Article

Geometric Inequalities of Bi-Warped Product Submanifolds of Nearly Kenmotsu Manifolds and Their Applications

Akram Ali ¹ and Fatemah Mofarreh ^{2,*} ¹ Department of Mathematics, College of Science, King Khalid University, Abha 62529, Saudi Arabia; akali@kku.edu.sa² Mathematical Science Department, Faculty of Science, Princess Nourah Bint Abdulrahman University, Riyadh 11546, Saudi Arabia

* Correspondence: fyalmofarrah@pnu.edu.sa

Received: 9 September 2020; Accepted: 9 October 2020; Published: 16 October 2020

Abstract: The present paper aims to construct an inequality for bi-warped product submanifolds in a special class of almost metric manifolds, namely nearly Kenmotsu manifolds. As geometric applications, some exceptional cases that generalized several other inequalities are discussed. We also deliberate some applications in the context of mathematical physics and derive a new relation between the Dirichlet energy and the second fundamental form. Finally, we present a constructive remark at the end of this paper which shows the motive of the study.

Keywords: bi-warped products; slant immersion; nearly Kenmotsu manifold; inequality; warping functions; Dirichlet energy

1. Background and Motivations

A new series of bi-warped product submanifolds is constructed and some examples about it first appeared in [1]. The concept of bi-warped product submanifolds in Kahler manifolds having holomorphic Ω_T , totally real $\Omega_\perp$, and pointwise slant Ω_ϕ submanifolds was studied in [1]. It is founded that the bi-warped product submanifolds of types $\Omega_\phi \times_{f_1} \Omega_T \times_{f_2} \Omega_\perp$, $\Omega_\perp \times_{f_1} \Omega_T \times_{f_2} \Omega_\phi$ and $\Omega_T \times_{f_1} \Omega_\phi \times_{f_2} \Omega_\perp$ do not exist in Kaehler manifolds. Additionally, an example was used to show the existence of a non-trivial bi-warped product submanifold of the type $\Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$, and the necessary and sufficient conditions are given for this submanifold to be locally trivial. In the same study, an inequality for the squared norm of the second fundamental form $\mathcal{B}$ regarding the warping functions f_1 and f_2 , is established as following

$$\|\mathcal{B}\|^2 \geq 2l\|\nabla \ln f_1\|^2 + 4k(\csc^2 \phi + \cot^2 \phi)\|\nabla \ln f_1\|^2 \quad (1)$$

where $l = \dim \Omega_\perp$ and $2k = \dim \Omega_\phi$. It is easily observed that Inequality (1) is generalized inequality for CR-warped product in [2] with $k = 0$ and warped product pointwise semi-slant submanifold in [3] with $l = 0$, respectively.

In [4], authors developed the sharp inequality in terms of the second fundamental form with its squared norm, for a bi-warped product submanifold $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ in a Kenmotsu manifold with giving a non-trivial example. They presented the following inequality and a few applications.

$$\|\mathcal{B}\|^2 \geq 2l\left\{\|\nabla(\ln f_1)\|^2 - 1\right\} + 4k\left(\csc^2 \phi + \cot^2 \phi\right)\left\{\|\nabla(\ln f_2)\|^2 - 1\right\}. \quad (2)$$

It is clear that Inequality (2) is an extension of inequalities in Theorem 3.1 [5] and Theorem 4.2 [6]. In a Kenmotsu manifold, further work of bi-warped product submanifolds of type $\Omega = \Omega_\phi \times_{f_1} \Omega_T \times_{f_2} \Omega_\perp$ is presented in [7], and the lower bound of the squared norm for the second fundamental form is obtained which is generalized to [8] with $\dim \Omega_T = 0$ and $\dim \Omega_\perp = 0$ [9] as well. More interesting is that the bi-warped product submanifolds of the form $\Omega = \Omega_\perp \times_{f_1} \Omega_T \times_{f_2} \Omega_\phi$ in Kenmotsu manifolds are discussed in [10] and the following inequality is presented

$$\|\mathcal{B}\|^2 \geq 2t \left\{ \|\nabla(\ln f_1)\|^2 - 1 \right\} + 2k \cos^2 \phi \left\{ \|\nabla(\ln f_2)\|^2 - 1 \right\} \quad (3)$$

where $2t = \dim \Omega_T$. However, we observed that Inequality (20) is a generalization of the inequalities which were derived in [11,12] with $\dim \Omega_\perp = 0$ and $\dim \Omega_\phi = 0$, respectively.

As an extension of CR-warped product submanifolds [13] and warped product semi-slant submanifolds [14] in the setting of a nearly Kaehler manifold, another interesting study focused on a bi-warped product submanifold of the form $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ in a nearly Kaehler manifold and provided the inequality

$$\|\mathcal{B}\|^2 \geq 2l \|\nabla \ln f_1\|^2 + 4k \left(1 + \frac{10}{9} \cot^2 \phi \right) \|\nabla \ln f_2\|^2 \quad (4)$$

where $l = \dim \Omega_\perp$ and $k = \frac{1}{2} \dim \Omega_\phi$, respectively. The equality cases and applications of this inequality are found in [15]. It shown that Inequality (4.1) [14] and Inequality (4.17) in [13] are special cases of Inequality (4). Recently, a more general study was done where it demonstrated that the bi-warped products in locally product Riemannian manifolds could be single-warped products or Riemannian products under specific considerations. Besides, the geometry of bi-warped product submanifold $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ in a locally product Riemannian manifold with non-trivial examples was studied [16]. They also proved a sharp general inequality for the second fundamental form in such settings and generalized all inequalities in [17–20]. For the general survey on warped product submanifolds of an almost Hermitian and almost contact setting see [21–29].

In this paper, we want to fill the gap in nearly Kenmotsu manifolds' study, the impressive structure of the almost contact manifolds. For example, it is well-known that a 6-dimensional sphere $\mathbb{S}^6$ with its canonical is a nearly Kaehler manifold, and if we define the warped product $\tilde{\Omega} = \mathbb{R} \times_f \mathbb{S}^6$, then $\tilde{\Omega}$ is a nearly Kenmotsu manifold with a warping function $f(t) = e^t$. Therefore, a nearly Kenmotsu manifold is a locally isometric to a warped product manifold with a base that is a real line, and fiber is a nearly Kaehler manifold [30]. If the sectional curvatures of all nearly Kaehler manifolds appear in the locally warped product of nearly Kenmotsu manifold are non-negative, the lower bound of the sectional curvatures of a nearly Kenmotsu manifold is -1 [31]. Moreover, nearly Kenmotsu hypersurfaces of nearly Kaehler manifolds do not exist, and a nearly Kenmotsu manifold with the normal condition $[\psi, \psi] + 2d\eta \otimes \zeta = 0$, is a Kenmotsu manifold where $[\psi, \psi]$ is the Nijenhuis torsion of ψ . After presenting some previous literature, we conclude that the nearly Kenmotsu manifold has different characteristics than other structures. The present paper's main ambition is to discuss the bi-warped product submanifold in a nearly Kenmotsu manifold. We established an inequality for the second fundamental form, which relates to the warping functions and slant immersions. It is shown that our inequality is generalized some inequalities in [32,33].

The manuscript is organized as follows: In Section 2, we collect the information about ambient manifolds and their submanifolds. We arrange some formulas and definitions to be used later. In Section 3, we consider bi-warped product submanifolds and derive appropriate lemmas. Then we give the proof of the main theorem, which includes inequality. In Section 4, we produce some geometrical consequences of our derived main theorem. In Section 5, some physical science applications are presented. In Section 6, we discuss the conclusion of the manuscript.

2. Notations and Formulas

The odd-dimensional C^∞ -manifold $(\tilde{\Omega}, g)$ associated to the almost-contact structure (ψ, ζ, η) is referred to as the almost contact metric manifold which fulfilling coming properties:

$$\psi^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1, \quad \psi(\zeta) = 0, \quad \eta \circ \psi = 0, \quad (5)$$

$$g(\psi U_1, \psi W_2) = g(U_1, W_2) - \eta(U_1)\eta(W_2), \quad \eta(U_1) = g(U_1, \zeta) \quad (6)$$

$\forall U_1, W_2 \in \Gamma(T\tilde{\Omega})$. A nearly Kenmotsu manifold [30] regarding Riemannian connection is contained the almost contact metric manifold which satisfies the next equation

$$(\tilde{\nabla}_{U_1}\psi)U_1 = -2\eta(U_1)\psi U_1. \quad (7)$$

It follows for a nearly Kenmotsu manifold

$$(\tilde{\nabla}_{U_1}\psi)W_2 + (\tilde{\nabla}_{W_2}\psi)U_1 = -\eta(W_2)\psi U_1 - \eta(U_1)\psi W_2 \quad (8)$$

for all vector fields U_1, W_2 tangent to $\tilde{\Omega}$. The Gauss and Weingarten formulas specifying the relation between the Levi-Civita connections ∇ on a submanifold Ω and $\tilde{\nabla}$ on a ambient manifold $\tilde{\Omega}$ are given by (for more detail see [28]).

$$\tilde{\nabla}_{U_1}W_2 = \nabla_{U_1}W_2 + \mathcal{B}(U_1, W_2) \quad (9)$$

$$\tilde{\nabla}_{U_1}\zeta = -A_\zeta U_1 + \nabla_{U_1}^\perp \zeta \quad (10)$$

for every $U_1, W_2 \in \Gamma(T\Omega)$ and $\zeta \in \Gamma(T^\perp\Omega)$. Here $\nabla^\perp$ is the normal connection defined on the normal bundle $T^\perp\Omega$ and $\mathcal{B}$ and A_ζ have the next relation

$$g(\mathcal{B}(U_1, W_2), \zeta) = g(A_\zeta U_1, W_2). \quad (11)$$

As well,

$$\psi U_1 = TU_1 + FU_1 \quad (12)$$

in which FU_1 and TU_1 are normal and tangential elements of ψU_1 , respectively. If Ω is invariant and anti-invariant then FU_1 as well as TU_1 are zero, respectively. Similarly, we have

$$\psi \zeta = t\zeta + f\zeta, \quad (13)$$

where $t\zeta$ (resp. $f\zeta$) are tangential (resp. normal) components of $\psi \zeta$. The covariant derivative of the endomorphism ψ is explained by

$$(\tilde{\nabla}_{U_1}\psi)W_2 = \tilde{\nabla}_{U_1}\psi W_2 - \psi \tilde{\nabla}_{U_1}W_2, \quad \forall U_1, W_2 \in \Gamma(T\tilde{\Omega}). \quad (14)$$

There is a motivating class of submanifolds presented as the slant submanifold class. For any non-zero vector U_1 tangential to Ω about p , in which U_1 is not proportional to ζ_p , $0 \leq \phi(U_1) \leq \pi/2$ is referred to the angle between ψU_1 and $T_p\Omega$ which is named the Wirtinger angle. If $\phi(U_1)$ is constant for any $U_1 \in T_p\Omega - \langle \zeta \rangle$ at point $p \in \Omega$, then Ω is referred to as the slant submanifold [34] and ϕ is then the slant angle of Ω . The following necessary and sufficient condition is important for this paper, known as the characterization slant submanifold, and was proved in [34,35]. A submanifold Ω is slant if and only if the equality holds:

$$T^2 = \lambda(-I + \eta \otimes \zeta) \quad (15)$$

for a constant $\lambda \in [0, 1]$ in which $\lambda = \cos^2 \phi$, where T is an endomorphism defined in (12). The following alliances are resulted from (15).

$$g(TU_1, TW_2) = \cos^2 \phi \left\{ g(U_1, W_2) - \eta(U_1)\eta(W_2) \right\} \quad (16)$$

$$g(FU_1, FW_2) = \sin^2 \phi \left\{ g(U_1, W_2) - \eta(U_1)\eta(W_2) \right\} \quad (17)$$

$$\forall U_1, W_2 \in \Gamma(T\Omega).$$

Definition 1. If we consider only two fibers of a multiple warped product $\Omega^1 \times_{f_1} \Omega^2 \times \cdots \times_{f_p} \Omega^p$ such that $\Omega = \Omega^1 \times_{f_1} \Omega^2 \times_{f_2} \Omega^3$, then Ω is called the bi-warped product submanifold and satisfies the following equation

$$\nabla_{U_1} Z = \sum_{i=2}^3 (U_1 \ln f_i) Z_i \quad (18)$$

where $U_1 \in \Gamma(T\Omega_1)$ and $Z \in \Gamma(T(\Omega^2 \times \cdots \times \Omega^p))$. For more details see [36–38].

3. Bi-Warped Product Submanifolds with Totally Real and Proper Slant Fibers

In this section, for a nearly Kenmotsu manifold $\tilde{\Omega}$, a bi-warped product submanifold in $\tilde{\Omega}$ that is introduced as $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$, where $\Omega_T, \Omega_\perp, \Omega_\phi$ indicates to holomorphic, totally real and proper slant submanifolds of $\tilde{\Omega}$, respectively. Assuming

$$\begin{aligned} T\Omega &= \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\phi \\ T^\perp \Omega &= \psi \mathcal{D}^\perp \oplus F\mathcal{D}^\phi \oplus \mu, \end{aligned}$$

where μ defines the ψ -invariant normal subbundle of the normal bundle $T^\perp \Omega$. Starting of this point, the coming conventions will be used: $U_1, U_2, \dots$ define vector fields at $\Gamma(\mathcal{D})$ and $W_1, W_2, \dots$ define vector fields at $\Gamma(\mathcal{D}^\phi)$, whereas $Z, W, \dots$ define vector fields at $\Gamma(\mathcal{D}^\perp)$.

The coming practical consequence will be used late in this paper.

Lemma 1. Suppose $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ is the bi-warped product submanifold of the nearly Kaehler manifold $\tilde{\Omega}$. Therefore, we got

$$g(\mathcal{B}(U_1, U_2), \psi Z) = 0 \quad (19)$$

$$g(\mathcal{B}(U_1, U_2), FW_1) = 0 \quad (20)$$

$$g(\mathcal{B}(U_1, Z), \psi W) = -\psi U_1(\ln f_1)g(Z, W) \quad (21)$$

$$g(\mathcal{B}(\psi U_1, Z), \psi W) = \{U_1(\ln f_1) - \eta(U_1)\}g(Z, W) \quad (22)$$

$$g(\mathcal{B}(\zeta, Z), \psi W) = 0 \quad (23)$$

for any $U_1, U_2 \in \Gamma(\mathcal{D})$, $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $W_1 \in \Gamma(\mathcal{D}^\phi)$.

Proof. For all $U_1, U_2 \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we got

$$g(\mathcal{B}(U_1, U_2), \psi Z) = g(\tilde{\nabla}_{U_1} U_2, \psi Z) = g((\tilde{\nabla}_{U_1} \psi)U_2, Z) - g(\tilde{\nabla}_{U_1} \psi U_2, Z)$$

by the use of (18), we have

$$g(\mathcal{B}(U_1, U_2), \psi Z) = g((\tilde{\nabla}_{U_1} \psi)U_2, Z) + U_1(\ln f_1)g(\psi U_2, Z) \quad (24)$$

Orthogonality of vector fields give the following

$$g(\mathcal{B}(U_1, U_2), \psi Z) = g((\tilde{\nabla}_{U_1} \psi)U_2, Z). \quad (25)$$

Replacing U_1 with U_2 at (25) gives

$$g(\mathcal{B}(U_1, U_2), \psi Z) = g((\tilde{\nabla}_{U_2} \psi)U_1, Z). \quad (26)$$

Combining Equations (25) and (26) leads to

$$2g(\mathcal{B}(U_1, U_2), \psi Z) = g((\tilde{\nabla}_{U_1} \psi)U_2 + (\tilde{\nabla}_{U_2} \psi)U_1, Z)$$

Thus using Equation (8), we get first part (19) of this lemma. Similarly, Equation (20) can be proved (20). For part three, we got

$$g(\mathcal{B}(U_1, Z), \psi W) = g(\tilde{\nabla}_Z U_1, \psi W) = g((\tilde{\nabla}_Z \psi)U_1, W) - g(\tilde{\nabla}_Z \psi U_1, W)$$

for all $U_1 \in \Gamma(\mathcal{D})$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$. From (8), (9) and (18) we have

$$\begin{aligned} g(\mathcal{B}(U_1, Z), \psi W) &= -g((\tilde{\nabla}_{U_1} \psi)Z, W) - \eta(U_1)g(\psi Z, W) - \eta(Z)g(\psi U_1, W) \\ &\quad - \psi U_1(\ln f_1)g(Z, W) \\ &= -g(\tilde{\nabla}_{U_1} \psi Z, W) + g(\psi \tilde{\nabla}_{U_1} Z, W) - \psi U_1(\ln f_1)g(Z, W). \end{aligned}$$

By the use of (8)–(10) and (12), we have

$$2g(\mathcal{B}(U_1, Z), \psi W) = g(\mathcal{B}(U_1, W), \psi Z) - \psi U_1(\ln f_1)g(Z, W). \quad (27)$$

Replacing Z with W at (27), leads to

$$2g(\mathcal{B}(U_1, W), \psi Z) = g(\mathcal{B}(U_1, Z), \psi W) - \psi U_1(\ln f_1)g(Z, W). \quad (28)$$

Consequently, part three follows from (27) and (28), which proves this lemma. $\square$

Lemma 2. Let $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ be a bi-warped product submanifold of a nearly Kenmotsu manifold $\tilde{\Omega}$. Therefore, we have

$$g(\mathcal{B}(U_1, Z), FW_1) = \frac{1}{2}g(\mathcal{B}(U_1, W_1), \psi Z) = 0 \quad (29)$$

$$g(\mathcal{B}(U_1, W_1), FW_2) = \frac{1}{3}\{(U_1 \ln f_2) - \eta(U_1)\}g(TW_1, W_2) - \psi U_1(\ln f_2)g(W_1, W_2) \quad (30)$$

$$g(\mathcal{B}(\psi U_1, W_1), FW_2) = \frac{1}{3}(\psi U_1 \ln f_2)g(TW_1, W_2) + \{U_1(\ln f_2) - \eta(U_1)\}g(W_1, W_2) \quad (31)$$

$$g(\mathcal{B}(U_1, W_1), FTW_2) = \frac{1}{3}\{(U_1 \ln f_2) - \eta(U_1)\}\cos^2 \phi g(W_1, W_2) - (\psi U_1 \ln f_2)g(W_1, TW_2) \quad (32)$$

$$g(\mathcal{B}(\psi U_1, W_1), FTW_2) = \frac{1}{3}(\psi U_1 \ln f_2)\cos^2 \phi g(W_1, W_2) + \{U_1(\ln f_2) - \eta(U_1)\}g(W_1, TW_2) \quad (33)$$

$$g(\mathcal{B}(U_1, TW_1), FW_2) = -\frac{1}{3}\{(U_1 \ln f_2) - \eta(U_1)\}\cos^2 \phi g(W_1, W_2) - (\psi U_1 \ln f_2)g(TW_1, W_2) \quad (34)$$

$$g(\mathcal{B}(\psi U_1, TW_1), FW_2) = -\frac{1}{3}(\psi U_1 \ln f_2)\cos^2 \phi g(W_1, W_2) + \{U_1(\ln f_2) - \eta(U_1)\}g(TW_1, W_2) \quad (35)$$

$$g(\mathcal{B}(U_1, TW_1), FTW_2) = -\frac{1}{3}\{U_1(\ln f_2) - \eta(U_1)\}\cos^2 \phi g(W_1, TW_2) - (\psi U_1 \ln f_2)\cos^2 \phi g(W_1, W_2) \quad (36)$$

$$g(\mathcal{B}(\psi U_1, TW_1), FTW_2) = -\frac{1}{3}(\ln f_2)\cos^2 \phi g(W_1, TW_2) + \{U_1(\ln f_2) - \eta(U_1)\}\cos^2 \phi g(W_1, W_2) \quad (37)$$

$$g(\mathcal{B}(\zeta, W_1), FW_1) = 0 \quad (38)$$

for all $U_1 \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $W_1, W_2 \in \Gamma(\mathcal{D}^\phi)$.

Proof. For all $U_1 \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $W_1 \in \Gamma(\mathcal{D}^\phi)$, we got

$$\begin{aligned} g(\mathcal{B}(U_1, Z), FW_1) &= g(\tilde{\nabla}_Z U_1, \psi W_1) - g(\tilde{\nabla}_Z U_1, TW_1) \\ &= g((\tilde{\nabla}_Z \psi)U_1, W_1) - g(\tilde{\nabla}_Z \psi U_1, W_1) - g(\tilde{\nabla}_Z U_1, TW_1) \\ &= -g((\tilde{\nabla}_{U_1} \psi)Z, W_1) - \eta(U_1)g(\psi Z, W_1) - \eta(Z)g(\psi U_1, W_1) \\ &\quad - \psi U_1 \ln f_1 g(Z, W_1) - U_1 \ln f_1 g(Z, TW_1). \end{aligned}$$

By the use of (8), (14) and the vector fields orthogonality, we have

$$\begin{aligned} g(\mathcal{B}(U_1, Z), FW_1) &= -g((\tilde{\nabla}_{U_1} \psi)Z, W_1) \\ &= -g(\tilde{\nabla}_{U_1} \psi Z, W_1) + g(\psi \tilde{\nabla}_{U_1} Z, W_1) \\ &= g(A_{\psi Z} U_1, W_1) - g(\tilde{\nabla}_{U_1} Z, TW_1) - g(\tilde{\nabla}_{U_1} Z, FW_1) \\ &= g(\mathcal{B}(U_1, W_1), \psi Z) - U_1 \ln f_1 g(Z, TW_1) - g(\mathcal{B}(U_1, Z), FW_1). \end{aligned}$$

Therefore, using (11)–(14) resulted in

$$g(\mathcal{B}(U_1, Z), FW_1) = \frac{1}{2}g(\mathcal{B}(U_1, W_1), \psi Z) \quad (39)$$

that is first equality of (i). At the same time, we take

$$\begin{aligned} g(\mathcal{B}(U_1, W_1), \psi Z) &= g(\tilde{\nabla}_{W_1} U_1, \psi Z) \\ &= g((\tilde{\nabla}_{W_1} \psi)U_1, Z) - g((\tilde{\nabla}_{W_1} \psi U_1), Z) \\ &= g((\tilde{\nabla}_{W_1} \psi)U_1, Z) - (\psi U_1 \ln f_2)g(W_1, Z). \end{aligned}$$

By the use of (8), (12) and the vector fields orthogonality, it is founded that

$$\begin{aligned} g(\mathcal{B}(U_1, W_1), \psi Z) &= -g((\tilde{\nabla}_{U_1} \psi)W_1, Z) \\ &= g((\tilde{\nabla}_{W_1} \psi)U_1, Z) + \eta(U_1)g(\psi W_1, Z) + \eta(W_1)g(\psi U_1, Z) \\ &= g(\tilde{\nabla}_{U_1} \psi W_1, Z) + g(\psi \tilde{\nabla}_{W_1} U_1, Z) \\ &= g(\tilde{\nabla}_{U_1} TW_1, Z) + g(\tilde{\nabla}_{U_1} FW_1, Z) - g(\tilde{\nabla}_{W_1} U_1, \psi Z). \end{aligned}$$

Therefore, it follows from (9), (10) and (18) that is

$$g(\mathcal{B}(U_1, W_1), \psi Z) = (U_1 \ln f_2)g(TW_1, Z) + g(A_{FW_1} U_1, Z) - g(\mathcal{B}(U_1, W_1), \psi Z).$$

Once more, by the use of (11), (14), (15) and the vector fields orthogonality, it is obtained that

$$g(\mathcal{B}(U_1, W_1), \psi Z) = \frac{1}{2}g(\mathcal{B}(U_1, Z), FW_1). \quad (40)$$

Thus, second part of (i) follows from (39) and (40). Part two of this lemma gives

$$\begin{aligned}
 g(\mathcal{B}(U_1, W_1), FW_2) &= g(\tilde{\nabla}_{W_1} U_1, \psi W_2) - g(\tilde{\nabla}_{W_1} U_1, TW_2) \\
 &= -g(\psi \tilde{\nabla}_{W_1} U_1, W_2) - (U_1 \ln f_2)g(W_1, TW_2) \\
 &= g((\tilde{\nabla}_{W_1} \psi)U_1, W_2) - g(\tilde{\nabla}_{W_1} \psi U_1, W_2) - (U_1 \ln f_2)g(W_1, TW_2) \\
 &= -g((\tilde{\nabla}_{U_1} \psi)W_1, W_2) - \eta(U_1)g(TW_1, W_2) - \eta(W_1)g(\psi U_1, W_2) \\
 &\quad - (\psi U_1 \ln f_2)g(W_1, W_2) - (U_1 \ln f_2)g(W_1, TW_2) \\
 &= g(\psi(\tilde{\nabla}_{U_1} W_1, W_2) - g(\tilde{\nabla}_{U_1} \psi W_1, W_2) - \eta(U_1)g(TW_1, W_2) \\
 &\quad - (\psi U_1 \ln f_2)g(W_1, W_2) - (U_1 \ln f_2)g(W_1, TW_2) \\
 &= -g(\tilde{\nabla}_{U_1} TW_1, W_2) - g(\tilde{\nabla}_{U_1} FW_1, W_2) - g(\tilde{\nabla}_{U_1} W_1, TW_2) - \eta(U_1)g(TW_1, W_2) \\
 &\quad - g(\tilde{\nabla}_{U_1} W_1, FW_2) - (\psi U_1 \ln f_2)g(W_1, W_2) - (U_1 \ln f_2)g(W_1, TW_2).
 \end{aligned}$$

By the use of (9), (10) and (12), we have

$$\begin{aligned}
 2g(\mathcal{B}(U_1, W_1), FW_2) &= g(\mathcal{B}(U_1, W_2), FW_1) - (U_1 \ln f_2)g(W_1, TW_2) \\
 &\quad - (\psi U_1 \ln f_2)g(W_1, W_2) - \eta(U_1)g(TW_1, W_2).
 \end{aligned} \quad (41)$$

Replacing W_1 with W_2 at (41), we have

$$\begin{aligned}
 2g(\mathcal{B}(U_1, W_1), FW_2) &= g(\mathcal{B}(U_1, W_1), FW_2) + (U_1 \ln f_2)g(W_1, TW_2) \\
 &\quad - (\psi U_1 \ln f_2)g(W_1, W_2) - \eta(U_1)g(TW_2, W_1).
 \end{aligned} \quad (42)$$

Part two follows from (41) and (42). The remaining connections are gained simply by replacing U_1 with ψU_1 , also W_1 and W_2 with TW_1 and TW_2 , in the same order. Thus the proof is completed. $\square$

Remark 1. The bi-warped product submanifold $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ in a nearly Kenmotsu manifold $\tilde{\Omega}$ named $\mathcal{D} \oplus \mathcal{D}^\perp$ -mixed totally geodesic (respectively, $\mathcal{D} \oplus \mathcal{D}^\phi$ -mixed totally geodesic) in case its second fundamental form $\mathcal{B}$ insures

$$\begin{aligned}
 \mathcal{B}(U_1, Z) &= 0 \\
 \mathcal{B}(U_1, W_1) &= 0
 \end{aligned}$$

for all $U_1 \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$ ((respectively, for all $U_1 \in \Gamma(\mathcal{D})$, $W_1 \in \Gamma(\mathcal{D}^\phi)$)).

Thus we provide the following propositions

Proposition 1. If a bi-warped product submanifold $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ of a nearly Kenmotsu manifold $\tilde{\Omega}$ is a $\mathcal{D} \oplus \mathcal{D}^\perp$ -mixed totally geodesic, then Ω is a Skew CR-warped product manifold.

Proof. If Ω is a $\mathcal{D} \oplus \mathcal{D}^\perp$ -mixed totally geodesic, then from (21), we have $\psi U_1 \ln f_1 = 0$. This implies that f_1 is a constant function and hence first two factor of Ω generate CR-submanifold structure. Following the definition in [39], Ω becomes Skew CR-warped product. $\square$

In similar way the next proposition is obtained.

Proposition 2. If a bi-warped product submanifold $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ of a nearly Kenmotsu manifold $\tilde{\Omega}$ is a $\mathcal{D} \oplus \mathcal{D}^\phi$ -mixed totally geodesic, then f_2 is constant function.

Proof. Adding Equations (30) and (33), we obtain

$$g(\mathcal{B}(U_1, W_1), FW_2) + g(\mathcal{B}(\psi U_1, W_1), FTW_2) = \frac{1}{3}(\psi U_1 \ln f_2)g(W_1, W_2) - \frac{2}{3}\{(U_1 \ln f_2) - \eta(U_1)\}g(TW_1, W_2).$$

As we considered in theorem that Ω is a $\mathcal{D} \oplus \mathcal{D}^\phi$ -mixed totally geodesic, thus proceeding equation gives

$$(\cos^2 \phi - 3)(\psi U_1 \ln f_2) = 0 \quad (43)$$

which is equivalent that either $\cos^2 \phi = \pm\sqrt{3}$ but it is not possible for $0 \leq \phi \leq \frac{\pi}{2}$ or $\psi U_1 \ln f_2 = 0$. The second condition gives that f_2 is a constant function. $\square$

Remark 2. Proposition 1 and Proposition 2 insure that there do not exists any non-trivial proper bi-warped product submanifold $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ of a nearly Kenmotsu manifold $\tilde{\Omega}$ with $\mathcal{D} \oplus \mathcal{D}^\phi$ -mixed totally geodesic and $\mathcal{D} \oplus \mathcal{D}^\perp$ -mixed totally geodesic restrictions.

Inequality for the Second Fundamental Form

Assuming that $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ is n -dimensional proper bi-warped product submanifold of the nearly Kenmotsu manifold $\tilde{\Omega}^{2m+1}$. Considering the local orthonormal frame field $u_1, \dots, u_n$ of $T\Omega$ that is

$$\begin{aligned} \mathcal{D} &= \text{Span}\{u_1, \dots, u_t, u_{t+1} = \psi u_1, \dots, u_{2t} = \psi u_t\} \\ \mathcal{D}^\perp &= \text{Span}\{u_{2t+1} = \hat{u}_1, \dots, u_{2t+l} = \hat{u}_l\} \\ \mathcal{D}^\phi &= \text{Span}\{u_{2t+l+1} = u_1^*, \dots, u_{2t+l+k} = u_k^*, u_{2t+l+k+1} = \sec \phi Tu_1^*, \dots, u_n = \sec \phi Tu_k^*\}. \end{aligned}$$

Therefore $\dim \Omega_T = 2t$, $\dim \Omega_\perp = l$ and $\dim \Omega_\phi = 2k$. Furthermore, the orthonormal frame fields $v_1, \dots, v_{2m+1-n-l-2k}$ of the normal subbundle $T^\perp \Omega$ are defined by

$$\begin{aligned} \psi \mathcal{D}^\perp &= \text{Span}\{v_1 = \psi \hat{u}_1, \dots, v_l = \psi \hat{u}_l\}, \\ F\mathcal{D}^\phi &= \text{Span}\{v_{l+1} = \tilde{v}_1 = \csc \phi Fu_1^*, \dots, v_{l+k} = \tilde{v}_k = \csc \phi Fu_k^*, \\ v_{l+k+1} &= \csc \phi \sec \phi FTu_1^*, \dots, v_{l+2k} = \tilde{v}_{2k} = \csc \phi \sec \phi FTu_k^*\}, \\ \mu &= \text{Span}\{v_{l+2k+1}, \dots, v_{2m+1-n-l-2k}\}. \end{aligned}$$

This paper's essential outcome is the following direct inequality for bi-warped product submanifolds of a nearly Kenmotsu manifold.

Theorem 1. Let $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ be a bi-warped product submanifold of a nearly Kenmotsu manifold $\tilde{\Omega}$, where $\zeta \in \Omega_T$, Ω_ϕ and $\Omega_\perp$ are holomorphic, proper slant and totally real submanifolds of $\tilde{\Omega}$, respectively. Then

(A) The second fundamental form $\mathcal{B}$ and warping functions f_1, f_2 insure

$$\|\mathcal{B}\|^2 \geq 2l \left\{ \|\nabla(\ln f_1)\|^2 - 1 \right\} + \frac{4k}{9} \left(10 \csc^2 \phi - 1 \right) \left\{ \|\nabla(\ln f_2)\|^2 - 1 \right\} \quad (44)$$

where $l = \dim \Omega_\perp$ and $k = \frac{1}{2} \dim \Omega_\phi$. Moreover, $\nabla(\ln f_i)$ is the gradient of $\ln f_i$.

(B) In case the equality sign at (44) holds identically, therefore, Ω_T is totally geodesic and $\Omega_\perp, \Omega_\phi$ are totally umbilical in $\tilde{\Omega}$. Furthermore, Ω has no $\mathcal{D} \oplus \mathcal{D}^\perp$ -mixed totally geodesic and $\mathcal{D} \oplus \mathcal{D}^\phi$ -mixed totally geodesic in $\tilde{\Omega}$.

Proof. Using the $\mathcal{B}$ meaning, we got

$$\|\mathcal{B}\|^2 = \sum_{i,j=1}^n g(\mathcal{B}(u_i, u_j), \mathcal{B}(u_i, u_j)).$$

Later, the previous relation is decomposed for the normal subbundles as next

$$\begin{aligned} \|\mathcal{B}\|^2 &= \sum_{r=1}^l \sum_{i,j=1}^n g^2(\mathcal{B}(u_i, u_j), \psi \hat{u}_r) + \sum_{r=l+1}^{l+2k} \sum_{i,j=1}^n g^2(\mathcal{B}(u_i, u_j), v_r) \\ &+ \sum_{r=l+2k+1}^{m+1-n-l-2k} \sum_{i,j=1}^n g^2(\mathcal{B}(u_i, u_j), v_r). \end{aligned} \quad (45)$$

Leaving the last term and remaining terms, and expressing according the orthonormal basis of $\mathcal{D}, \mathcal{D}^\perp$ and $\mathcal{D}^\phi$, we have

$$\begin{aligned} \|\mathcal{B}\|^2 &\geq \sum_{r=1}^l \sum_{i,j=1}^{2t+1} g^2(\mathcal{B}(u_i, u_j), \psi \hat{u}_r) + \sum_{r=1}^l \sum_{i,j=1}^l g^2(\mathcal{B}(\hat{u}_i, \hat{u}_j), \psi \hat{u}_r) \\ &+ \sum_{r=1}^l \sum_{i,j=1}^{2k} g^2(\mathcal{B}(u_i^*, u_j^*), \psi \hat{u}_r) + \sum_{r=1}^{2k} \sum_{i,j=1}^{2t+1} g^2(\mathcal{B}(u_i, u_j), \tilde{v}_r) \\ &+ \sum_{r=1}^{2k} \sum_{i,j=1}^l g^2(\mathcal{B}(\hat{u}_i, \hat{u}_j), \tilde{v}_r) + \sum_{r=1}^{2k} \sum_{i,j=1}^{2k} g^2(\mathcal{B}(u_i^*, u_j^*), \tilde{v}_r) \\ &+ 2 \sum_{r=1}^l \sum_{i=1}^{2t+1} \sum_{j=1}^l g^2(\mathcal{B}(u_i, \hat{u}_j), \psi \hat{u}_r) + 2 \sum_{r=1}^l \sum_{i=1}^{2t+1} \sum_{j=1}^{2k} g^2(\mathcal{B}(u_i, u_j^*), \psi \hat{u}_r) \\ &+ 2 \sum_{r=1}^l \sum_{i=1}^{2k} \sum_{j=1}^l g^2(\mathcal{B}(u_i^*, \hat{u}_j), \psi \hat{u}_r) + 2 \sum_{r=1}^{2k} \sum_{i=1}^l \sum_{j=1}^{2k} g^2(\mathcal{B}(\hat{u}_i, u_j^*), \tilde{v}_r) \\ &+ 2 \sum_{r=1}^{2k} \sum_{i=1}^{2t+1} \sum_{j=1}^{2k} g^2(\mathcal{B}(u_i, u_j^*), \tilde{v}_r) + 2 \sum_{r=1}^{2k} \sum_{i=1}^{2t+1} \sum_{j=1}^l g^2(\mathcal{B}(u_i, \hat{u}_j), \tilde{v}_r). \end{aligned}$$

Utilizing Equations (19), (20) and (29) in the above equation, we arrive at

$$\begin{aligned} \|\mathcal{B}\|^2 &\geq \sum_{r=1}^l \sum_{i,j=1}^l g(\mathcal{B}(\hat{u}_i, \hat{u}_j), \psi \hat{u}_r) + \sum_{r=1}^l \sum_{i,j=1}^{2k} g(\mathcal{B}(u_i^*, u_j^*), \psi \hat{u}_r) \\ &+ \sum_{r=1}^{2k} \sum_{i,j=1}^{2k} g^2(\mathcal{B}(u_i^*, u_j^*), \tilde{v}_r) + \sum_{r=1}^{2k} \sum_{i,j=1}^l g^2(\mathcal{B}(\hat{u}_i, \hat{u}_j), \tilde{v}_r) \\ &+ 2 \sum_{r=1}^{2k} \sum_{i=1}^l \sum_{j=1}^{2k} g^2(\mathcal{B}(\hat{u}_i, u_j^*), \tilde{v}_r) + 2 \sum_{r=1}^l \sum_{i=1}^{2k} \sum_{j=1}^l g^2(\mathcal{B}(u_i^*, \hat{u}_j), \psi \hat{u}_r) \\ &+ 2 \sum_{r=1}^l \sum_{i=1}^{2t+1} \sum_{j=1}^l g^2(\mathcal{B}(u_i, \hat{u}_j), \psi \hat{u}_r) + 2 \sum_{r=1}^{2k} \sum_{i=1}^{2t+1} \sum_{j=1}^{2k} g^2(\mathcal{B}(u_i, u_j^*), \tilde{v}_r). \end{aligned} \quad (46)$$

Considering only last two terms from the above equation, we get

$$\begin{aligned}\|\mathcal{B}\|^2 &\geq 2 \sum_{r=1}^l \sum_{i=1}^{2t} \sum_{j=1}^l g^2(\mathcal{B}(u_i, \hat{u}_j), \psi \hat{u}_r) + 2 \sum_{r=1}^l \sum_{j=1}^l g^2(\mathcal{B}(\zeta, \hat{u}_j), \psi \hat{u}_r) \\ &\quad + 2 \sum_{r=1}^{2k} \sum_{i=1}^{2t} \sum_{j=1}^{2k} g^2(\mathcal{B}(u_i, u_j^*), \tilde{v}_r) + 2 \sum_{r=1}^{2k} \sum_{j=1}^{2k} g^2(\mathcal{B}(\zeta, u_j^*), \tilde{v}_r).\end{aligned}\quad (47)$$

Releasing the latest μ -components part at (45) and by the use of the frame fields of tangent and normal subbundles of Ω , we obtain

$$\begin{aligned}\|\mathcal{B}\|^2 &\geq 2 \sum_{r=1}^l \sum_{i=1}^t \sum_{j=1}^l g^2(\mathcal{B}(u_i, \hat{u}_j), \psi \hat{u}_r) + 2 \sum_{r=1}^l \sum_{i=1}^t \sum_{j=1}^l g^2(\mathcal{B}(\psi u_i, \hat{u}_j), \psi \hat{u}_r) \\ &\quad + 2 \sum_{r=1}^l \sum_{j=1}^l g^2(\mathcal{B}(\zeta, \hat{u}_j), \psi \hat{u}_r) + 2 \sum_{r=1}^{2k} \sum_{j=1}^{2k} g^2(\mathcal{B}(\zeta, u_j^*), \tilde{v}_r) \\ &\quad + 2 \csc^2 \phi \sum_{r,j=1}^k \sum_{i=1}^t \left\{ g^2(\mathcal{B}(u_i, u_j^*), Fu_r^*) + g^2(\mathcal{B}(\psi u_i, u_j^*), Fu_r^*) \right\} \\ &\quad + 2 \csc^2 \phi \sec^2 \phi \sum_{r,j=1}^k \sum_{i=1}^t \left\{ g^2(\mathcal{B}(u_i, Tu_j^*), Fu_r^*) + g^2(\mathcal{B}(\psi u_i, Tu_j^*), Fu_r^*) \right\} \\ &\quad + 2 \csc^2 \phi \sec^2 \phi \sum_{r,j=1}^k \sum_{i=1}^t \left\{ g^2(\mathcal{B}(\hat{u}_i, u_j^*), FTu_r^*) + g^2(\mathcal{B}(\psi u_i, u_j^*), FTu_r^*) \right\} \\ &\quad + 2 \csc^2 \phi \sec^4 \phi \sum_{r,j=1}^k \sum_{i=1}^t \left\{ g^2(\mathcal{B}(\hat{u}_i, Tu_j^*), FTu_r^*) + g^2(\mathcal{B}(\psi u_i, Tu_j^*), FTu_r^*) \right\}.\end{aligned}\quad (48)$$

Utilizing Equations (21) and (22) in the first terms, and then substitute (29)–(37) to give

$$\begin{aligned}\|\mathcal{B}\|^2 &\geq 2l \sum_{i=1}^t \left\{ (-\psi u_i \ln f_1)^2 + (u_i \ln f_1)^2 \right\} + 2k \csc^2 \phi \sum_{i=1}^t \left\{ (-\psi u_i \ln f_2)^2 + (u_i \ln f_2)^2 \right\} \\ &\quad + \frac{2k}{9} \csc^2 \phi \sec^2 \phi \cos^4 \phi \sum_{i=1}^t \left\{ (-\psi u_i \ln f_2)^2 + (u_i \ln f_2)^2 \right\} \\ &\quad + \frac{2k}{9} \csc^2 \phi \sec^2 \phi \cos^4 \phi \sum_{i=1}^t \left\{ (\psi u_i \ln f_2)^2 + (u_i \ln f_2)^2 \right\} \\ &\quad + 2k \csc^2 \phi \sec^4 \phi \cos^4 \phi \sum_{i=1}^t \left\{ (-\psi u_i \ln f_1)^2 + (u_i \ln f_1)^2 \right\} \\ &\quad + 2 \sum_{r=1}^l \sum_{j=1}^l g^2(\mathcal{B}(\zeta, \hat{u}_j), \psi \hat{u}_r) + 2 \sum_{r=1}^{2k} \sum_{j=1}^{2k} g^2(\mathcal{B}(\zeta, u_j^*), \tilde{v}_r).\end{aligned}$$

From (23) and (38), last two terms in the above equation should be zero. With some rearrangement in remaining terms, we arrive at

$$\|\mathcal{B}\|^2 \geq 2l \sum_{i=1}^{2t} (u_i \ln f_1)^2 + 4k \csc^2 \phi \sum_{i=1}^{2t} (u_i \ln f_1)^2 + \frac{4k}{9} \cot^2 \phi \sum_{i=1}^{2t} (u_i \ln f_1)^2.$$

Adding and subtracting some terms and using trigonometric identifies, we derive

$$\begin{aligned}\|\mathcal{B}\|^2 &\geq 2l\|\nabla \ln f_1\|^2 - 2l(u_{2t+1} \ln f_1)^2 + \frac{4k}{9}\left(10 \csc^2 \phi - 1\right)\|\nabla \ln f_2\|^2 \\ &\quad - \frac{4k}{9}\left(10 \csc^2 \phi - 1\right)(u_{2t+1} \ln f_2)^2.\end{aligned}$$

It is well-known that for a nearly Kenmotsu manifold, the two conditions are satisfied such that $\zeta \ln f_1 = 1$ and $\zeta \ln f_2 = 1$, for structure vector field ζ tangent to the first factor Ω_T . Inserting the previous values in proceeding equation, we get

$$\|\mathcal{B}\|^2 \geq 2l\|\nabla \ln f_1\|^2 - 2l + \frac{4k}{9}\left(10 \csc^2 \phi - 1\right)\|\nabla \ln f_2\|^2 - \frac{4k}{9}\left(10 \csc^2 \phi - 1\right).$$

The above equation implies Inequality (44). In case of equality holds in (44), part number three that is ignored at (45) gives

$$\mathcal{B}(TM, TM) \perp \mu. \quad (49)$$

Using term number one that vanishes by (19), (20) and (46) gives

$$\mathcal{B}(\mathcal{D}, \mathcal{D}) \perp \psi \mathcal{D}^\perp \quad \text{and} \quad \mathcal{B}(\mathcal{D}, \mathcal{D}) \perp F\mathcal{D}^\phi. \quad (50)$$

Therefore using (49) and (50), we have

$$\mathcal{B}(\mathcal{D}, \mathcal{D}) = 0. \quad (51)$$

From another point of view, ignoring the 1st and 4th terms in Equation (46) leads to

$$\mathcal{B}(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp \psi \mathcal{D}^\perp \quad \text{and} \quad \mathcal{B}(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp F\mathcal{D}^\phi. \quad (52)$$

One more time, using (49) and (52), it is concluded that

$$\mathcal{B}(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0. \quad (53)$$

In addition, terms number two and three that are ignored at (46) give

$$\mathcal{B}(\mathcal{D}^\phi, \mathcal{D}^\phi) \perp \psi \mathcal{D}^\perp \quad \text{and} \quad \mathcal{B}(\mathcal{D}^\phi, \mathcal{D}^\phi) \perp F\mathcal{D}^\phi. \quad (54)$$

Therefore, using (49) and (54) leads to

$$\mathcal{B}(\mathcal{D}^\phi, \mathcal{D}^\phi) = 0. \quad (55)$$

Furthermore, 5th and 6th terms that are ignored at (46) give

$$\mathcal{B}(\mathcal{D}^\perp, \mathcal{D}^\phi) \perp \psi \mathcal{D}^\perp \quad \text{and} \quad \mathcal{B}(\mathcal{D}^\perp, \mathcal{D}^\phi) \perp F\mathcal{D}^\phi. \quad (56)$$

Then, using (49) and (56) leads to

$$\mathcal{B}(\mathcal{D}^\perp, \mathcal{D}^\phi) = 0. \quad (57)$$

In contrast, using the second term that vanishes in (29) along with (49) gives

$$\mathcal{B}(\mathcal{D}, \mathcal{D}^\perp) \subset \psi \mathcal{D}^\perp. \quad (58)$$

In similar way, from the zero first term in (29) along with (49), we have

$$\mathcal{B}(\mathcal{D}, \mathcal{D}^\phi) \subset F\mathcal{D}^\phi. \quad (59)$$

Because of Ω_T is totally geodesic in $\tilde{\Omega}$ follows from [22,25,31]), by the use of this point along with (51), (53) and (57), it is known that Ω_T is totally geodesic in $\tilde{\Omega}$. In addition, because of $\Omega_\perp$ and Ω_ϕ are totally umbilical in Ω and by the use of this point along with (53), (55), (58) and (59), it is concluded that $\Omega_\perp$ and Ω_ϕ together are totally umbilical in $\tilde{\Omega}$. Moreover, using Remark 2, (58) and (59) gives that Ω is neither $\mathcal{D} \oplus \mathcal{D}^\perp$ -mixed totally geodesic nor $\mathcal{D} \oplus \mathcal{D}^\phi$ -mixed totally geodesic at $\tilde{\Omega}$. As a result, the proof is completed. $\square$

4. Some Geometric Applications

Remark 3. If we substitute $\dim \Omega_\phi = 2k = 0$ in Theorem 1, then $\Omega = \Omega_T \times_f \Omega_\perp$ becomes CR-warped product submanifold in a nearly Kenmotsu manifold and Inequality (44) is reduced to the following inequality

$$\|\mathcal{B}\|^2 \geq 2l \left(\|\nabla \ln f\|^2 - 1 \right) \quad (60)$$

where $l = \dim \Omega_\perp$. Furthermore, the equality sign in (60) remains identically, at that point $\Omega_\perp$ is totally umbilical and Ω_T is totally geodesic in $\tilde{\Omega}$. Now, it should be noticed that Inequality (60) coincides with inequality (7) of Theorem 3.1 in [5]. Therefore, our derived Theorem 1 is a generalization of Theorem 3.1 in [5]. On the other hand, if we consider $\alpha = 0$ and $\beta = 1$ of Theorem 4.1 in [32], then the nearly trans Sasakian manifold becomes a nearly Kenmotsu manifold. Hence, Inequality (4.1) of Theorem 4.1 in [32] is associated with Inequality (60). It can be easily seen that Theorem 4.1 [32] is a special case of our Theorem 1.

Remark 4. Let the dimension of a totally real submanifold $\Omega_\perp$ vanish, i.e., $\dim \Omega_\perp = l = 0$. Then the bi-warped submanifold Ω is formed a warped product semi-slant submanifold of type $\Omega = \Omega_T \times_f \Omega_\phi$ in a nearly Kenmotsu manifold $\tilde{\Omega}$, and then Inequality (44) implies the following inequality

$$\|\mathcal{B}\|^2 \geq 4k \left(\frac{1}{9} \cot^2 \phi + \csc^2 \phi \right) \left\{ \|\nabla \ln f\|^2 - 1 \right\} \quad (61)$$

with $\dim \Omega_\phi = 2k$. Again, we have acknowledged that if $\alpha = 0$ and $\beta = 1$ in Theorem 4.1 [33], then Inequality (4.1) of Theorem 4.1 in [33] and Inequality (61) are accorded to each other. Therefore, Theorem 1 is an extension of Theorem 4.1 [33] with some conditions.

Remark 5. As we know that the following inequality holds in general

$$\cot^2 \phi \geq \frac{\cot^2 \phi}{9} \implies \left(\cot^2 \phi + \csc^2 \phi \right) \geq \left(\frac{\cot^2 \phi}{9} + \csc^2 \phi \right) \quad (62)$$

for $\phi \in [0, \frac{\pi}{2}]$ as a definition of slant submanifold, then Inequality (61) for warped product semi-slant submanifold $\Omega = \Omega_T \times_f \Omega_\phi$ with help of (62) is equal to the following

$$\|\mathcal{B}\|^2 \geq 4k \left(\cot^2 \phi + \csc^2 \phi \right) \left\{ \|\nabla \ln f\|^2 - 1 \right\}. \quad (63)$$

It is recognized that Inequality (63) is exactly Inequality (24) of Theorem 3 in [6]. Now we reached the confirmation that Theorem 3 in [6] can be generalized from Theorem 1.

Remark 6. A multiple warped product $\Omega_T \times_{f_1} \Omega_\perp^1 \times \cdots \times_{f_p} \Omega_\perp^p$ of a nearly Kenmotsu manifold is said to be a multiple CR-warped product if $\phi = \frac{\pi}{2}$, then Ω_ϕ is equal to a totally real submanifold $\Omega_\perp$ and $\csc(\frac{\pi}{2}) = 1$. Now,

we calculate $\frac{4k}{9}(10-1) = 4k$. Inserting these values into Inequality (44) of Theorem 1, we get the bi-warped product submanifold of kind $\Omega = \Omega_T \times_{f_1} \Omega_\perp^1 \times_{f_2} \Omega_\perp^2$ of a nearly Kenmotsu manifold and Inequality (44) gives

$$\|\mathcal{B}\|^2 \geq 2l_1 \left\{ \|\nabla(\ln f_1)\|^2 - 1 \right\} + 2l_2 \left\{ \|\nabla(\ln f_2)\|^2 - 1 \right\} \quad (64)$$

with $\dim \Omega_\perp^1 = l_1$ and $\dim \Omega_\perp^2 = l_2$. Of course, Inequality (64) is defined for a multiple CR-warped product with two fibers. By rule of mathematical induction with p fibers, Inequality (64) can be extended for a multiple CR-warped product submanifold $\Omega_T \times_{f_1} \Omega_\perp^1 \times \cdots \times_{f_p} \Omega_\perp^p$ of a nearly Kenmotsu manifold and get the following

$$\|\mathcal{B}\|^2 \geq 2 \sum_{i=1}^p l_i \left\{ \|\nabla(\ln f_i)\|^2 - 1 \right\} \quad (65)$$

where $l_p = \dim \Omega_\perp^p$. Thus, Theorem 1.2 of [36] and Theorem 4.1 [40] are particular cases of Theorem 4.1 for two fibers.

Remark 7. From the trigonometric relation between $\cot \phi$ and $\csc \phi$ with combining the condition (62), thus Theorem 1 is reconstructed as

Theorem 2. Let $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ be a bi-warped product submanifold of a nearly Kenmotsu manifold $\tilde{\Omega}$. Then

(A) The second fundamental form $\mathcal{B}$ and warping functions f_1, f_2 insure

$$\|\mathcal{B}\|^2 \geq 2l \left\{ \|\nabla(\ln f_1)\|^2 - 1 \right\} + 4k \left(\cot^2 \phi + \csc^2 \phi \right) \left\{ \|\nabla(\ln f_2)\|^2 - 1 \right\} \quad (66)$$

where $l = \dim \Omega_\perp, k = \frac{1}{2} \dim \Omega_\phi$. Moreover, $\nabla(\ln f_i)$ is the gradient of $\ln f_i$.

(B) In case the equality sign at (44) holds identically, therefore Ω_T is totally geodesic and $\Omega_\perp, \Omega_\phi$ are totally umbilical in $\tilde{\Omega}$. Furthermore, Ω is neither $\mathcal{D} \oplus \mathcal{D}^\perp$ -mixed totally geodesic nor $\mathcal{D} \oplus \mathcal{D}^\phi$ -mixed totally geodesic at $\tilde{\Omega}$.

The Equation (66) is the same as Equation (2). Finally we distinguish that Theorem 4.1 [4] is an exceptional case of Theorem 1.

5. Some Physical Applications

In this section, we investigate the Dirichlet energy that satisfies the following for a compact submanifold Ω and differentiable function $\theta : \Omega \rightarrow \mathbb{R}$, that is

$$E(\theta) = \frac{1}{2} \int_{\Omega} \|\nabla \theta\|^2 dV \quad (67)$$

where dV is a volume element. It was discovered that the minimal Dirichlet energy with boundary conditions on warping function has the solution of variation problems by using the minimum principle $\Delta \theta = 0$, as Δ denotes the Laplacian operator. For example, if θ is an electric potential in bounded domain Ω and ρ is a charge distribution, then they are related by the Poisson equation $\nabla^2 \theta = -4\pi\rho$. These types of applications for Dirichlet energy $E(\theta)$ are presented in [41]. They show that due to charge distribution and electric field, the whole energy of the system contained two components; one $\frac{1}{2} \int_{\Omega} \theta \rho dV$ and the other is a Dirichlet energy $\frac{1}{2} \int_{\Omega} \|\nabla \theta\|^2 dV$. Now we raise the question of under what boundary condition does the Poisson equation have a unique solution under the bounded domain? It is a simple answer, that is, by designation of the potential on a surface (the conductor

system gripped at various potentials) classified unique potential problems which are called Dirichlet problems. From this motivation, we give the following Theorem by combining (44) and (67)

Theorem 3. Let $\Omega = \Omega_T \times_{f_1} \Omega_\perp \times_{f_2} \Omega_\phi$ be a compact bi-warped product submanifold of a nearly Kenmotsu manifold $\tilde{\Omega}$ with $\zeta \in \Omega_T$. Then we have

$$4lE(\ln f_1) + \frac{8k}{9} \left(10 \csc^2 \phi - 1 \right) E(\ln f_2) \leq \int_{\Omega} \left\{ \|\mathcal{B}\|^2 + 2l + \frac{4k}{9} \left(10 \csc^2 \phi - 1 \right) \right\} dV \quad (68)$$

where $E(\ln f_1)$ and $E(\ln f_2)$ are Dirichlet energies of the warping functions f_1 and f_2 , respectively.

As an immediate applications of Theorem 3, we give corollaries as follow

Corollary 1. Assuming that $\Omega = \Omega_T \times_{f_1} \Omega_\perp$ is a compact CR-warped product submanifold of a nearly Kenmotsu manifold $\tilde{\Omega}$ with $\zeta \in \Omega_T$. Then we have

$$E(\ln f_1) \leq \frac{1}{4l} \int_{\Omega} \left(\|\mathcal{B}\|^2 + 2l \right) dV.$$

Corollary 2. Assuming that $\Omega = \Omega_T \times_{f_2} \Omega_\phi$ is a compact warped product semi-slant submanifold of a nearly Kenmotsu manifold $\tilde{\Omega}$ with $\zeta \in \Omega_T$. Then we have

$$E(\ln f_2) \leq \left(\frac{9 \sin^2 \phi}{8k(10 - \sin^2 \phi)} \right) \int_{\Omega} \left\{ \|\mathcal{B}\|^2 + \frac{4k}{9} \left(10 \csc^2 \phi - 1 \right) \right\} dV.$$

From (65), we obtain

Corollary 3. A compact multiple CR-warped product submanifold $\Omega_T \times_{f_1} \Omega_\perp^1 \times \cdots \times_{f_p} \Omega_\perp^p$ of a nearly Kenmotsu manifold gives

$$\sum_{i=1}^p l_i E(\ln f_i) \leq \frac{1}{4} \int_{\Omega} \left(\|\mathcal{B}\|^2 + 2 \sum_{i=1}^p l_i \right) dV. \quad (69)$$

where $l_p = \dim \Omega_\perp^p$.

The vanishing of Dirichlet energies is equivalent to the Dirichlet condition with the unique solution of the Poisson equation $\nabla^2 \theta = -4\pi\rho$. It implies that Neumann or Dirichlet boundary conditions classified to electrostatic problems.

6. Conclusions Remark

In brief, the warped product submanifolds study attracted more attention recently because of its importance in mathematics and its contribution to other related fields as mathematical physics. For example, if $\mathbb{S}^3$ indicates a three-dimensional manifold with constant curvature $\kappa = -1, 0, 1$ and $\mathbb{I}$ denotes an open interval in real line $\mathbb{R}$, then a warped product of the form $\Omega(\kappa, f) = \mathbb{I} \times_f \mathbb{S}^3$ with its metric $ds^2 = -dt^2 + f^2 ds_{\mathbb{S}^3}^2$ is a Robertson–Walker spacetime. It is famous that a cosmological model of the universe consists of a perfect fluid whose molecules are galaxies—Robertson–Walker spacetime. In the present work, we considered the bi-warped product submanifolds and more specifically, in a nearly Kenmotsu manifold. The introduction of some basics, inequality of the second fundamental form, is given and proved as a general case of some previous studies regarding warping functions. Moreover, we provided some geometrical and physical applications, and show that several results in [4–6,32,33,36,40,42] can be evaluated as particular cases of the main results of this work. Other future works are recommended for the same topic under different considerations.

Author Contributions: Writing and original draft, A.A.; funding acquisition, editing and draft, F.M.; review and editing, A.A.; methodology, project administration, F.M.; formal analysis, resources, F.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

Acknowledgments: This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Taştan, H.M. Biwarped product submanifolds of a Kähler manifold. *Filomat* **2018**, *32*, 2349–2365. [\[CrossRef\]](#)
2. Chen, B.Y. Geometry of warped product CR-submanifolds in Kaehler manifolds. *Monatsh. Math.* **2001**, *133*, 177–195. [\[CrossRef\]](#)
3. Sahin, B. Warped product pointwise semi-slant submanifolds of Kähler manifolds. *Port. Math.* **2013**, *70*, 251–268. [\[CrossRef\]](#)
4. Uddin, S.; Al-Solamy, F.R.; Shahid, M.H.; Saloom, A.B.-Y. Chen's inequality for bi-warped products and its applications in Kenmotsu manifolds. *Mediterr. J. Math.* **2018**, *15*, 193. [\[CrossRef\]](#)
5. Arslan, K.; Ezentas, R.; Mihai, I.; Murathan, C. Contact CR-warped product submanifolds in Kenmotsu space forms. *J. Korean Math. Soc.* **2005**, *42*, 1101–1110. [\[CrossRef\]](#)
6. Uddin, S. Geometry of warped product semi-slant submanifolds of Kenmotsu manifolds. *Bull. Math. Sci.* **2018**, *8*, 435–451. [\[CrossRef\]](#)
7. Uddin, S.; Alkhaldi, A.H. Bi-warped product submanifolds of Kenmotsu manifolds and their applications. *Int. J. Geom. Methods Mod. Phys.* **2019**, *16*, 1950001. [\[CrossRef\]](#)
8. Al-Solamy, F.R.; Naghi, M.F.; Uddin, S. Geometry of warped product pseudo-slant submanifolds of Kenmotsu manifolds. *Quaest. Math.* **2019**, *42*, 373–389. [\[CrossRef\]](#)
9. Uddin, S.; Naghi, M.F.; Al-Solamy, F.R. Another class of warped product submanifolds of Kenmotsu manifolds. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mater.* **2018**, *112*, 1141–1155. [\[CrossRef\]](#)
10. Naghi, M.F.; Uddin, S.S.; Stankovic, M.S. Another general inequality for bi-warped products in Kenmotsu manifolds. *C. R. Acad. Bulg. Sci.* **2019**, *72*, 1458–1467.
11. Naghi, M.F.; Uddin, S.; Al-Solamy, F.R. Warped product submanifolds of Kenmotsu manifolds with slant fiber. *Filomat* **2018**, *32*, 2115–2126. [\[CrossRef\]](#)
12. Uddin, S.; Alghanemi, A.; Naghi, M.F.; Al-Solamy, F.R. Another class of warped product CR-submanifolds in Kenmotsu manifolds. *J. Math. Comput. Sci.* **2017**, *17*, 148–157. [\[CrossRef\]](#)
13. Khan, V.A.; Khan, K.A. Generic warped product submanifolds in nearly Kaehler manifolds. *Beiträge Algebra Geom.* **2009**, *50*, 337–352.
14. Al-Solamy, F.R.; Khan, V.A.; Uddin, S. Geometry of warped product semi-slant submanifolds of nearly Kaehler manifolds. *Results Math.* **2017**, *71*, 783–799. [\[CrossRef\]](#)
15. Uddin, S.; Chen, B.Y.; Awatif, A.J.; Alghanemi, A. Bi-warped product submanifolds of nearly Kähler manifolds. *Bull. Malays. Math. Sci. Soc.* **2020**, *42*, 1945–1958. [\[CrossRef\]](#)
16. Uddin, S.; Mihai, A.; Mihai, I.; Awatif, A.J. Geometry of bi-warped product submanifolds of locally product Riemannian manifolds. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mater.* **2020**, *144*, 42. [\[CrossRef\]](#)
17. Al-Solamy, F.R.; Uddin, S. An inequality for warped product submanifolds of a locally product Riemannian manifold. *Hacet. J. Math. Stat.* **2019**, *48*, 351–358. [\[CrossRef\]](#)
18. Atçeken, M. Warped product semi-invariant submanifolds in locally decomposable Riemannian manifolds. *Hacet. J. Math. Stat.* **2011**, *40*, 401–407.
19. Sahin, B. Warped product semi-invariant submanifolds of a locally product Riemannian manifold. *Bull. Math. Soc. Sci. Math. Roum.* **2006**, *49*, 383–394.
20. Uddin, S.; Ali, A.; Al-Asmari, N.M.; Othman, W.A. Warped product pseudo-slant submanifolds in a locally product Riemannian manifold. *Diff. Geom. Dyn. Syst.* **2016**, *18*, 147–158.
21. Ali, A.; Othman, W.A.M.; Ozel, C. Some inequalities for warped product pseudo-slant submanifolds of nearly Kenmotsu manifolds. *J. Inequal. Appl.* **2015**, *2015*, 1–12.

22. Ali, A.; Othman, W.A.M.; Ozel, C.; Hajjari, T. A geometric inequality for warped product pseudo-slant submanifolds of nearly Sasakian manifolds. *C. R. Acad. Bulg. Sci.* **2017**, *70*, 175–182.
23. Ali, A.; Ozel, C. Geometry of warped product pointwise semi-slant submanifolds of cosymplectic manifolds and its applications. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 1750042. [\[CrossRef\]](#)
24. Ali, A.; Laurian-Ioan, P. Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions. *J. Geom. Phys.* **2017**, *114*, 276–290. [\[CrossRef\]](#)
25. Ali, A.; Laurian-Ioan, P. Geometric classification of warped products isometrically immersed into Sasakian space forms. *Math. Nachr.* **2019**, *292*, 234–251.
26. Alkhaldi, A.H.; Ali, A. Classification of warped product submanifolds in Kenmotsu space forms admitting gradient Ricci solitons. *Mathematics* **2019**, *7*, 112. [\[CrossRef\]](#)
27. Ali, R.; Alkhaldi, A.H.; Ali, A.; Othman, W.A.M. The first eigenvalue estimates of warped product pseudo-slant submanifolds. *Mathematics* **2019**, *7*, 162. [\[CrossRef\]](#)
28. Chen, B.Y. *Differential Geometry of Warped Product Manifolds and Submanifolds*; World Scientific: Singapore, 2017.
29. Alodan, H.; Deshmukh, S.; Turki, N.B.; Vilcu, G.E. Hypersurfaces of a Sasakian manifold. *Mathematics* **2020**, *8*, 877. [\[CrossRef\]](#)
30. Küpeli, E.; Dacko, P.; Murathan, C. On the existence of proper nearly Kenmotsu manifolds. *Mediterr. J. Math.* **2016**, *16*, 4497–4507.
31. Heidari, N.; Kashani, N.H.P.; Najafi, B. Nearly Kähler and nearly Kenmotsu manifolds. *Turk. J. Math.* **2018**, *42*, 67–976.
32. Mustafa, A.; Uddin, S.; Khan, V.A.; Wong, R.B. Contact CR-warped product submanifolds of nearly trans-Sasakian manifolds. *Taiwan. J. Math.* **2013**, *17*, 1473–1486. [\[CrossRef\]](#)
33. Mustafa, A.; Uddin, S.; Wong, R.B. Generalized inequalities on warped product submanifolds in nearly trans-Sasakian manifolds. *J. Inequal. Appl.* **2014**, *2014*, 346. [\[CrossRef\]](#)
34. Cabrerizo, J.L.; Carriazo, A.; Fernandez, L.M.; Fernandez, M. Slant submanifolds in Sasakian manifolds. *Glasg. Math. J.* **2000**, *42*, 125–138. [\[CrossRef\]](#)
35. Cabrerizo, J.L.; Carriazo, A.; Fernandez, L.M.; Fernandez, M. Semi-slant submanifolds of a Sasakian manifold. *Geom. Dedicata* **1999**, *78*, 183–199. [\[CrossRef\]](#)
36. Chen, B.Y.; Dillen, F. Optimal inequalities for multiply warped product submanifolds. *Int. Electron. J. Geom.* **2008**, *1*, 1–11.
37. Ünal, B. Multiply warped products. *J. Geom. Phys.* **2000**, *34*, 287–301. [\[CrossRef\]](#)
38. Bishop, R.L.; O'Neill, B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* **1969**, *145*, 1–49. [\[CrossRef\]](#)
39. Sahin, B. Skew CR-warped products of Kaehler manifolds. *Math. Commun.* **2010**, *15*, 189–204.
40. Olteanu, A. Multiply warped product submanifolds in Kenmotsu space forms. *Bull. Inst. Math. Acad. Sin.* **2010**, *5*, 201–214.
41. Jackson, J.D. *Classical Electrodynamics*; Wiley: New York, NY, USA, 1962.
42. Chen, B.Y. Geometry of warped product CR-submanifolds in Kaehler manifolds, II. *Monatsh. Math.* **2001**, *134*, 103–119. [\[CrossRef\]](#)

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).