $v = 0$ or $v = 1$. Good results for the composite operation between it and the canonical projections are that $(r^*(x))_v = (r_v(x))^*$ and $(r^*(x))_w = (r_w(x))^*$. Based on these facts, the polynomials $r_v^*(x)$ and $r_w^*(x)$ can be expressed explicitly.*

Proposition 7. *Let $r(x), s(x) \in R[x]$ with $s(x)|r(x)$. Then, $(\frac{r(x)}{s(x)})^* = \frac{r^*(x)}{s^*(x)}$.*

Proof. Firstly, it is easy to verify that $(f(x)g(x))^* = f^*(x)g^*(x)$, $f(x), g(x) \in \mathbb{F}_q[x]$. Secondly, the polynomial can be decompose into a combination of $\{v, w\}$ over $\mathbb{F}_q[x]$, which translates into the proof of the polynomial over traditional finite fields. These finish the proof. □

Remark 8. *Like as the case of finite fields, there is $r^{**}(x) = r(x)$ for each $r(x) \in R[x]$. In the following sections, Proposition 1, 2 and 7 will be used directly without explanation.*

Based on the extended inner product forms, the following significant theorem is obtained.

Theorem 3. *Let C be a double cyclic code of length (m, n) over R with*

$$C = \langle(\xi(x)|0), (0|o(x))\rangle = \langle(\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w)\rangle.$$

Then, $C^{\perp} \in \mathcal{C}_{m,n}(R)$ and set

$$C^{\perp} = \left\langle (\overline{\xi}(x)|0), (\overline{\ell}(x)|\overline{o}(x)) \right\rangle = \left\langle (\overline{\xi}_v(x)v + \overline{\xi}_w(x)w|0), (\overline{\ell}_v(x)v + \overline{\ell}_w(x)w|\overline{o}_v(x)v + \overline{o}_w(x)w) \right\rangle.$$

Proof. First of all, let us $C^{\perp} \in \mathcal{C}_{m,n}(R)$. Set $d = (d'_0, \ldots, d'_{m-1}|d''_0, \ldots, d''_{n-1}) \in C^{\perp}$. From the definition to R-double cyclic codes, it must be proved that $\tau(d) \in C^{\perp}$ in the following. For any codeword $c \in C$, it just needs to show $\langle \tau(d), c \rangle = 0$. Notice that $\tau^l(c) = c$, where $l = lcm(m,n)$. Since C is an R-double cyclic code, which implies that $\tau^{l-1}(c) = \tau^{l-2}\tau(c) \in C$. Taking $c = (c'_0, \ldots, c'_{m-1}|c''_0, \ldots, c''_{n-1})$ and analyzing the detail of $\tau^{l-1}(c)$, this gives $\tau^{l-1}(c) = (c'_1, \ldots, c'_{m-1}, c'_0|c''_1, \ldots, c''_{n-1}, c''_0)$. Since $d \in C^{\perp}$, $c \in C$,

$$0 = \left\langle d, \tau^{l-1}(c) \right\rangle = d'_0c'_1 + \cdots + d'_{m-2}c'_0 + d'_{m-1}c'_0 + d''_0c''_1 + \cdots + d''_{m-2}c''_0 + d''_{m-1}c''_0 = \langle \tau(d), c \rangle$$

shows that $\tau(d) \in C^{\perp}$. Therefore, $C^{\perp}$ is also an R-double cyclic code of length (m,n). It follows from Theorem 1 that the second results of Theorem 3 is yielded. □

Corollary 3. *Just like as the station to Proposition 5, let*

$$C = \langle (\xi_v(x)v + \xi_w(x)w|0), (0|o_v(x)v + o_w(x)w) \rangle$$

be a separable double cyclic code of length (m,n) over R. Then, $C^{\perp}$ is also a separable double cyclic code over R and

$$C^{\perp} = \left\langle (\frac{x^m-1}{\xi^*(x)}|0), (0|\frac{x^n-1}{o^*(x)}) \right\rangle = \left\langle (\frac{x^m-1}{\xi^*_v(x)}v + \frac{x^m-1}{\xi^*_w(x)}w|0), (0|\frac{x^n-1}{o^*_v(x)}v + \frac{x^n-1}{o^*_w(x)}w) \right\rangle.$$

Proof. If C is separable, then $C = C_m \times C_n$. Thus, it is easy to verify that $C^{\perp} = C_m^{\perp} \times C_n^{\perp}$. By the related results about cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$ (see Lemmas 4, 5, 6), one can acquire that

$$C^{\perp} = \left\langle (\frac{x^m-1}{\xi^*(x)}|0), (0|\frac{x^n-1}{o^*(x)}) \right\rangle = \left\langle (\frac{x^m-1}{\xi^*_v(x)}v + \frac{x^m-1}{\xi^*_w(x)}w|0), (0|\frac{x^n-1}{o^*_v(x)}v + \frac{x^n-1}{o^*_w(x)}w) \right\rangle.$$

□

4.2. Some Propositions

Let $\theta_m(x)$ represent the polynomial $\sum_{i=0}^{m-1} x^i$. Using this symbol, the following proposition holds.

Proposition 8. *Let $m, n \in \mathbb{N}$, then $x^{mn} - 1 = (x^m - 1)\theta_n(x^m)$.*

Proof. It is obvious that $t^n - 1 = (t-1)\theta_n(t)$. The desired result is achieved by replacing t with x^m. □

From now on, l denotes the least common multiple of m and n.

Definition 6. *Let* $\begin{cases} c(x) = (c'_v(x)v + c'_w(x)w|c'_v(x)v + c''_w(x)w), \\ d(x) = (d'_v(x)v + d'_w(x)w|d''_v(x)v + d''_w(x)w), \end{cases}$ *be two elements in $R_{m,n}[x]$. Define the map $\circ : R_{m,n}[x] \times R_{m,n}[x] \to R_l[x]$ with*

$$\begin{aligned} \circ(c(x), d(x)) = {} & (c'_v(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_v(x))}d'^*_v(x) + c''_v(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d''_v(x))}d''^*_v(x))v + \\ & (c'_w(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_w(x))}d'^*_w(x) + c''_w(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d''_w(x))}d''^*_w(x))w \mod (x^l - 1). \end{aligned}$$

It is easy to verify that the map $\circ$ is linear in each of its arguments. Then, $\circ$ is a bilinear map between the two $R[x]$-modules. For the convenience of writing, denote $\circ(c(x), d(x))$ by $c(x) \circ d(x)$.

Proposition 9. *Let* $c = (c'_0, \ldots, c'_{m-1} | c''_0, \ldots, c''_{n-1}), d = (d'_0, \ldots, d'_{m-1} | d''_0, \ldots, d''_{n-1})$ *be two vectors in* $R^m \times R^n$ *with associated polynomials*

$$c(x) = (c'_v(x)v + c'_w(x)w | c''_v(x)v + c''_w(x)w), d(x) = (d'_v(x)v + d'_w(x)w | d''_v(x)v + d''_w(x)w),$$

respectively. Then, c is orthogonal to d and all of its shifts if and only if $c(x) \circ d(x) \equiv 0$.

Proof. Denote by $d^{(s)} = (d'_{0+s}, \ldots, d'_{m-1+s} \mid d''_{0+s}, \ldots, d''_{n-1+s})$ the s-th cyclic shift of vector d, where $0 \leq s \leq l-1$. By the definition of inner product, $\left\langle c, d^{(s)} \right\rangle = 0$ if and only if $\sum_{k_1=0}^{m-1} c'_{k_1} d'_{k_1+s} + \sum_{k_2=0}^{n-1} c''_{k_2} d''_{k_2+s} = 0$. Set $\Delta_s = \sum_{k_1=0}^{m-1} c'_{k_1} d'_{k_1+s} + \sum_{k_2=0}^{n-1} c''_{k_2} d''_{k_2+s}$, one can check that

$$\begin{aligned} c(x) \circ d(x) &= \sum_{i=0}^{m-1} (\theta_{\frac{l}{m}}(x^m) \sum_{k_1=0}^{m-1} c'_{k_1} d'_{k_1+i} x^{l-1-i}) + \sum_{j=0}^{n-1} (\theta_{\frac{l}{n}}(x^n) \sum_{k_2=0}^{n-1} c''_{k_2} d''_{k_2+j} x^{l-1-j}) \\ &= (\theta_{\frac{l}{m}}(x^m))[\sum_{i=0}^{m-1} \sum_{k_1=0}^{m-1} c'_{k_1} d'_{k_1+i} x^{l-1-i}] + (\theta_{\frac{l}{n}}(x^n)[\sum_{j=0}^{m-1} \sum_{k_2=0}^{n-1} c''_{k_2} d''_{k_2+j} x^{l-1-j}] \\ &= \sum_{s=0}^{l-1} \Delta_s x^{l-1-s} \end{aligned}$$

in $R[x]/(x^l - 1)$. Thus, $c(x) \circ d(x) = 0$ if and only if $\Delta_s = 0$ for all $0 \leq s \leq l-1$. □

Proposition 10. *Let* $c(x) = (c'(x) | c''(x))$ *and* $d(x) = (d'(x) | d''(x))$ *be two elements in* $R_{m,n}[x]$ *such that* $c(x) \circ d(x) = 0$ *mod* $(x^l - 1)$. *Then:*

(i) If $c'(x) \equiv 0$ *or* $d'(x) \equiv 0$, *we have* $c''(x)d''^*(x) = 0 \mod (x^n - 1)$;
(ii) if $c''(x) \equiv 0$ *or* $d''(x) \equiv 0$, *we have* $c'(x)d'^*(x) = 0 \mod (x^m - 1)$.

Proof. This only prove case (ii), case (i) can be proved in a similar way. Let $c''(x)$ or $d''(x)$ equal to 0 module $x^n - 1$. This means that $c''_v(x) \equiv c''_w(x) \equiv 0$ or $d''_v(x) \equiv d''_w(x) \equiv 0$. From the specific definition of $\circ$ associated to each two elements in $R[x]$,

$$\begin{aligned} &c(x) \circ d(x) = (c'_v(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_v(x))}d'^*_v(x))v + (c'_w(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_w(x))}d'^*_w(x))w \\ &= 0 \mod (x^l - 1). \end{aligned}$$

Thus, there exists a polynomial $\delta(x) \in R[x]$ such that

$$(c'_v(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_v(x))}d'^*_v(x))v + (c'_w(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_w(x))}d'^*_w(x))w = \delta(x)(x^l - 1).$$

Let $\delta(x) = \delta_v(x)v + \delta_w(x)w$, then

$$\begin{aligned} &(c'_v(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_v(x))}d'^*_v(x))v + (c'_w(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_w(x))}d'^*_w(x))w \\ &= \delta_v(x)(x^l - 1)v + \delta_w(x)(x^l - 1)w. \end{aligned}$$

Further,

$$c'_v(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_v(x))}d'^*_v(x) = \delta_v(x)(x^l - 1), c'_w(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(d'_w(x))}d'^*_w(x) = \delta_w(x)(x^l - 1).$$

Let $\Lambda(x) = \Lambda_v(x)v + \Lambda_w(x)w$ with $\Lambda_v(x) = \delta_v(x)x^{\deg(d'_v(x))+1}, \Lambda_w(x) = \delta_w(x)x^{\deg(d'_w(x))+1}$, then

$$c'_v(x)\theta_{\frac{l}{m}}(x^m)x^l d'^*_v(x) = \Lambda_v(x)(x^l-1), c'_w(x)\theta_{\frac{l}{m}}(x^m)x^l d'^*_w(x) = \Lambda_w(x)(x^l-1).$$

While $x^l - 1 = \theta_{\frac{l}{m}}(x^m)(x^m-1)$ is obtained by Proposition 8. Hence, $c'_v(x)d'^*_v(x)x^l = \Lambda_v(x)(x^m - 1)$ and $c'_w(x)d'^*_w(x)x^l = \Lambda_w(x)(x^m-1)$. This means that $x^m-1|c'_v(x)d'^*_v(x)x^l$ and $x^m-1|c'_w(x)d'^*_w(x)x^l$. It is obvious that x^m-1 is prime to x^l, which yields that $x^m-1|c'_v(x)d'^*_v(x)$ and $x^m-1|c'_w(x)d'^*_w(x)$. Therefore, $c'_v(x)d'^*_v(x)v + c'_w(x)d'^*_w(x)w = c'(x)d'^*(x) = 0 \mod (x^m-1)$. □

4.3. Main Results

Proposition 11. *Let $C \in \mathcal{C}_{m,n}(R)$ with*

$$\begin{cases} C = \langle(\xi(x)|0), (\ell(x)|o(x))\rangle = \langle(\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w)\rangle, \\ C^\perp = \left\langle(\overline{\xi}(x)|0), (\overline{\ell}(x)|\overline{o}(x))\right\rangle = \left\langle(\overline{\xi}_v(x)v + \overline{\xi}_w(x)w|0), (\overline{\ell}_v(x)v + \overline{\ell}_w(x)w|\overline{o}_v(x)v + \overline{o}_w(x)w)\right\rangle. \end{cases}$$

Then,

$$\begin{cases} |C_m| = q^{2m+k_v+k_w-\deg(\xi_v(x))-\deg(\xi_w(x))}, & |C_n| = q^{2n-\deg(o_v(x))-\deg(o_w(x))}, \\ |(C^\perp)_m| = q^{\deg(o_v(x))+\deg(o_w(x))}, & |(C^\perp)_m| = q^{\deg(o_v(x))+\deg(o_w(x))+k_v+k_w}, \end{cases}$$

where $k_v = \deg(\xi_v(x)) - \deg(\gcd(\xi_v(x), \ell_v(x)))$ and $k_w = \deg(\xi_w(x)) - \deg(\gcd(\xi_w(x), \ell_w(x)))$ are two integers.

Proof. By the generating matrix of C in Theorem 2 and using the projection on the first m coordinates of it,

$$|C_m| = |(C_m)_v||(C_m)_w| = q^{m-\deg(\xi_v(x))+k_v+m-\deg(\xi_w(x))+k_w} = q^{2m+k_v+k_w-\deg(\xi_v(x))-\deg(\xi_w(x))}.$$

Similarly to C_n, it has$|C_n| = q^{2n-\deg(o_v(x))-\deg(o_w(x))}$.

Through the calculation, the parity check matrix of C can be obtained. It is

$$H = \left(\begin{array}{ccc|ccc} \dot{A}^t_v v & I_{k_v} v & 0 & 0 & \dot{B}^t_v v & \dot{B}^t_v \ddot{M}^t_v v \\ \ddot{A}^t_v v & 0 & I_{\deg(\xi_v(x))-k_v} v & 0 & \ddot{B}^t_v v & \ddot{B}^t_v \dot{M}^t_v v \\ 0 & 0 & 0 & I_{\deg(o_v(x))} & \dddot{B}^t_v v & (\dot{M}^t_v + \dddot{B}^t_v \ddot{M}^t_v)v \\ \dot{A}^t_w w & I_{k_w} w & 0 & 0 & \dot{B}^t_w w & \dot{B}^t_w \ddot{M}^t_w w \\ \ddot{A}^t_w w & 0 & I_{\deg(\xi_w(x))-k_w} w & 0 & \ddot{B}^t_w w & \ddot{B}^t_w \dot{M}^t_w w \\ 0 & 0 & 0 & I_{\deg(o_w(x))} & \dddot{B}^t_w w & (\dot{M}^t_w + \dddot{B}^t_w \ddot{M}^t_w)w \end{array}\right).$$

Utilizing the relationship between cyclic codes and their dual codes about the generating matrix, one can use the same method as above to derive the following results:

$$\begin{cases} |(C^\perp)_m| = |((C^\perp)_m)_v||((C^\perp)_m)_w| = q^{\deg(o_v(x))+\deg(o_w(x))}, \\ |(C^\perp)_n| = |((C^\perp)_n)_v||((C^\perp)_n)_w| = q^{\deg(o_v(x))+\deg(o_w(x))+k_v+k_w}. \end{cases}$$

□

Remark 9. *Similarly to the cardinality relationship between cyclic codes and their dual codes over traditional finite fields, $|C||C^\perp| = q^{2n}$ for $C \in \mathcal{C}_n(R)$. Consequently, $|(C_m)^\perp| = q^{\deg(\xi_v(x))+\deg(\xi_w(x))-k_v-k_w}$ and $|(C_n)^\perp| = q^{\deg(o_v(x))+\deg(o_w(x))}$. Of course, someone can also refer to [13].*

Corollary 4. *Let C and $C^{\perp}$ be defined as above. Then,*

$$\begin{cases} \deg(\overline{\xi}_v(x)) = m - \deg(\gcd(\xi_v(x), \ell_v(x))), \\ \deg(\overline{\xi}_w(x)) = m - \deg(\gcd(\xi_w(x), \ell_w(x))), \\ \deg(\overline{o}_v(x)) = n - \deg(\xi_v(x)) - \deg(o_v(x)) + \deg(\gcd(\xi_v(x), \ell_v(x))), \\ \deg(\overline{o}_w(x)) = n - \deg(\xi_w(x)) - \deg(o_w(x)) + \deg(\gcd(\xi_w(x), \ell_w(x))). \end{cases}$$

Proof. It is easy to check that $(C_m)^{\perp}$ is a single cyclic code generated by $\overline{\xi}(x)$. From the conclusion about single cyclic codes over R, $|((C_m)^{\perp})_v| = q^{m-\deg(\xi_v(x))}$. Moreover, by Proposition 11, $|((C_m)^{\perp})_v| = q^{\deg(\xi_v(x))-k_v}$. Thus $\deg(\overline{\xi}_v(x)) = m - \deg(\gcd(\xi_v(x), \ell_v(x)))$. The same method can obtain that $\deg(\overline{\xi}_w(x)) = m - \deg(\gcd(\xi_w(x), \ell_w(x)))$.

Since $C^{\perp}$ is also an R-double cyclic code of the same length to C, $(C^{\perp})_m$ is a cyclic code generated by $\overline{o}(x)$. Thus, $|((C^{\perp})_m)_v| = q^{n-\deg(\overline{o}_v(x))}$. Moreover, by Proposition 8, $|((C^{\perp})_m)_v| = q^{\deg(o_v(x))+k_v}$. Hence, $\deg(\overline{o}_v(x)) = n - \deg(\xi_v(x)) - \deg(o_v(x)) + \deg(\gcd(\xi_v(x), \ell_v(x)))$. The same proof works for $\deg(\overline{\ell}_w(x)) = n - \deg(\xi_w(x)) - \deg(o_w(x)) + \deg(\gcd(\xi_w(x), \ell_w(x)))$. □

Theorem 4. *Let $C \in \mathcal{C}_{m,n}(R)$ with*

$$\begin{cases} C = \langle (\xi(x)|0), (\ell(x)|o(x)) \rangle = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle, \\ C^{\perp} = \left\langle (\overline{\xi}(x)|0), (\overline{\ell}(x)|\overline{o}(x)) \right\rangle = \left\langle (\overline{\xi}_v(x)v + \overline{\xi}_w(x)w|0), (\overline{\ell}_v(x)v + \overline{\ell}_w(x)w|\overline{o}_v(x)v + \overline{o}_w(x)w) \right\rangle. \end{cases}$$

Then, $\overline{\xi}(x) = \frac{x^m-1}{\gcd^*(\xi(x),\ell(x))} = \frac{x^m-1}{\gcd^*(\xi_v(x),\ell_v(x))}v + \frac{x^m-1}{\gcd^*(\xi_w(x),\ell_w(x))}w$.

Proof. By the generators to the dual code of C, $(\overline{\xi}(x)|0)$ belongs to $C^{\perp}$. According to Proposition 9,

$$\begin{cases} (\overline{\xi}(x)|0) \circ (\xi(x)|0) & = 0 \mod (x^l - 1), \\ (\overline{\xi}(x)|0) \circ (\ell(x)|o(x)) & = 0 \mod (x^l - 1). \end{cases}$$

Therefore, by Proposition 10,

$$\begin{cases} \overline{\xi}^*(x)\xi(x) & = 0 \mod (x^m - 1) \Longleftrightarrow (x^m - 1)|\overline{\xi}^*(x)\xi(x), \\ \overline{\xi}^*(x)\ell(x) & = 0 \mod (x^m - 1) \Longleftrightarrow (x^m - 1)|\overline{\xi}^*(x)\ell(x). \end{cases}$$

Furthermore, $x^m - 1|\overline{\xi}^*(x)\gcd(\xi(x), \ell(x))$. While $x^m - 1|\overline{\xi}^*(x)\gcd(\xi(x), \ell(x))$ if and only if

$$\begin{cases} x^m - 1|\overline{\xi}_v^*(x)\gcd_v(\xi(x), \ell(x)) = \overline{\xi}_v^*(x)\gcd(\xi_v(x), \ell_v(x)), \\ x^m - 1|\overline{\xi}_w^*(x)\gcd_w(\xi(x), \ell(x)) = \overline{\xi}_w^*(x)\gcd(\xi_w(x), \ell_w(x)). \end{cases}$$

Since $\overline{\xi}_v^*(x), \gcd(\xi_v(x), \ell_v(x)), \overline{\xi}_w^*(x), \gcd(\xi_w(x), \ell_w(x))$ are all factors of $x^m - 1$, it follows from Corollary 4 that

$$\begin{cases} \deg(\overline{\xi}_v^*(x)) = \deg(\overline{\xi}_v(x)) = m - \deg(\gcd(\xi_v(x), \ell_v(x))), \\ \deg(\overline{\xi}_w^*(x)) = \deg(\overline{\xi}_w(x)) = m - \deg(\gcd(\xi_w(x), \ell_w(x))). \end{cases}$$

Then,

$$\begin{cases} x^m - 1 = \overline{\xi}_v^*(x)\gcd_v(\xi(x), \ell(x)) = \overline{\xi}_v^*(x)\gcd(\xi_v(x), \ell_v(x)), \\ x^m - 1 = \overline{\xi}_w^*(x)\gcd_w(\xi(x), \ell(x)) = \overline{\xi}_w^*(x)\gcd(\xi_w(x), \ell_w(x)). \end{cases}$$

Hence,

$$\overline{\xi}^*(x)\gcd(\xi(x),\ell(x)) = \overline{\xi}_v^*(x)\gcd(\xi_v(x),\ell_v(x))v + \overline{\xi}_w^*(x)\gcd(\xi_w(x),\ell_w(x))w = x^m - 1.$$

Therefore, $\overline{\xi}(x) = \frac{x^m-1}{\gcd^*(\xi(x),\ell(x))} = \frac{x^m-1}{\gcd^*(\xi_v(x),\ell_v(x))}v + \frac{x^m-1}{\gcd^*(\xi_w(x),\ell_w(x))}w$. □

Theorem 5. *Let $C \in \mathcal{C}_{m,n}(R)$ with*

$$\begin{cases} C = \langle(\ell(x) \mid 0),(\ell(x)|o(x))\rangle = \langle(\xi_v(x)v + \xi_w(x)w|0),(\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w)\rangle, \\ C^\perp = \left\langle(\overline{\xi}(x)|0),(\overline{\ell}(x)|\overline{o}(x))\right\rangle = \left\langle(\overline{\xi}_v(x)v + \overline{\xi}_w(x)w|0),(\overline{\ell}_v(x)v + \overline{\ell}_w(x)w|\overline{o}_v(x)v + \overline{o}_w(x)w)\right\rangle. \end{cases}$$

Then, $\overline{o}(x) = \frac{(x^n-1)\gcd^*(\xi(x),\ell(x))}{\xi^*(x)o^*(x)} = \frac{(x^n-1)\gcd^*(\xi_v(x),\ell_v(x))}{\xi_v^*(x)o_v^*(x)}v + \frac{(x^n-1)\gcd^*(\xi_w(x),\ell_w(x))}{\xi_w^*(x)o_w^*(x)}w$.

Proof. Concentrate on the codeword of

$$\left(0|\frac{\xi(x)}{\gcd(\xi(x),\ell(x))}o(x)\right) = \frac{\xi(x)}{\gcd(\xi(x),\ell(x))}(\ell(x)|o(x)) - \frac{\ell(x)}{\gcd(\xi(x),\ell(x))}(\xi(x)|0).$$

Then, $\left(0|\frac{\xi(x)}{\gcd(\xi(x),\ell(x))}o(x)\right) \in C$. From Proposition 9,

$$(\overline{\ell}(x)|\overline{o}(x)) \circ \left(0|\frac{\xi(x)}{\gcd(\xi(x),\ell(x))}o(x)\right) = 0 \mod (x^l - 1).$$

Thus, by Proposition 10,

$$\overline{o}(x)\frac{\xi^*(x)o^*(x)}{\gcd^*(\xi(x),\ell(x))} = 0 \mod (x^n-1) \iff x^n - 1|\overline{o}(x)\frac{\xi^*(x)o^*(x)}{\gcd^*(\xi(x),\ell(x))}.$$

However, one can check that $x^n - 1|\overline{o}(x)\frac{\xi^*(x)o^*(x)}{\gcd^*(\xi(x),\ell(x))}$ if and only if

$$\begin{cases} x^n - 1|\overline{o}_v(x)\frac{\xi_v^*(x)o_v^*(x)}{\gcd_v^*(\xi(x),\ell(x))} = \overline{o}_v(x)\frac{\xi_v^*(x)o_v^*(x)}{\gcd^*(\xi_v(x),\ell_v(x))}, \\ x^n - 1|\overline{o}_w(x)\frac{\xi_w^*(x)o_w^*(x)}{\gcd_w^*(\xi(x),\ell(x))} = \overline{o}_w(x)\frac{\xi_w^*(x)o_w^*(x)}{\gcd^*(\xi_w(x),\ell_w(x))}. \end{cases}$$

$\overline{o}_v(x)|(x^n-1)$ and $\overline{o}_w(x)|(x^n-1)$ follow from Theorem 3. At the same time, from Corollary 1, $\frac{\xi_v^*(x)o_v^*(x)}{\gcd(\xi_v(x),\ell_v(x))}|(x^n-1)$, $\frac{\xi_w^*(x)o_w^*(x)}{\gcd(\xi_w(x),\ell_w(x))}|(x^n-1)$. By Corollary 4,

$$\begin{cases} \deg(\overline{o}_v(x)) = n - \deg(\xi_v(x)) - \deg(o_v(x)) + \deg(\gcd(\xi_v(x),\ell_v(x)), \\ \deg(\overline{o}_w(x)) = n - \deg(\xi_w(x)) - \deg(o_w(x)) + \deg(\gcd(\xi_w(x),\ell_w(x)). \end{cases}$$

Hence, $\deg(\frac{\overline{o}_v(x)\xi_v^*(x)o_v^*(x)}{\gcd^*(\xi_v(x),\ell_v(x))}) = n = \deg(x^n-1), \deg(\frac{\overline{o}_w(x)\xi_w^*(x)o_w^*(x)}{\gcd^*(\xi_w(x),\ell_w(x))}) = n = \deg(x^n-1)$. These mean that $x^n - 1 = \frac{\overline{o}_v(x)\xi_v^*(x)o_v^*(x)}{\gcd^*(\xi_v(x),\ell_v(x))}, x^n - 1 = \frac{\overline{o}_w(x)\xi_w^*(x)o_w^*(x)}{\gcd^*(\xi_w(x),\ell_w(x))}$. Therefore,

$$\begin{aligned} \overline{o}(x)\frac{\xi^*(x)o^*(x)}{\gcd^*(\xi(x),\ell(x))} &= \overline{o}_v(x)\frac{\xi_v^*(x)o_v^*(x)}{\gcd^*(\xi_v(x),\ell_v(x))}v + \overline{o}_w(x)\frac{\xi_w^*(x)o_w^*(x)}{\gcd^*(\xi_w(x),\ell_w(x))}w \\ &= (x^n-1)v + (x^n-1)w = (x^n-1)(v+w) = x^n - 1. \end{aligned}$$

Therefore, $\overline{o}(x) = \frac{(x^n-1)\gcd^*(\xi(x),\ell(x))}{\xi^*(x)o^*(x)} = \frac{(x^n-1)\gcd^*(\xi_v(x),\ell_v(x))}{\xi_v^*(x)o_v^*(x)}v + \frac{(x^n-1)\gcd^*(\xi_w(x),\ell_w(x))}{\xi_w^*(x)o_w^*(x)}w$. □

Remark 10. *In the process of the above proof, the fact that $deg(f^*(x)) = deg(f(x))$ holds for all $f(x) \in \mathbb{F}_q[x]$ has been applied.*

Theorem 6. *Let $C \in \mathcal{C}_{m,n}(R)$ with*

$$\begin{cases} C = \langle (\xi(x) \mid 0), (\ell(x)|o(x)) \rangle = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle, \\ C^{\perp} = \left\langle (\overline{\xi}(x)|0), (\overline{\ell}(x)|\overline{o}(x)) \right\rangle = \left\langle (\overline{\xi}_v(x)v + \overline{\xi}_w(x)w|0), (\overline{\ell}_v(x)v + \overline{\ell}_w(x)w|\overline{o}_v(x)v + \overline{o}_w(x)w) \right\rangle. \end{cases}$$

Then, $\overline{\ell}(x) = (\frac{x^m-1}{\xi_v^*(x)}v + \frac{x^m-1}{\xi_w^*(x)}w)\rho(x)$, where

$$\rho(x) = (-x^{l-\deg(o_v(x))+\deg(\xi_v(x))}v - x^{l-\deg(o_w(x))+\deg(\xi_w(x))}w)(\frac{\xi^*(x)}{\gcd^*(\xi(x),\ell(x))})^{-1} mod \frac{\xi^*(x)}{\gcd^*(\xi(x),\ell(x))}.$$

Proof. Since $(\overline{\ell}(x)|\overline{o}(x)) \in C^{\perp}$ and $(\xi(x)|0) \in C$, it follows from Proposition 9 that

$$(\overline{\ell}(x)|\overline{(o)}(x)) \circ (\xi(x)|0) \equiv 0 \mod (x^l - 1).$$

Then, $\overline{\ell}(x)\xi^*(x) = 0 \mod (x^m - 1)$ because of Proposition 10. Hence, there exists a polynomial $\rho(x) \in R[x]$ such that $\overline{\ell}(x) = \frac{x^m-1}{\xi^*(x)}\rho(x) = (\frac{x^m-1}{\xi_v^*(x)}v + \frac{x^m-1}{\xi_w^*(x)}w)\rho(x)$. The remainder of this proof is to show the concrete expression of $\rho(x)$.

Computing $(\overline{\ell}(x) \mid \overline{o}(x)) \circ (\ell(x)|o(x))$, it follows that

$$\begin{aligned} &(\overline{\ell}(x)|\overline{o}(x)) \circ (\ell(x)|o(x)) = (\frac{x^m-1}{\xi^*(x)}\rho(x)|\frac{(x^n-1)\gcd^*(\xi(x),\ell(x))}{\xi^*(x)o^*(x)}) \circ (\ell(x)|o(x)) = \\ &((\frac{x^m-1}{\xi_v^*(x)}\rho(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(\ell_v(x))}\ell_v^*(x) + \frac{(x^n-1)\gcd_v^*(\xi(x),\ell(x))}{\xi_v^*(x)o_v^*(x)}\theta_{\frac{l}{n}}(x^n)x^{l-1-\deg(o_v(x))}o_v^*(x))v + \\ &((\frac{x^m-1}{\xi_w^*(x)}\rho(x)\theta_{\frac{l}{m}}(x^m)x^{l-1-\deg(\ell_w(x))}\ell_w^*(x) + \frac{(x^n-1)\gcd_w^*(\xi(x),\ell(x))}{\xi_w^*(x)o_w^*(x)}\theta_{\frac{l}{n}}(x^n)x^{l-1-\deg(o_w(x))}o_w^*(x))w. \end{aligned}$$

While $(x^m-1)\theta_{\frac{l}{m}}(x^m) = x^l - 1$ and $(x^n-1)\theta_{\frac{l}{n}}(x^n) = x^l - 1$. Then,

$$\begin{aligned} &\frac{(x^l-1)\gcd_v^*(\xi(x),\ell(x))}{\xi_v^*(x)}(\rho_v(x)x^{l-\deg(\ell_v(x))}\frac{\ell_v^*(x)}{\gcd_v^*(\xi(x),\ell(x))} + x^{l-\deg(o_v(x)-1)})v + \\ &\frac{(x^l-1)\gcd_w^*(\xi(x),\ell(x))}{\xi_w^*(x)}(\rho_w(x)x^{l-\deg(\ell_w(x))}\frac{\ell_w^*(x)}{\gcd_w^*(\xi(x),\ell(x))} + x^{l-\deg(o_w(x)-1)})w \\ &= 0 \mod (x^l - 1). \end{aligned}$$

This means that

$$\begin{aligned} &\frac{x^l-1}{\xi_v^*(x)/\gcd_v^*(\xi(x),\ell(x))}(\rho_v(x)x^{l-\deg(\ell_v(x))}\frac{\ell_v^*(x)}{\gcd_v^*(\xi(x),\ell(x))} + x^{l-\deg(o_v(x)-1)})v + \\ &\frac{x^l-1}{\xi_w^*(x)/\gcd_w^*(\xi(x),\ell(x))}(\rho_w(x)x^{l-\deg(\ell_w(x))}\frac{\ell_w^*(x)}{\gcd_w^*(\xi(x),\ell(x))} + x^{l-\deg(o_w(x)-1)})w = 0 \mod (x^l - 1). \end{aligned}$$

To simplify the length and complexity of the above equations, let

$$\hat{\xi}(x) = \frac{\xi(x)}{\gcd(\xi(x),\ell(x))}, \hat{\ell}(x) = \frac{\ell(x)}{\gcd(\xi(x),\ell(x))}.$$

Hence,

$$\begin{aligned} &\frac{x^l-1}{\hat{\xi}_v^*(x)}(\rho_v(x)x^{l-\deg(\ell_v(x))}\hat{\ell}_v^*(x) + x^{l-\deg(o_v(x)-1)})v + \frac{x^l-1}{\hat{\xi}_w^*(x)}(\rho_w(x)x^{l-\deg(\ell_w(x))}\hat{\ell}_w^*(x) \\ &+ x^{l-\deg(o_w(x)-1)})w = 0 \mod (x^l - 1). \end{aligned}$$

Then,

$$(\rho_v(x)x^{l-\deg(\ell_v(x))}\hat{\ell}_v^*(x)+x^{l-\deg(o_v(x)-1)})v+(\rho_w(x)x^{l-\deg(\ell_w(x))}\hat{\ell}_w^*(x)+x^{l-\deg(o_w(x)-1)})w \\ =0 \mod (x^l-1)$$

or

$$(\rho_v(x)x^{l-\deg(\ell_v(x))}\hat{\ell}_v^*(x)+x^{l-\deg(o_v(x)-1)})v+(\rho_w(x)x^{l-\deg(\ell_w(x))}\hat{\ell}_w^*(x)+x^{l-\deg(o_w(x)-1)})w \\ =0 \mod (\hat{\xi}^*(x)).$$

Note that these two equations are closely related. In fact, the former can deduce the latter according to $\hat{o}^*(x)|(x^l-1)$. Therefore, one can assume that

$$(\rho_v(x)x^{l-\deg(\ell_v(x))}\hat{\ell}_v^*(x)+x^{l-\deg(o_v(x)-1)})v+(\rho_w(x)x^{l-\deg(\ell_w(x))}\hat{\ell}_w^*(x)+x^{l-\deg(o_w(x)-1)})w \\ =0 \mod (\hat{\xi}^*(x)).$$

From the setting of the abbreviation, $\gcd(\hat{\xi}(x),\hat{\ell}(x))=1$. Furthermore, $x^l=1 \mod \xi^*(x)$. Then, $\hat{\ell}^*(x)$ is an invertible element modulo $\hat{\xi}^*(x)$. Thus,

$$\rho(x)=(-x^{l-\deg(o_v(x))+\deg(\xi_v(x))}v-x^{l-\deg(o_w(x))+\deg(\xi_w(x))}w)\left(\frac{\ell^*(x)}{\gcd^*(\xi(x),\ell(x))}\right)^{-1} \mod \frac{\xi^*(x)}{\gcd^*(\xi(x),\ell(x))}.$$

□

As an application to this section, consider the dual codes of the codes in Section 3.3.

Example 3. *Continue to use the parameters in Example 1. From Theorem 3, $C^\perp$ is also a $\mathbb{F}_2+v\mathbb{F}_2$-double cyclic code. According to the results in this section, an easy computation shows that*

$$\begin{cases} \overline{\xi}(x) &= (x^4+x^2+x+1)v+(x^4+x^3+x^2+1)w, \\ \overline{\ell}(x) &= (x^3+x)v+(x^3+x+1)w, \\ \overline{o}(x) &= v+(x^3+x^2+1)w. \end{cases}$$

Similarly, the minimal generating set of $C^\perp$ is $\overline{G}_1^v \cup \overline{G}_1^w \cup \overline{G}_2^v \cup \overline{G}_2^w$, where

$$\begin{cases} \overline{G}_1^v = \left\{v(x^4+x^2+x+1|0), vx(x^4+x^2+x+1|0), vx^2(x^4+x^2+x+1|0)\right\}, \\ \overline{G}_1^w = \left\{w(x^4+x^3+x^2+1|0), wx(x^4+x^3+x^2+1|0), wx^2(x^4+x^3+x^2+1|0)\right\}, \\ \overline{G}_2^v = \Big\{v(x^3+x|1), vx(x^3+x|1), vx^2(x^3+x|1), vx^3(x^3+x|1), \\ \qquad vx^4(x^3+x|1), vx^5(x^3+x|1), vx^6(x^3+x|1)\Big\}, \\ \overline{G}_2^w = \Big\{w(x^3+x+1|x^3+x^2+1), wx(x^3+x+1|x^3+x^2+1), \\ \qquad wx^2(x^3+x+1|x^3+x^2+1), wx^3(x^3+x+1|x^3+x^2+1)\Big\}. \end{cases}$$

Therefore, the generating matrix of $C^{\perp}$ *is*

$$\left(\begin{array}{ccccccc|ccccccc}
v & v & v & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v & v & v & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v & v & v & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v & 0 & v & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v & 0 & v & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v & 0 & v & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v & 0 & v & 0 & 0 & 0 & v & 0 & 0 & 0 \\
v & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 \\
0 & v & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & v & 0 \\
v & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \\
w & 0 & w & w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & w & 0 & w & w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & w & 0 & w & w & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
w & w & 0 & w & 0 & 0 & 0 & w & 0 & w & w & 0 & 0 & 0 \\
0 & w & w & 0 & w & 0 & 0 & 0 & w & 0 & w & w & 0 & 0 \\
0 & 0 & w & w & 0 & w & 0 & 0 & 0 & w & 0 & w & w & 0 \\
0 & 0 & 0 & w & w & 0 & w & 0 & 0 & 0 & w & 0 & w & w
\end{array}\right).$$

This means that the generating matrices of $P_v(C^{\perp})$, $P_w(C^{\perp})$ *are*

$$\left(\begin{array}{ccccccc|ccccccc}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right), \left(\begin{array}{ccccccc|ccccccc}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}\right).$$

Hence,

$$\begin{cases} \text{the length of } P_v(C^{\perp}) \text{ is } 14,\ \dim_{\mathbb{F}_2}(P_v(C^{\perp})) = 10,\ d_{min}(P_v(C^{\perp})) = 3; \\ \text{the length of } P_w(C^{\perp}) \text{ is } 14,\ \dim_{\mathbb{F}_2}(P_w(C^{\perp})) = 7,\ d_{min}(P_w(C^{\perp})) = 4. \end{cases}$$

Consequently, the codes $P_v(C^{\perp})$ *and* $P_w(C^{\perp})$ *have the parameters* $[14, 10, 3]$ *and* $[14, 7, 4]$, *respectively. Then, the* $\mathbb{F}_2 + v\mathbb{F}_2$*-double cyclic code* $C^{\perp}$ *has the parameter of* $[28, 17, 3]$.

Example 4. *Applying the parameters in Example 2. By the results about the dual code of R-double cyclic code in this section, then*

$$\begin{cases} \overline{\xi}(x) &= (x^4 + x^3 + x^2 + 1)v + (x^7 + 1)w, \\ \overline{\ell}(x) &= xv + (x^4 + x^3 + x^2 + x + 1)w, \\ \overline{o}(x) &= (x + 1)v + (x^3 + x + 1)w. \end{cases}$$

As in Example 3, $P_v(C^{\perp})$, $P_w(C^{\perp})$, $C^{\perp}$ *have the parameters of* $[21, 16, 3]$, $[21, 11, 6]$, $[42, 27, 3]$, *respectively.*

More examples about $\mathbb{F}_2 + v\mathbb{F}_2$-double cyclic codes and their duals have below in Tables 2 and 3.

Table 2. Table of $\mathbb{F}_2 + v\mathbb{F}_2$-double cyclic codes.

Code	Generators	[m,n]	Parameters
C_1	$\xi(x) = vx^7 + (1+v)x^6 + (1+v)x^5 + (1+v)x^4 + (1+v)x^3 + (1+v)x^2 + (1+v)x + 1$, $\ell(x) = vx^4 + (1+v)x^3 + x^2 + 1$, $o(x) = x^4 + x^2 + x + 1$	[7, 7]	[28, 7, 7]
C_2	$\xi(x) = vx^7 + (1+v)x^4 + (1+v)x^3 + (1+v)x^2 + 1$, $\ell(x) = vx^4 + (1+v)x^3 + vx^2 + x + 1$, $o(x) = vx^4 + (1+v)x^3 + x^2 + vx + 1$	[7, 7]	[28, 10, 4]
C_3	$\xi(x) = vx^7 + (1+v)x^6 + (1+v)x^5 + (1+v)x^4 + (1+v)x^3 + (1+v)x^2 + (1+v)x + 1$, $\ell(x) = vx^3 + x + 1$, $o(x) = vx^9 + vx^8 + vx^6 + vx^5 + vx^4 + vx^3 + (1+v)x^2 + 1$	[7, 14]	[42, 17, 5]
C_4	$\xi(x) = vx^7 + (1+v)x^3 + (1+v)x^2 + 1$, $\ell(x) = vx^3 + vx + 1$, $o(x) = vx^9 + vx^8 + vx^6 + vx^5 + vx^4 + vx^3 + (1+v)x^2 + 1$	[7, 14]	[42, 21, 3]
C_5	$\xi(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$, $\ell(x) = vx^4 + vx^3 + (1+v)x + 1$, $o(x) = vx^5 + (1+v)x^3 + x^2 + vx + 1$	[7, 14]	[42, 22, 5]
C_6	$\xi(x) = vx^6 + vx^5 + vx^4 + x^3 + x^2 + vx + 1$, $\ell(x) = vx^4 + vx^3 + 1$, $o(x) = vx^5 + x^2 + vx + 1$	[7, 14]	[42, 22, 3]
C_7	$\xi(x) = vx^6 + vx^5 + vx^4 + x^3 + x^2 + vx + 1$, $\ell(x) = vx + 1$, $o(x) = vx^3 + x^2 + 1$	[7, 14]	[42, 38, 3]

Table 3. The dual codes of Table 2.

Code	Generators	[m, n]	Paramaters
$C_1^\perp$	$\overline{\xi}(x) = (1+v)x^4 + vx^3 + x^2 + (1+v)x + 1$, $\overline{\ell}(x) = (1+v)x^3 + (1+v)x + v$, $\overline{o}(x) = 1$	[7, 7]	[28, 21, 2]
$C_2^\perp$	$\overline{\xi}(x) = (1+v)x^4 + x^3 + x^2 + 1$, $\overline{\ell}(x) = (1+v)x^3 + (1+v)x + 1$, $\overline{o}(x) = (1+v)x^3 + (1+v)x^2 + 1$	[7, 7]	[28, 18, 2]
$C_3^\perp$	$\overline{\xi}(x) = (1+v)x^7 + vx^4 + vx^3 + vx^2 + 1$, $\overline{\ell}(x) = (1+v)x^6 + (1+v)x^4 + (1+v)x^3 + (1+v)x^2 + x + (1+v)$, $\overline{o}(x) = (1+v)x^5 + (1+v)x^4 + (1+v)x^3 + vx + 1$	[7, 14]	[42, 25, 3]
$C_4^\perp$	$\overline{\xi}(x) = (1+v)x^7 + vx^4 + vx^3 + vx^2 + 1$, $\overline{\ell}(x) = (1+v)x^6 + (1+v)x^5 + (1+v)x^2 + vx + 1$, $\overline{o}(x) = (1+v)x^9 + (1+v)x^6 + (1+v)x^5 + (1+v)x^4 + (1+v)x^3 + x + 1$	[7, 14]	[42, 21, 3]
$C_5^\perp$	$\overline{\xi}(x) = x^7 + 1$, $\overline{\ell}(x) = (1+v)x^6 + x^4 + x^3 + x^2 + x + (1+v)$, $\overline{o}(x) = (1+v)x^5 + (1+v)x^4 + x^3 + vx + 1$	[7, 14]	[42, 20, 6]
$C_6^\perp$	$\overline{\xi}(x) = x^7 + 1$, $\overline{\ell}(x) = (1+v)x^6 + (1+v)x^5 + vx^4 + vx^3 + x^2 + vx + 1$, $\overline{o}(x) = (1+v)x^9 + (1+v)x^6 + (1+v)x^5 + (1+v)x^4 + (1+v)x^3 + x + 1$	[7, 14]	[42, 16, 6]
$C_7^\perp$	$\overline{\xi}(x) = x^7 + 1$, $\overline{\ell}(x) = x^6 + (1+v)x^5 + vx^4 + vx^3 + x^2 + vx + 1$, $\overline{o}(x) = (1+v)x^9 + (1+v)x^6 + x^5 + x^4 + x^3 + (1+v)x + 1$	[7, 14]	[42, 14, 6]

5. Summaries

Owing to the nice algebraic structure of $\mathbb{F}_q + v\mathbb{F}_q$, some results about double cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$ are provided in a convenient way. Section 3 gives the basic forms of the double cyclic codes, which include the forms of generating polynomials, minimal generating sets and generating matrices. Section 4 explores the relation between the generating polynomials of double cyclic codes and their duals. The main results of this article are summarized as follows.

Conclusions 1. Let C be a double cyclic code of length (m, n) over R, then C has the forms of

$$C = \langle (\xi(x)|0), (\ell(x)|o(x)) \rangle = \langle (\xi_v(x)v + \xi_w(x)w|0), (\ell_v(x)v + \ell_w(x)w|o_v(x)v + o_w(x)w) \rangle,$$

where $\xi_v(x), \xi_w(x)|x^m - 1, o_v(x), o_w(x)|x^n - 1$ and

If C is a separable R-double cyclic code, then $\ell_v(x) = \ell_w(x) = 0$ (i.e., $\ell(x) = 0$).

If C is a free R-double cyclic code, then $\begin{cases} (1)\deg(\ell_v(x)) < \deg(\xi_v(x)), \deg(\ell_w(x)) < \deg(\xi_w(x)); \\ (2)\iota_v(x)|\frac{x^n-1}{o_v(x)}\ell_v(x), \iota_w(x)|\frac{x^n-1}{o_w(x)}\ell_w(x); \\ (3)\xi_v(x)|\frac{x^n-1}{o_v(x)}\gcd(\xi_v(x),\ell_v(x)), \xi_w(x)|\frac{x^n-1}{o_w(x)}\gcd(\xi_w(x),\ell_w(x)). \end{cases}$

Conclusions 2. Let $C \in \mathcal{C}_{m,n}(R)$ as mentioned above, then $C^\perp \in \mathcal{C}_{m,n}(R)$. Let

$$C^\perp = \left\langle (\overline{\xi}(x)|0), (\overline{\ell}(x)|\overline{o}(x)) \right\rangle = \left\langle (\overline{\xi}_v(x)v + \overline{\xi}_w(x)w|0), (\overline{\ell}_v(x)v + \overline{\ell}_w(x)w|\overline{o}_v(x)v + \overline{o}_w(x)w) \right\rangle.$$

Then:

(1) $\overline{\xi}(x) = \frac{x^m-1}{\gcd^*(\xi_v(x),\ell_v(x))}v + \frac{x^m-1}{\gcd^*(\xi_w(x),\ell_w(x))}w$;
(2) $\overline{o}(x) = \frac{(x^n-1)\gcd^*(\xi_v(x),\ell_v(x))}{\xi_v^*(x)o_v^*(x)}v + \frac{(x^n-1)\gcd^*(\xi_w(x),\ell_w(x))}{\xi_w^*(x)o_w^*(x)}w$;
(3) $\overline{\ell}(x) = (\frac{x^m-1}{\xi_v^*(x)}v + \frac{x^m-1}{\xi_w^*(x)}w)\rho(x)$, where

$$\begin{cases} \rho(x) = 0 \text{ if C is separable, or otherwise} \\ \rho(x) = (-x^{l-\deg(o_v(x))+\deg(\xi_v(x))}v - x^{l-\deg(o_w(x))+\deg(\xi_w(x))}w)\left(\frac{\ell^*(x)}{\gcd^*(\xi(x),\ell(x))}\right)^{-1} \bmod \frac{\xi^*(x)}{\gcd^*(\xi(x),\ell(x))}. \end{cases}$$

While letting $v = 0$ (i.e., $w = 1$) or $v = 1$ (i.e., $w = 0$), the above conclusions become to

Conclusions 1′. Let C be a $\mathbb{F}_q$-double cyclic code of length (m, n), then C has the forms of

$$C = \langle (\xi(x)|0), (\ell(x)|o(x)) \rangle,$$

where $\xi(x)|x^m - 1$, $o(x)|x^n - 1$ and:

If C is a separable $\mathbb{F}_q$-double cyclic code, then $\ell(x) = 0$.

If C is a free $\mathbb{F}_q$-double cyclic code, then $\begin{cases} (1)\deg(\ell(x)) < \deg(\xi(x)); \\ (2)\iota(x)|\frac{x^n-1}{o(x)}\ell(x); \\ (3)\xi(x)|\frac{x^n-1}{o(x)}\gcd(\xi(x),\ell(x)). \end{cases}$

Conclusions 2′. Let $C \in \mathcal{C}_{m,n}(\mathbb{F}_q)$ as mentioned above, then $C^\perp \in \mathcal{C}_{m,n}(\mathbb{F}_q)$. Let

$$C^\perp = \left\langle (\overline{\xi}(x)|0), (\overline{\ell}(x)|\overline{o}(x)) \right\rangle.$$

Then:

(1) $\overline{\xi}(x) = \frac{x^m-1}{\gcd^*(\xi(x),\ell(x))}$;
(2) $\overline{o}(x) = \frac{(x^n-1)\gcd^*(\xi(x),\ell(x))}{\xi^*(x)o^*(x)}$;
(3) $\overline{\ell}(x) = \frac{x^m-1}{\xi^*(x)}\rho(x)$, where $\begin{cases} \rho(x) = 0 \text{ if C is separable, or otherwise} \\ \rho(x) = -x^{l-\deg(o(x))+\deg(\xi(x))}\left(\frac{\ell^*(x)}{\gcd^*(\xi(x),\ell(x))}\right)^{-1} \bmod \frac{\xi^*(x)}{\gcd^*(\xi(x),\ell(x))}. \end{cases}$

These are the main results about $\mathbb{F}_q$-double cyclic codes in [9,12]. Therefore, the double cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$ investigated in this paper are the generalization of those over finite fields.

Throughout this paper, one can seen that cyclic codes are a special class of double cyclic codes. On the other side, double cyclic codes are permutation equivalent to generalized quasi-cyclic codes of index 2. Consequently, the study of double cyclic codes can help us to realize various generalized and deformed structures of cyclic codes. While it must also point out that although $\mathbb{F}_q + v\mathbb{F}_q$-double cyclic codes can be directly regarded as linear codes over finite fields, $\mathbb{F}_q$-linear codes obtained in this way are usually not optimal codes. However, the main motivation of this paper was to provide an underlying theoretical framework for considering the weight distribution of $\mathbb{F}_q + v\mathbb{F}_q$-double cyclic codes. Meanwhile, possible acquisition of quantum codes based on such $\mathbb{F}_q + v\mathbb{F}_q$-double cyclic codes also requires the conclusions of this paper as the theoretical foundation. Therefore, possible

further research is to consider the weight distributions or the case of quantum codes from these $\mathbb{F}_q + v\mathbb{F}_q$-double cyclic codes.

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