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Fixed Point Results of Expansive Mappings in Metric Spaces

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Abstract: In this study, we introduce the concept of θ-expansive mapping in ordered metric spaces and prove a fixed point theorem for such mappings. We give some fixed point results for θ-expansive mapping in metric spaces and prove fixed point theorems for such mappings. These results extend the main results of many comparable results from the current literature. We also obtain a common fixed point theorem of two weakly compatible mappings in metric spaces. Finally, the examples are presented to support the new theorems and results proved.

Keywords: metric spaces; weakly compatible; ordered metric spaces; θ-expansive mapping

MSC: 47H10; 54H25

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1. Introduction

The study of expansive mappings is a very interesting research area in the fixed point theory. Wang et al. [1] proved some fixed point theorems for expansion mappings, which correspond to some contractive mappings in metric spaces. Rhoades [2] and Taniguchi [3] generalized the results of Wang for pair of mappings. Thereafter, several authors obtained many fixed point theorems for expansive mappings. For more details see [4–8]. Sessa [9] defined weak commutativity and proved a common fixed point theorem for weakly commuting maps. Further, Jungck [10] introduced the concept of weakly compatible maps by giving the notion of compatibility.

Definition 1 ([9])**.** *Let A and B be self mappings of a set Y. A point $y \in Y$ is called a coincidence point of A and B iff $Ay = By$. In this case, $s = Ay = By$ is called a point of coincidence of A and B.*

Definition 2 ([10])**.** *Two self mappings A and B of a metric space (Y, d) are said to be weakly compatible iff there is a point $y \in Y$ which is a coincidence point of A and B at which A and B commute; that is, $ABy = BAy$.*

Theorem 1 ([1])**.** *Let (Y, d) be a complete metric space and A a self mapping on Y. If A is surjective and satisfies*

$$d(Ax, Az) \geqslant qd(x, z) \tag{1}$$

for all $x, z \in Y$, with $q > 1$ then A has a unique fixed point in Y.

Ran and Reurings [11] proved a fixed point theorem on a partially ordered metric space as follow:

Theorem 2 ([11])**.** *Let $(Y, \preceq)$ be an ordered set and d be a metric on Y such that (Y, d) is a complete metric space. Let $A : Y \to Y$ be a nondecreasing mapping such that there exists $x_0 \in Y$ with $x_0 \preceq Ax_0$. Suppose that there exists $L \in [0, 1)$ such that*

$$d(Ax, Ay) \leqslant Ld(x, y) \quad \forall x, y \in Y \;\; with \;\; x \preceq y.$$

If A is continuous then has a fixed point in Y.

Then several authors considered the problem of the existence of a fixed point for contraction type operators on partially ordered sets. Some of these works may be noted in [12–16].

Lately, Jleli and Samet [17] introduced a new type of contractions called θ-contraction. They denote by Θ the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;
(Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;
(Θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Lemma 1 ([7])**.** *Let (Y, d) be a metric spaces and $A : Y \to Y$ a surjective mapping. Then, A has a right inverse mapping i.e., a mapping $A^* : Y \to Y$, such that $A \circ A^* = I_A$.*

Let $(Y, \preceq)$ be an ordered set and d be a metric on Y. Then we say that the tripled $(Y, \preceq, d)$ is an ordered metric space. We will say that Y is regular, if the ordered metric spaces $(Y, \preceq, d)$ provides the following condition:

If $\{x_n\} \subseteq Y$ is an increasing sequence with $x_n \to a \in Y$, then $x_n \preceq a$ for all $n \in \mathbb{N}$.

In Section 1, some basic definitions, lemma, and theorems in the literature that will be used later in the paper are given. In Section 2, following by Wang et al. [1], Jungck [10], Sessa [9], Jleli and Samet [17], we introduce a new approach to expansion mappings in fixed point theory and establish some fixed point theorems. We show illustrative examples where the theorems are applicable. In Section 3, we give conclusions.

2. Main Results

In this section, we introduce a fixed point theorem for θ-expansive mapping on ordered metric spaces. Then, we give some fixed point results for θ-expansive mappings in metric spaces. We also obtain a common fixed point theorem of two weakly compatible mappings in metric spaces. First, let us start with the definition of θ-expansive mappings on ordered metric spaces.

Definition 3. *Let $(Y, \preceq, d)$ be an ordered metric space. A mapping $A : Y \to Y$ is said to be surjective θ-expansive if there exists $\theta \in \Theta$ and $\eta > 1$ such that*

$$\theta(d(Ax, Az)) \geqslant [\theta(d(x, z))]^{\eta}, \tag{2}$$

for all $(x, z) \in M$, where

$$M = \{(x, z) \in Y \times Y : x \preceq z, \;\; d(Ax, Az) > 0\}. \tag{3}$$

Theorem 3. *Let $(Y, \preceq, d)$ be an ordered complete metric space, $A : Y \to Y$ a surjective θ-expansive mapping and A^* a right inverse of A such that A^* is $\preceq$ increasing. Suppose that there exists $x_0 \in Y$ such that $x_0 \preceq A^*x_0$. If A is continuous or Y is regular, then A has a fixed point.*

Proof. Let x_0 be an arbitrary point in Y. Since A is surjective, there exists $x_1 \in Y$ such that $x_0 = Ax_1$. In general, having chosen $x_n \in Y$, we choose $x_{n+1} \in Y$ such that $x_n = Ax_{n+1}$ for all $n = 0, 1, 2, ...$ If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_{n+1} is a fixed point of A. Now assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $x_0 \preceq A^*x_0$ and A^* is $\preceq$ increasing, we obtain

$$x_0 \preceq x_1 \preceq x_3 \preceq \cdots \preceq x_n \preceq \ldots$$

Now since $x_{n-1} \preceq x_n$ and $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$, then $(x_{n-1}, x_n) \in M$ and so, from (2) we obtain

$$\theta(d(x_{n-1}, x_n)) = \theta(d(Ax_n, Ax_{n+1})) \geqslant [\theta(d(x_n, x_{n+1}))]^{\eta} \tag{4}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (4), we get

$$\theta(d(x_{n-1}, x_n)) \geqslant [\theta(d(x_n, x_{n+1}))]^{\eta}. \tag{5}$$

Let $s = \frac{1}{\eta}$. Since $\eta > 1$, we obtain $s < 1$. Therefore since $x_{n-2} \preceq x_{n-1}$ and $d(x_{n-2}, x_{n-1}) > 0$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\theta(d(x_n, x_{n+1})) &\leqslant [\theta(d(x_{n-1}, x_n))]^{s} \\
&\leqslant [\theta(d(x_{n-2}, x_{n-1}))]^{s^2} \\
&\;\;\vdots \\
&\leqslant [\theta(d(x_0, x_1))]^{s^n}.
\end{aligned} \tag{6}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1, \tag{7}$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0^+.$$

Let

$$k_n = d(x_n, x_{n+1}).$$

From condition (Θ_3), there exists $c \in (0,1)$ and $G \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(k_n) - 1}{(k_n)^c} = G. \tag{8}$$

Suppose that $G < \infty$. In this case, let $J = \frac{G}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(k_n) - 1}{(k_n)^c} - G \right| \leqslant J, \text{ for all } n \geqslant n_0.$$

This implies that

$$\frac{\theta(k_n) - 1}{(k_n)^c} \geqslant G - J = J, \text{ for all } n \geqslant n_0.$$

Then for all $n \geqslant n_0$, we obtain

$$n(k_n)^c \leqslant Hn[\theta(k_n) - 1],$$

where $H = \frac{1}{J}$. Now assume that $G = \infty$. Let $J > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(k_n) - 1}{(k_n)^c} \geqslant J,$$

for all $n \geqslant n_0$. This implies that for all $n \geqslant n_0$,

$$n(k_n)^c \leqslant Hn[\theta(k_n) - 1],$$

where $H = \frac{1}{J}$. Therefore, in all cases, there exists $H > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$n(k_n)^c \leqslant Hn[\theta(k_n) - 1].$$

Using (6), we have

$$n(k_n)^c \leqslant Hn([\theta(k_0)]^{s^n} - 1), \tag{9}$$

for all $n \geqslant n_0$. Letting $n \to \infty$ in (9), we obtain

$$\lim_{n\to\infty} n(k_n)^c = 0.$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that

$$k_n \leqslant \frac{1}{n^{\frac{1}{c}}}, \quad \text{for all} \quad n \geqslant n_1. \tag{10}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $n, m \in \mathbb{N}$ such that $m > n \geqslant n_1$. Using the triangular inequality for the metric and from (10), we have

$$\begin{aligned} d(x_n, x_m) \leqslant & d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ \leqslant & k_n + k_{n+1} + \cdots + k_{m-1} \\ \leqslant & \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{c}}} \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{c}}}. \end{aligned}$$

By the convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{c}}}$, in the limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (Y, d). Since (Y, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $a \in Y$, that is,

$$\lim_{n\to\infty} x_n = a. \tag{11}$$

Now we shall show that a is a fixed point of A. If A is continuous, then we have

$$a = \lim_{n\to\infty} x_n = A(\lim_{n\to\infty} x_{n+1}) = Aa.$$

Therefore $a = Aa$, that is, a is a fixed point of A. Now we suppose that Y is regular, then $x_n \preceq a$ for all $n \in \mathbb{N}$. We consider the following two cases:

Case 1. If there exists $b \in \mathbb{N}$ for which $x_b = a$, then, we obtain

$$A^*a = A^*x_b = x_{b+1} \preceq a.$$

We also get

$$a = x_b \preceq x_{b+1} = A^*x_b = A^*a,$$

then we obtain, $a = A^*a$.

Case 2. Suppose that $x_n \neq a$ for every $n \in \mathbb{N}$ and $d(a, A^*a) > 0$. Therefore, from (2) we obtain

$$\theta(d(x_n, a)) = \theta(d(Ax_{n+1}, AA^*a)) \geqslant [\theta(d(x_{n+1}, A^*a))]^\eta \geqslant \theta(d(x_{n+1}, A^*a)),$$

which yields

$$d(x_n, a) > d(x_{n+1}, A^*a).$$

Taking limit as $n \to \infty$, we obtain that

$$d(a, A^*a) < d(a, a),$$

a contradiction. Thus, conclude that $d(a, A^*a) = 0$, that is, $a = A^*a$. Therefore, we obtain $Aa = A(A^*a) = a$. This concludes the proof. □

Remark 1. *In Theorem 3, if every pair of elements has a lower bound and upper bound, then the fixed point of A is unique. To see this, it is sufficient to show that for every $x \in Y$, $\lim_{n\to\infty} A^n x = a$ where a is the fixed point of A such that $\lim_{n\to\infty} A^n x_0 = a$. Let $x_0 \in Y$. So, here two cases arise.*

Case 1. If $x \preceq x_0$ or $x_0 \preceq x$, then $A^n x \preceq A^n x_0$ or $A^n x_0 \preceq A^n x$ for all $n \in \mathbb{N}$. If $A^{n_0} x = A^{n_0} x_0$ for some $n_0 \in \mathbb{N}$ then, $A^n x \to a$. Afterwards, let $A^n x \neq A^n x_0$ for all $n \in \mathbb{N}$ then, $d(A^n x, A^n x_0) > 0$ and so $(A^n x, A^n x_0) \in M$ for all $n \in \mathbb{N}$. Accordingly, from (2), we obtain

$$\theta(d(A^n x, A^n x_0)) \geqslant [\theta(d(A^{n+1} x, A^{n+1} x_0))]^\eta. \tag{12}$$

Let $s = \frac{1}{\eta}$. Since $\eta > 1$, we obtain $s < 1$. Subsequently, we have

$$\theta(d(A^n x, A^n x_0)) \leqslant [\theta(d(x, x_0))]^{s^n}. \tag{13}$$

Letting $n \to \infty$ in (13) we obtain

$$\lim_{n\to\infty} \theta(d(A^n x, A^n x_0)) = 1. \tag{14}$$

Taking into account (Θ_2), we have $\lim_{n\to\infty} d(A^n x, A^n x_0) = 0^+$. Thus we obtain $\lim_{n\to\infty} A^n x = \lim_{n\to\infty} A^n x_0 = a$.

Case 2. If $x \npreceq x_0$ or $x_0 \npreceq x$, then, from every pair of elements has a lower bound and upper bound that there exists $x_1, x_2 \in Y$ such that $x_2 \preceq x \preceq x_1$ and $x_2 \preceq x_0 \preceq x_1$. Thus, as in the case 1 we show that

$$\lim_{n\to\infty} A^n x_1 = \lim_{n\to\infty} A^n x_2 = \lim_{n\to\infty} A^n x = \lim_{n\to\infty} A^n x_0 = a.$$

Example 1. *Let $Y = \{x_r = \frac{1}{r+1},\ r \in \mathbb{N}\} \cup \{0\}$ be endowed with the usual metric d. Define an order relation $\preceq$ on Y as*

$$x \preceq z \Leftrightarrow [x = z \ \text{ or } \ x \leqslant z \ \text{ with } \ x, z \in Y],$$

where $\leqslant$ is usual order. Clearly, $(Y, \preceq, d)$ be an ordered complete metric spaces. Define a mapping $A : Y \to Y$ by

$$Ax = \begin{cases} \frac{1}{r}, & x = x_r \\ 0, & x = 0. \end{cases}$$

Then, A^* is $\preceq$ increasing. We claim that A is a θ-expanding mapping with $\theta(p) = e^{pe^p}$ and $\eta = \frac{3}{2} > 1$. To see this, we have to show that A satisfies the condition (2). Then we obtain

$$e^{d(Ax,Az)e^{d(Ax,Az)}} \geqslant e^{\eta d(x,z)e^{d(x,z)}}$$

for $\eta = \frac{3}{2}$. Let $x = x_r$, $z = x_{r+1}$. So, we obtain

$$\frac{d(Ax, Az)}{d(x,z)} e^{d(Ax,Az)-d(x,z)} = \frac{r+2}{r} e^{\frac{2}{r(r+1)(r+2)}} \geqslant \eta.$$

Thus, Theorem 3 is satisfied with $\eta = \frac{3}{2} > 1$. Therefore, implies that A has a unique fixed point. On the other hand, it is not an expansive mapping in metric spaces. To see this, we obtain

$$\lim_{r\to\infty} \frac{d(Ax, Az)}{d(x,z)} = \lim_{r\to\infty} \frac{r+2}{r} = 1.$$

Then

$$d(Ax, Az) \geqslant q d(x,z) \tag{15}$$

does not hold for $q > 1$. Hence the condition of Theorem 1 is not satisfied. This example shows that the new class of θ-expanding mapping is not included in expanding mapping known in literature.

Example 2. *Let $Y = (0, \infty)$ be endowed with the usual metric d. Define a mapping $A : Y \to Y$ by*

$$Ax = \begin{cases} \frac{x}{2}, & 0 < x \leqslant 1 \\ 2x - \frac{3}{2}, & 1 \leqslant x < \infty. \end{cases}$$

Then, A is a surjective and having a right inverse A^* given by

$$A^*x = \begin{cases} 2x, & 0 < x \leqslant \frac{1}{2} \\ \frac{x}{2} + \frac{3}{4}, & \frac{1}{2} \leqslant x < \infty. \end{cases}$$

Define an order relation $\preceq$ on Y as $x \preceq z$ if and only if either $x = z$ or $1 \leqslant x \leqslant z$ with $x, z \in \mathbb{Q}$, where $\leqslant$ is usual order. Clearly, $(Y, \preceq, d)$ is an ordered complete metric space. Then, A^* is $\preceq$ increasing. A is a θ-expanding mapping with $\theta(p) = e^p$ and $\eta = \frac{3}{2}$. Thus, the hypotheses of Theorem 3 is satisfied and A has a fixed point (namely $x = \frac{3}{2}$).

Now, we give some fixed point results for θ-expansive mapping in metric spaces and prove fixed point theorems for such mappings. Using the same argument as in the proof of the Theorem 3, we prove the following result.

Corollary 1. *Let (Y, d) be a complete metric space and $A : Y \to Y$ a continuous surjective θ-expansive mapping. If there exists a constant $\eta > 1$ such that*

$$\theta(d(Ax, Az)) \geqslant [\theta(d(x,z))]^\eta \tag{16}$$

for all $x, z \in Y$, then A has a unique fixed point in Y.

Note that Θ contains a large class of functions. For example, if we take

$$\theta(w) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{w^\lambda}\right),$$

where $\lambda \in (0,1)$, $w > 0$ and from Corollary 1 we obtain the following result.

Corollary 2. *Let (Y,d) be a complete metric space and A be a self mapping on Y. If there exists $\lambda \in (0,1)$ and a constant $\eta > 1$ such that*

$$2 - \frac{2}{\pi}\arctan\left(\frac{1}{d(Ax,Az)^{\lambda}}\right) \geqslant \left[2 - \frac{2}{\pi}\arctan\left(\frac{1}{d(x,z)^{\lambda}}\right)\right]^{\eta} \tag{17}$$

for all $x,z \in Y$ with $Ax \neq Az$, then A has a fixed point.

Theorem 4. *Let (Y,d) be a complete metric space and $A : Y \to Y$ a continuous surjective θ-expansive mapping. If there exists a constant $\eta > 1$ such that*

$$\theta(d(Ax,Az)) \geqslant [\theta(\min\{d(x,z), d(x,Ax), d(z,Az)\})]^{\eta} \tag{18}$$

for all $x,z \in Y$, then A has a fixed point.

Proof. Let x_0 be an arbitrary point in Y. Since A is surjective, there exists $x_1 \in Y$ such that $x_0 = Ax_1$. In general, having chosen $x_n \in Y$, we choose $x_{n+1} \in Y$ such that $x_n = Ax_{n+1}$ for all $n = 0,1,2,\ldots$ If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_{n+1} is a fixed point of A. Now assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then from (18) for $x = x_n$ and $z = x_{n+1}$ we obtain

$$\theta(d(x_{n-1},x_n)) = \theta(d(Ax_n,Ax_{n+1})) \geqslant [\theta(\min\{d(x_n,x_{n+1}), d(x_n,Ax_n), d(x_{n+1},Ax_{n+1})\}))]^{\eta}, \tag{19}$$

where $\min\{d(x_n,x_{n+1}), d(x_n,Ax_n), d(x_{n+1},Ax_{n+1})\} = \min\{d(x_n,x_{n+1}), d(x_n,x_{n-1})\}$.

Thus, here two cases arise.

Case 1. Let $\min\{d(x_n,x_{n+1}), d(x_n,x_{n-1})\} = d(x_n,x_{n-1})$. So, from (18),

$$\theta(d(x_{n-1},x_n)) \geqslant [\theta(d(x_{n-1},x_n)))]^{\eta},$$

which is a contradiction, since $\eta > 0$, so, $\min\{d(x_n,x_{n+1}), d(x_n,x_{n-1})\} = d(x_n,x_{n+1})$. Then, by using (18), we obtain

$$\theta(d(x_{n-1},x_n)) \geqslant [\theta(d(x_n,x_{n+1})))]^{\eta}.$$

Let $s = \frac{1}{\eta}$. Since $\eta > 1$, we obtain $s < 1$. The rest of the proof can be completed as in the proof of Theorem 3. □

Theorem 5. *Let (Y,d) be a complete metric space. Let A and B be weakly compatible self mappings of Y and $B(Y) \subseteq A(Y)$. Suppose that $\theta \in \Theta$ and there exists a constant $\eta > 1$ such that*

$$\theta(d(Ax,Az)) \geqslant [\theta(d(Bx,Bz))]^{\eta}, \tag{20}$$

for all $x,z \in Y$. If one of the subspaces $B(Y)$ or $A(Y)$ is complete, then A and B have a unique common fixed point in Y.

Proof. Let x_0 be an arbitrary point in Y. Since $B(Y) \subseteq A(Y)$, choose $x_1 \in Y$ such that $\rho_1 = Ax_1 = Bx_0$. In general, choose x_{n+1} such that $\rho_{n+1} = Ax_{n+1} = Bx_n$. Let $s = \frac{1}{\eta}$. Since $\eta > 1$ we obtain $s < 1$. Then from (20) we obtain

$$\begin{aligned}\theta(d(\rho_{n+1}, \rho_{n+2})) = \theta(d(Bx_n, Bx_{n+1})) &\leqslant [\theta(d(Ax_n, Ax_{n+1}))]^s \\ &= [\theta(d(Bx_{n-1}, Bx_n))]^s \\ &= [\theta(d(\rho_n, \rho_{n+1}))]^s.\end{aligned}$$

Therefore, it can be seen that the (ρ_n) is Cauchy with similar operations in Theorem 3. Since $B(Y) \subseteq A(Y)$ and $B(Y)$ or $A(Y)$ is a complete subspace of Y then from Corollary 1, $(A(Y), d)$ is complete and so, the sequence $\rho_n = Bx_{n-1} \subseteq A(Y)$ is converges in the metric spaces $(A(Y), d)$, that is, there exists a w in $A(Y)$ such that $\lim_{n\to\infty} d(\rho_n, w) = 0$.

So, we can find $k \in Y$ such that $Ak = w$. Accordingly, from Corollary 1, we have

$$d(Ak, w) = d(w, w) = \lim_{n\to\infty} d(\rho_n, w) = \lim_{n\to\infty} d(\rho_n, \rho_m).$$

This yields that (ρ_n) is a Cauchy sequence in the metric spaces $(A(Y), d)$.

Now, we show that $Bk = w$. According to (20) we obtain

$$\begin{aligned}\theta(d(Bx_{n-1}, Ak)) = \theta(d(Ax_n, Ak)) &\geqslant [\theta(d(Bx_n, Bk))]^\eta \\ &= [\theta(d(\rho_{n+1}, Bk))]^\eta\end{aligned}$$

that is, we can write

$$\theta(d(w, Ak)) \geqslant [\theta(d(w, Bk))]^\eta.$$

Thus, we have $\theta(d(w, Bk)) \leqslant [\theta(d(w, w))]^s$ then, we obtain $w = Ak = Bk$. Since A and B are weakly compatible, $ABk = BAk$, that is, $Aw = Bw$.

Now, we shall show that w is a common fixed point of A and B. In view of (20), we obtain

$$\theta(d(Aw, Ax_n)) \geqslant [\theta(d(Bw, Bx_n))]^\eta = [\theta(d(Bw, \rho_{n+1}))]^\eta$$

the limit as $n \to \infty$, we obtain

$$\theta(d(Aw, w)) \geqslant [\theta(d(Bw, w))]^\eta = [\theta(d(Aw, w))]^\eta$$

which implies that $d(Aw, w) = 0$, that is, $Aw = Bw = w$. To prove uniqueness, suppose that $t \neq w$ is also another common fixed point of A and B, that is, $At = Bt = t$. Then, we obtain

$$\theta(d(t, w)) = \theta(d(At, Aw)) \geqslant [\theta(d(Bt, Bw))]^\eta = [\theta(d(t, w))]^\eta,$$

which is a contraction. Therefore, w is a unique common fixed point of A and B. This complete the proof. □

Example 3. *Let $Y = [0,1]$ and define $d(x,z) = |x - z|$, for all $x, z \in Y$. (Y, d) is a complete metric space. Define $Ax = \frac{x}{4}$ and $Bx = \frac{x}{12}$, then $B(Y) \subseteq A(Y)$ and $A(Y)$ is complete. $\theta(p) = e^p$ belong to Θ, for all $x \in [0,1]$ with $x \geqslant z$ from (20), we obtain*

$$e^{\frac{1}{4}|x-z|} \geqslant e^{\frac{\eta}{12}|x-z|},$$

for $1 < \eta < 3$ and (20) is satisfied. In this example, A and B are weakly compatible mappings and 0 is the unique common fixed point. Therefore, Theorem 5 is satisfied.

Example 4. *Let $Y = [0,1]$ and define $d(x,z) = |x - z|$, for all $x, z \in Y$. (Y,d) is a complete metric space. Let $Ax = \frac{1-x}{2}$, $Bx = \frac{1-x}{4}$. Then $B(Y) \subseteq A(Y)$ and $A(Y)$ is complete. $\theta(p) = e^{pe^p}$ belong to Θ, for all $x \in [0,1]$ with $x \geqslant z$. From (20), we obtain*

$$2e^{\frac{1}{4}|x-z|} \geqslant \eta$$

for $1 < \eta < 2$ and (20) is satisfied. $A1 = B1 = 0$ but $AB1 = \frac{1}{2}$ and $BA1 = \frac{1}{4}$, then A and B are not weakly compatible. It follows that, except for the weak compatibility of A and B, all other hypotheses of Theorem 5 are satisfied. However, they do not have a common fixed point. This example shows that the weak compatible condition of Theorem 5 cannot be removed.

3. Conclusions

Wang et al. [1], proved some fixed point theorems for expansive mappings, which correspond to some contractive mappings in metric spaces. Jleli and Samet [17] introduced a new type of contractions called θ-contraction. In the present article, we introduce a new approach to expansive mappings in fixed point theory by combining the ideas of Wang, Jleli and Samet. We introduce the concept of θ-expansive mappings in ordered metric spaces and prove a fixed point theorem for such mappings. We give some fixed point results for θ-expansive mappings in metric spaces and prove fixed point theorems for such mappings. Some examples are presented to support the new theorems and results proved. Further, these examples show that the new class of θ-expansive mapping is not included in expansive mappings known in the literature.

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