mathematics

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Existence and Multiplicity of Solutions to a Class of Fractional p-Laplacian Equations of Schrödinger-Type with Concave-Convex Nonlinearities in $\mathbb{R}^N$

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Received: 22 September 2020; Accepted: 8 October 2020; Published: 15 October 2020

Abstract: We are concerned with the following elliptic equations: $(-\Delta)_p^s v + V(x)|v|^{p-2}v = \lambda a(x)|v|^{r-2}v + g(x,v)$ in $\mathbb{R}^N$, where $(-\Delta)_p^s$ is the fractional p-Laplacian operator with $0 < s < 1 < r < p < +\infty$, $sp < N$, the potential function $V : \mathbb{R}^N \to (0,\infty)$ is a continuous potential function, and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. By employing the mountain pass theorem and a variant of Ekeland's variational principle as the major tools, we show that the problem above admits at least two distinct non-trivial solutions for the case of a combined effect of concave–convex nonlinearities. Moreover, we present a result on the existence of multiple solutions to the given problem by utilizing the well-known fountain theorem.

Keywords: fractional p-Laplacian; variational methods; critical point theory

MSC: 35R11; 35A15; 35J60; 49R05

1. Introduction

The study of problems of elliptic type involving nonlocal fractional Laplacian or more general integro-differential operators has extensively been considered in light of the pure or applied mathematical theory to explain some concrete phenomena arising from the thin obstacle problem, crystal dislocation, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, soft thin films, phase transition phenomena, multiple scattering, image process, minimal surfaces and the Levy process [1–6], and the references therein. In particular, the fractional Schrödinger equation which was originally introduced by Laskin [5] has received significant attention in recent years (see, e.g., [7–9]). The Schrödinger equation plays a basic role in quantum theory, analogous to the role of Newton's laws of conservation of energy in classical mechanics. The linear Schrödinger equation describes the evolution of a free non-relativistic quantum particle. This is one of the main consequences in quantum mechanics. The structure of the nonlinear Schrödinger equation is substantially complicated and requires more sophisticated analysis; see [10]. This equation has been studied greatly in accordance with the pure or applied mathematical theory, because it stands out as a prototypical system that has proven to be essential in modeling and understanding the characteristics of numerous areas in nonlinear physics. In particular, the considerable developments of the Bose-Einstein condensate activated the studies on the nonlinear waveforms of the nonlinear Schrödinger equations with external potentials and associated nonlinear partial differential equations. For further applications and more details, we infer the reader to [11–17]. The remarkable mathematical model for the Bose-Einstein condensate with effectively attractive interactions between particles under a magnetic trap is the nonlinear Schrödinger equation, which is sometimes called the Gross-Pitaevskii equation [18,19].

Motivated by huge interest in the current literature, exploiting variational methods, we investigate the existence of nontrivial weak solutions for the fractional p-Laplacian problems. To be more precise, we consider the existence results of nontrivial weak solutions for the following nonlinear elliptic equations of the fractional p-Laplace type involving the concave–convex nonlinearities:

$$(-\Delta)_p^s v + V(x)|v|^{p-2}v = \lambda a(x)|v|^{r-2}v + g(x,v) \quad \text{in} \quad \mathbb{R}^N, \tag{1}$$

where λ is a real parameter, $0 < s < 1 < r < p < +\infty$, $sp < N$, $V : \mathbb{R}^N \to (0,\infty)$ is potential function continuous, $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $(-\Delta)_p^s$ is the fractional p-Laplacian operator defined as

$$(-\Delta)_p^s v(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon^N(x)} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+ps}}\, dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon^N(x) := \{y \in \mathbb{R}^N : |y-x| \leq \varepsilon\}$. Many researchers have extensively studied the fractional p-Laplacian type problems in various ways; see [2,3,9,20–26] and the references therein.

Since the pioneer work of Ambrosetti and Rabinowitz in [27], the critical point theory has become one of the most effectual analytic tools to look for solutions to elliptic equations of variational type. Afterward, lots of important results on the existence and multiplicity of nontrivial solutions to nonlinear elliptic problems involving the nonlocal operators have been obtained; see, for example, [7,20–24,26,28–31]. The key ingredient for achieving these results is the Ambrosetti and Rabinowitz condition (the (AR)-condition, in brief) in [27];

(AR) There are $C_0 > 0$ and $\eta > 0$, such that $\eta > p$ and

$$0 < \eta G(x,t) \leq g(x,t)t, \quad \text{for} \quad x \in \Omega \quad \text{and} \quad |t| \geq C_0,$$

where $G(x,t) = \int_0^t g(x,s)\, ds$, and Ω is a bounded domain in $\mathbb{R}^N$.

As we are well aware, the (AR)-condition is indispensable to ensure the compactness condition of the Euler-Lagrange functional, which plays a fundamental role in employing the critical point theory. However, this condition is very restrictive and removes many nonlinearities. In this direction, Liu [32] studied the existence and multiplicity of weak solutions for the p-Laplacian equation in case of the whole space $\mathbb{R}^N$ under the following assumption:

(Je) There exists $\eta \geq 1$, such that

$$\eta \mathcal{G}(x,t) \geq \mathcal{G}(x,\tau t)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $\tau \in [0,1]$, where $\mathcal{G}(x,t) = g(x,t)t - pG(x,t)$ and $G(x,t) = \int_0^t g(x,s)ds$.

Recently, by utilizing the mountain pass theorem under this condition, the existence result for the fractional p-Laplacian problem was obtained by Torres in [30]. Indeed, the condition above was initially proposed by L. Jeanjean [33] in the case of $p = 2$. In the last few decades, there were extensive studies dealing with p-Laplacian problems by assumption (Je); see [32,34] for the p-Laplacian and [35–38] for the $p(x)$-Laplacian. In particular, the authors of [34] provided many examples that did not fulfill the condition of the nonlinear term g given in [23,32,35,38]; for instance,

$$g(x,t) = m(x)\,|t|^{p-2}\, t(4\,|t|^3 + 2t\sin t - 4\cos t),$$

where $m \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < \inf_{\mathbb{R}^N} m \leq \sup_{\mathbb{R}^N} m < \infty$. In this respect, authors in [7,28] extended the existence of infinitely many weak solutions to the fractional Laplacian problems.

The main aim of the present paper is to establish the existence of multiple solutions for Schrödinger-type problems in the case where the nonlinear term is concave-convex, by making use of the variational methods. The concave-convex-type elliptic problems have been extensively investigated (see [39–44]) since the seminal work of Ambrosetti, Brezis, and Cerami [45] for the Laplacian problem:

$$\begin{cases} -\triangle v = \lambda |v|^{q-2}v + |v|^{h-2}v & \text{in} \quad \Omega, \\ v > 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $1 < q < 2 < h < 2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N = 1, 2. \end{cases}$ Particularly, the existence of two nontrivial nonnegative solutions and a sequence of solutions to degenerate $p(x)$-Laplacian problems involving the concave-convex nonlinearities with two parameters has been established in [42]. For elliptic problems driven by a nonlocal integro-differential operator with Dirichlet boundary conditions, by utilizing the Nehari manifold method, the authors in [41] obtained the existence of multiple solutions to the following problem

$$\begin{cases} -\mathcal{L}_K v = \lambda \, |v|^r + |v|^q \,, \quad v > 0 \quad \text{in } \Omega; \\ v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, the exponents r and q satisfy $0 < r < 1 < q \le 2_s^* - 1$, $N > 2s$ with $s \in (0,1)$, $2_s^* = 2N/(N-2s)$, λ is a positive parameter. Here, the operator $\mathcal{L}_K$ is the non-local operator of the fractional type, defined as follows:

$$\mathcal{L}_K v(x) = 2 \int_{\mathbb{R}^N} |v(x) - v(y)|^{p-2}(v(x) - v(y))K(x-y)dy \quad \text{for all } x \in \mathbb{R}^N, \tag{2}$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a kernel function satisfying some suitable conditions; see [41]. Additionally, the existence of two non-trivial entire solutions for a non-homogeneous fractional p-Kirchhoff type problem involving concave-convex nonlinear terms was built in [44]. Very recently, Kim et al. [46] established the existence of at least two distinct nontrivial solutions for a Schrödinger-Kirchhoff type problem driven by the non-local fractional $p(\cdot)$-Laplacian with the concave-convex nonlinearities when the convex term fulfilled the assumption (AR) and (Je), respectively. In order to get the multiplicity result, they considered the mountain pass theorem in [27] and a variant of Ekeland's variational principle (see [47]) as primary tools. In that sense, the first aim of the present article is to get the existence of two distinct nontrivial solutions for problem (1) for the case of a combined effect of concave-convex nonlinearities, provided that the condition on convex term g is weaker than (AR) and different from (Je), which is originally given in [48] even if the considered domain is bounded. The second one is to prove the result on the existence of multiple solutions to (1) by utilizing the well-known fountain theorem in [49]. As far as we are aware, the present paper is the first attempt to study the multiplicity of nontrivial weak solutions to Schrödinger-type problems with the concave-convex nonlinearity in these circumstances.

This paper is structured as follows. In Section 2, we recall briefly some fundamental results for the fractional Sobolev spaces. Under appropriate conditions on g, we also obtain several existence results of nontrivial weak solutions for problem (1) by utilizing the variational principle as the major tools.

2. Preliminaries and Main Results

In this section, we briefly recall some definitions and basic properties of the fractional Sobolev spaces. We refer the reader to [4,25,50,51] for further references. Then, we deal with the existence of a nontrivial weak solution for the problem (1) under suitable assumptions.

Let $s \in (0,1)$ and $p \in (1,+\infty)$. We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ as follows:

$$W^{s,p}(\mathbb{R}^N) := \left\{ v \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\, dxdy < +\infty \right\},$$

endowed with the norm

$$\|v\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|v\|^p_{L^p(\mathbb{R}^N)} + |v|^p_{W^{s,p}(\mathbb{R}^N)} \right)^{\frac{1}{p}},$$

where

$$\|v\|^p_{L^p(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |v|^p\, dx \qquad \text{and} \qquad |v|^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\, dxdy.$$

Let $s \in (0,1)$ and $1 < p < +\infty$. Then, $W^{s,p}(\mathbb{R}^N)$ is a separable and reflexive Banach space. Additionally, the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, so that is $W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$ (see, e.g., [50]).

Lemma 1. ([51]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary $s \in (0,1)$ and $p \in (1,+\infty)$. Then, we have the following continuous embeddings:*

$$\begin{aligned}
&W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for all } q \in [1, p_s^*], && \text{if } sp < N;\\
&W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for every } q \in [1, \infty), && \text{if } sp = N;\\
&W^{s,p}(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) && \text{for all } \lambda < s - N/p && \text{if } sp > N,
\end{aligned}$$

where p_s^* is the fractional critical Sobolev exponent, that is,

$$p_s^* := \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \geq N. \end{cases}$$

In particular, the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [p, p_s^*)$.

Lemma 2. ([25,50]) *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Then, there exists a positive constant $C = C(N,p,s)$, such that for all $u \in W^{s,p}(\mathbb{R}^N)$,*

$$\|v\|_{L^{p_s^*}(\mathbb{R}^N)} \leq C\, |v|_{W^{s,p}(\mathbb{R}^N)}.$$

Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^*]$.

For our analysis, we assume that

(V) $V \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0$ and $\lim_{x\to 0} V(x) = \lim_{|x|\to\infty} V(x) = +\infty$.

When V satisfies (V), the basic space

$$X_s(\mathbb{R}^N) := \left\{ v \in W^{s,p}(\mathbb{R}^N) : V|v|^p \in L^1(\mathbb{R}^N) \right\}$$

denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|v\|_{X_s(\mathbb{R}^N)} := \left(|v|^p_{W^{s,p}(\mathbb{R}^N)} + \|v\|^p_{L^p(V,\mathbb{R}^N)} \right)^{\frac{1}{p}}.$$

With the aid of Lemmas 1 and 2, we get the following consequence.

Lemma 3. ([30]) *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Then, there is a compact embedding $X_s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $q \in [p, p_s^*)$.*

Definition 1. *Let $0 < s < 1 < p < +\infty$. We say that $u \in X_s(\mathbb{R}^N)$ is a weak solution of the problem* (1) *if*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dxdy$$
$$+ \int_{\mathbb{R}^N} V(x)\,|v(x)|^{p-2}\,v\varphi\,dx = \lambda \int_{\mathbb{R}^N} a(x)|v|^{r-2}v\varphi\,dx + \int_{\mathbb{R}^N} g(x,v)\varphi\,dx$$

for all $\varphi \in X_s(\mathbb{R}^N)$.

Let us define a functional $\Phi_{s,p} : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi_{s,p}(v) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v|^p\,dx.$$

Then, from Lemma 3.2 of [30], the functional $\Phi_{s,p}$ is well-defined on $X_s(\mathbb{R}^N)$, $\Phi_{s,p} \in C^1(X_s(\mathbb{R}^N),\mathbb{R})$ and its Fréchet derivative is given by, for any $\varphi \in X_s(\mathbb{R}^N)$,

$$\langle \Phi'_{s,p}(v), \varphi\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dxdy$$
$$+ \int_{\mathbb{R}^N} V(x)\,|v|^{p-2}\,v\varphi\,dx.$$

Lemma 4. ([25,30]) *Let $0 < s < 1 < p < +\infty$ and let the assumption* (V) *hold. Then, the functional $\Phi'_{s,p}$ is of type (S_+), that is, if $v_n \rightharpoonup v$ in $X_s(\mathbb{R}^N)$ and $\limsup_{n\to\infty} \left\langle \Phi'_{s,p}(v_n) - \Phi'_{s,p}(v), v_n - v \right\rangle \le 0$, then, $v_n \to v$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$.*

Denoting $G(x,t) = \int_0^t g(x,s)\,ds$ and when we assume that for $1 < r < p < q < p_s^*$ and $x \in \mathbb{R}^N$,

(A) $0 \le a \in L^{\frac{p}{p-r}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathrm{meas}\{x \in \mathbb{R}^N : a(x) \neq 0\} > 0$.

(G1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.

(G2) There exists a nonnegative function $b \in L^{q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, such that

$$|g(x,t)| \le b(x)\,|t|^{q-1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ where $1/q + 1/q' = 1$.

(G3) $\lim_{|t|\to\infty} \frac{G(x,t)}{|t|^p} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$.

(G4) There exist $\nu > p$ and $M > 0$, such that

$$g(x,t)t - \nu G(x,t) \geq -\varrho\,|t|^p - \eta(x) \quad \text{for all} \quad x \in \mathbb{R}^N \quad \text{and} \quad |t| \geq M,$$

where $\varrho \geq 0$ and $\eta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\eta(x) \geq 0$.

Some examples for g satisfying the above assumptions can be found in [34,48]. Under the assumptions (G1) and (G2), we define the functional $\Psi_\lambda : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Psi_\lambda(v) = \frac{\lambda}{r}\int_{\mathbb{R}^N} a(x)|v|^r\,dx + \int_{\mathbb{R}^N} G(x,v)\,dx.$$

Then, it follows from the similar arguments as those of Proposition 1.12 in [49] that $\Psi_\lambda \in C^1(X_s(\mathbb{R}^N),\mathbb{R})$ and its Fréchet derivative is

$$\left\langle \Psi'_\lambda(v),\varphi \right\rangle = \lambda\int_{\mathbb{R}^N} a(x)|v|^{r-2}v\varphi\,dx + \int_{\mathbb{R}^N} g(x,v)\varphi\,dx$$

for any $v,\varphi \in X_s(\mathbb{R}^N)$. Next, we define a functional $\mathcal{I}_\lambda : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(v) = \Phi_{s,p}(v) - \Psi_\lambda(v).$$

Then, we know that the functional $\mathcal{I}_\lambda \in C^1(X_s(\mathbb{R}^N),\mathbb{R})$ and its Fréchet derivative is

$$\begin{aligned}\langle \mathcal{I}'_\lambda(v),\varphi\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dxdy \\ &+ \int_{\mathbb{R}^N} V(x)\,|v|^{p-2}\,u\varphi\,dx - \lambda\int_{\mathbb{R}^N} a(x)|v|^{r-2}v\varphi\,dx - \int_{\mathbb{R}^N} g(x,v)\varphi\,dx\end{aligned}$$

for any $v,\varphi \in X_s(\mathbb{R}^N)$.

The proof of the following Lemma can be regarded as modifications of those of Lemma 3.3 in [46]. For the convenience of the readers, we will present the proof.

Lemma 5. *Let $\mathcal{I}_\lambda$ be defined above. Assume that* (V), (A) *and* (G1)–(G3) *hold. In addition, we assume that*

(G5) $G(x,t) \geq 0$ *for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Then, we have the following:

(i) *There is a constant $\lambda^* > 0$, such that for any $\lambda \in (0,\lambda^*)$, we can choose $R > 0$ and $0 < \delta < 1$, such that $\mathcal{I}_\lambda(v) \geq R$ for all $v \in X_s(\mathbb{R}^N)$ with $\|v\|_{X_s(\mathbb{R}^N)} = \delta$.*
(ii) *There exists an element ϕ in $X_s(\mathbb{R}^N)$, $\phi > 0$, such that $\mathcal{I}_\lambda(t\phi) \to -\infty$ as $t \to +\infty$.*
(iii) *There is an element ψ in $X_s(\mathbb{R}^N)$, $\psi > 0$, such that $\mathcal{I}_\lambda(t\psi) < 0$ for all $t \to 0^+$.*

Proof. Let us prove the condition (i). By Lemma 1, there is a constant $C_1 > 0$, such that $\|v\|_{L^\gamma(\mathbb{R}^N)} \le C_1\|v\|_{X_s(\mathbb{R}^N)}$ for $p \le \gamma < p_s^*$. Assume that $\|v\|_{X_s(\mathbb{R}^N)} < 1$. Then, it follows from (A) and (G2) that

$$\begin{aligned}
\mathcal{I}_\lambda(v) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v|^p\,dx - \frac{\lambda}{r}\int_{\mathbb{R}^N} a(x)\,|v|^r\,dx - \int_{\mathbb{R}^N} G(x,v)\,dx \\
&\ge \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v|^p\,dx \\
&\qquad - \frac{\lambda C_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{X_s(\mathbb{R}^N)} - \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\|v\|^q_{L^q(\mathbb{R}^N)} \\
&\ge \frac{1}{p}\|v\|^p_{X_s(\mathbb{R}^N)} - \frac{\lambda C_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v\|^r_{X_s(\mathbb{R}^N)} - \frac{C_1}{q}\|b\|_{L^\infty(\mathbb{R}^N)}\|v\|^q_{X_s(\mathbb{R}^N)} \\
&\ge \left(\frac{1}{p} - \frac{\lambda C_2}{r}\|v\|^{r-p}_{X_s(\mathbb{R}^N)} - \frac{C_3}{q}\|v\|^{q-p}_{X_s(\mathbb{R}^N)}\right)\|v\|^p_{X_s(\mathbb{R}^N)}
\end{aligned} \tag{3}$$

for positive constants C_2, C_3. Let us define the function $f_\lambda : (0,\infty) \to \mathbb{R}$ by

$$f_\lambda(t) = \frac{\lambda C_2}{r}t^{r-p} + \frac{C_3}{q}t^{q-p}.$$

Then, it is trivial that f_λ has a local minimum at the point $t_0 = \left(\frac{\lambda q C_2(p-r)}{rC_3(q-p)}\right)^{\frac{1}{q-r}}$, and so

$$\lim_{\lambda\to 0^+} f_\lambda(t_0) = 0.$$

Thus, there is a positive constant λ^*, such that for each $\lambda \in (0,\lambda^*)$, there are $R > 0$ and $\delta > 0$ small enough, such that $\mathcal{I}_\lambda(v) \ge R > 0$ for any $v \in X_s(\mathbb{R}^N)$ with $\|v\|_{X_s(\mathbb{R}^N)} = \delta$.

Next, we show the statement (ii). By the condition (G3), for any $K > 0$, there is a constant $t_0 > 0$, such that

$$G(x,t) \ge K\,|t|^p \tag{4}$$

for $|t| > t_0$ and for almost all $x \in \mathbb{R}^N$. Take $\phi \in X_s(\mathbb{R}^N)\setminus\{0\}$. Then, the relation (4) implies that

$$\begin{aligned}
\mathcal{I}_\lambda(t\phi) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{t^p|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|t\phi|^p\,dx - \frac{\lambda}{r}\int_{\mathbb{R}^N} a(x)|t\phi|^r\,dx - \int_{\mathbb{R}^N} G(x,t\phi)\,dx \\
&\le t^p\left(\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|\phi|^p\,dx - K\int_{\mathbb{R}^N}|\phi|^p\,dx\right)
\end{aligned}$$

for sufficiently large $t > 1$. If K is large enough, then we infer that $\mathcal{I}_\lambda(t\phi) \to -\infty$ as $t \to \infty$. Hence, the functional $\mathcal{I}_\lambda$ is unbounded from below.

Next, we remain to show (iii). Choose $\psi \in X_s(\mathbb{R}^N)$, such that $\psi > 0$. For sufficiently small $t > 0$, from (A) and (G5), we obtain

$$\begin{aligned}
\mathcal{I}_\lambda(t\psi) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{t^p|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|t\psi|^p\,dx - \frac{\lambda}{r}\int_{\mathbb{R}^N} a(x)|t\psi|^r\,dx - \int_{\mathbb{R}^N} G(x,t\psi)\,dx \\
&\le \frac{t^p}{p}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V(x)\,|\psi|^p\,dx\right) - \frac{\lambda t^r}{r}\int_{\mathbb{R}^N} a(x)|\psi|^r\,dx.
\end{aligned}$$

Since $r < p$, it follows that $\mathcal{I}_\lambda(t\psi) < 0$ as $t \to 0^+$. This completes the proof. □

The proof of the following consequence proceeds in the analogous way, as that of Lemma 3.3 in [46].

Lemma 6. *Assume that* (V), (A) *and* (G1)–(G2) *hold. Then,* Ψ *and* Ψ' *are weakly strongly continuous on* $X_s(\mathbb{R}^N)$.

With the aid of Lemmas 4 and 6, we show that the energy functional $\mathcal{I}_\lambda$ satisfies the Cerami condition ((C)-condition for brevity), i.e., any sequence $\{v_n\} \subset X_s(\mathbb{R}^N)$, such that

$$\{\mathcal{I}_\lambda(v_n)\} \text{ is bounded} \quad \text{and} \quad \|\mathcal{I}'_\lambda(v_n)\|_{X_s^*(\mathbb{R}^N)}(1+\|v_n\|_{X_s(\mathbb{R}^N)}) \to 0 \quad \text{as} \quad n \to \infty$$

has a convergent subsequence. The basic idea of the proofs for the following assertion comes from the paper [34]. This plays a fundamental role in showing the existence of nontrivial weak solutions for problem (1).

Lemma 7. *Let* $0 < s < 1 < p < +\infty$ *with* $ps < N$. *Assume that* (V), (A) *and* (G1)–(G4) *hold. For any* $\lambda > 0$, *the functional* $\mathcal{I}_\lambda$ *satisfies the* (C)*-condition.*

Proof. Let $\{v_n\}$ be a (C)-sequence in $X_s(\mathbb{R}^N)$, that is,

$$\sup_{n\in\mathbb{N}} |\mathcal{I}_\lambda(v_n)| \le \mathcal{K} \quad \text{and} \quad \langle \mathcal{I}'_\lambda(v_n), v_n \rangle = o(1), \tag{5}$$

where $o(1) \to 0$ as $n \to \infty$ and $\mathcal{K}$ is a positive constant. It follows from Lemmas 4 and 6 that $\Phi'_{s,p}$ and Ψ' are mappings of type (S_+). Since $\mathcal{I}'_\lambda$ is of type (S_+) and $X_s(\mathbb{R}^N)$ is reflexive, it suffices to assure that the sequence $\{v_n\}$ is bounded in $X_s(\mathbb{R}^N)$. Suppose on the contrary that the sequence $\{v_n\}$ is unbounded in $X_s(\mathbb{R}^N)$. Then, we may suppose that

$$\|v_n\|_{X_s(\mathbb{R}^N)} > 1 \quad \text{and} \quad \|v_n\|_{X_s(\mathbb{R}^N)} \to \infty \quad \text{as} \quad n \to \infty.$$

Define a sequence $\{w_n\}$ by $w_n = v_n/\|v_n\|_{X_s(\mathbb{R}^N)}$. Then, it is clear that $\{w_n\} \subset X_s(\mathbb{R}^N)$ and $\|w_n\|_{X_s(\mathbb{R}^N)} = 1$. Hence, up to a subsequence, still denoted by $\{w_n\}$, we obtain $w_n \rightharpoonup w$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$, and by Lemma 3, we have

$$w_n(x) \to w(x) \quad \text{a.e. in} \quad x \in \mathbb{R}^N \quad \text{and} \quad w_n \to w \quad \text{in} \quad L^\kappa(\mathbb{R}^N) \quad \text{as} \quad n \to \infty \tag{6}$$

for any κ with $p \le \kappa < p_s^*$. Set $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. Due to the relation (5), we have that

$$\begin{aligned}
\mathcal{K} \ge \mathcal{I}_\lambda(v_n) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx \\
&\quad - \frac{\lambda}{p}\int_{\mathbb{R}^N} a(x)|v_n|^r\,dx - \int_{\mathbb{R}^N} G(x,v_n)\,dx \\
&\ge \frac{1}{p}\|v_n\|^p_{X_s(\mathbb{R}^N)} - \frac{\lambda}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v_n\|^r_{L^p(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(x,v_n)dx.
\end{aligned}$$

Since $\|v_n\|_{X_s(\mathbb{R}^N)} \to \infty$ as $n \to \infty$, we arrive at

$$\int_{\mathbb{R}^N} G(x,v_n) \ge \frac{1}{p}\|v_n\|^p_{X_s(\mathbb{R}^N)} - \frac{\lambda}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v_n\|^r_{L^p(\mathbb{R}^N)} - \mathcal{K} \to \infty \tag{7}$$

as $n \to \infty$. Moreover,

$$\begin{aligned}\mathcal{I}_\lambda(v_n) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} a(x)|v_n|^r\,dx - \int_{\mathbb{R}^N} G(x,v_n)\,dx \\ &\le \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx - \int_{\mathbb{R}^N} G(x,v_n)\,dx.\end{aligned}$$

Then, one has

$$\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx \ge \int_{\mathbb{R}^N} G(x,v_n)\,dx + \mathcal{I}_\lambda(v_n). \tag{8}$$

The assumption (G3) implies that we can choose $t_0 > 1$, such that $G(x,t) > |t|^p$ for all $x \in \mathbb{R}^N$ and $|t| > t_0$. From (G1) and (G2), there is a constant $\mathcal{C} > 0$, such that $|G(x,t)| \le \mathcal{C}$ for all $(x,t) \in \mathbb{R}^N \times [-t_0, t_0]$. Therefore, we can choose a real number K_0, such that $G(x,t) \ge K_0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and thus,

$$\frac{G(x,v_n) - K_0}{\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx} \ge 0,$$

for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. Using the convergence (6), we know that $|v_n(x)| = |w_n(x)|\,\|v_n\|_{X_s(\mathbb{R}^N)} \to \infty$ as $n \to \infty$ for all $x \in \Omega$. In addition, we obtain from (G3) that for all $x \in \Omega$,

$$\begin{aligned}&\lim_{n\to\infty}\frac{G(x,v_n)}{\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx} \\ &\qquad \ge \lim_{n\to\infty}\frac{G(x,v_n)}{\frac{1}{p}\|v_n\|^p_{X_s(\mathbb{R}^N)}} = \lim_{n\to\infty}\frac{pG(x,v_n)}{|v_n(x)|^p}\,|w_n(x)|^p = \infty.\end{aligned} \tag{9}$$

Hence, we get that $|\Omega| = 0$. Indeed, if $|\Omega| \neq 0$, then taking into account (7)–(9) and the Fatou lemma, we deduce that

$$\begin{aligned}1 &= \liminf_{n\to\infty}\frac{\int_{\mathbb{R}^N} G(x,v_n)\,dx}{\int_{\mathbb{R}^N} G(x,v_n)\,dx + \mathcal{I}_\lambda(v_n)} \\
&\ge \liminf_{n\to\infty}\int_\Omega \frac{G(x,v_n)}{\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx}\,dx \\
&\quad - \limsup_{n\to\infty}\int_\Omega \frac{K_0}{\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx}\,dx \\
&= \liminf_{n\to\infty}\int_\Omega \frac{G(x,v_n) - K_0}{\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx}\,dx \\
&\ge \int_\Omega \liminf_{n\to\infty}\frac{G(x,v_n) - K_0}{\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx}\,dx \\
&= \int_\Omega \liminf_{n\to\infty}\frac{G(x,v_n)}{\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx}\,dx \\
&\quad - \int_\Omega \limsup_{n\to\infty}\frac{K_0}{\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|v_n|^p\,dx}\,dx = \infty,\end{aligned}$$

which is a contradiction. Thus, $w(x) = 0$ for almost all $x \in \mathbb{R}^N$.

Notice that $V(x) \to +\infty$ as $|x| \to \infty$, then

$$\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} V(x)\,|v_n|^p\;dx - C_4\int_{|v_n|\le M}(|v_n|^p + b(x)\,|v_n|^q)\,dx$$
$$\ge \frac{1}{2}\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} V(x)\,|v_n|^p\;dx - \mathcal{K}_0,$$

where C_4 and $\mathcal{K}_0$ are positive constants. Combining this with (G2), (G3), and (G4), one has

$$\begin{aligned}
\mathcal{K}_1 + o(1) &\ge \mathcal{I}_\lambda(v_n) - \frac{1}{\nu}\left\langle \mathcal{I}'_\lambda(v_n), v_n\right\rangle \\
&\ge \left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} V(x)\,|v_n|^p\;dx \\
&\quad - \lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} a(x)|v_n|^r\,dx + \int_{\mathbb{R}^N}\left(\frac{1}{\nu}g(x,v_n)v_n - G(x,v_n)\right)\,dx \\
&\ge \left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} V(x)\,|v_n|^p\;dx \\
&\quad - \lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} a(x)|v_n|^r\,dx + \int_{|v_n|>M}\left(\frac{1}{\nu}g(x,v_n)v_n - G(x,v_n)\right)\,dx \\
&\quad - C_4\int_{|v_n|\le M}(|v_n|^p + b(x)\,|v_n|^q)\,dx \\
&\ge \left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{2}\left(\frac{1}{p}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} V(x)\,|v_n|^p\;dx \\
&\quad - \lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} a(x)|v_n|^r\,dx - \frac{1}{\nu}\int_{\mathbb{R}^N}\left(\varrho\,|v_n|^p + \eta(x)\right)\,dx - \mathcal{K}_0 \\
&\ge \frac{1}{2}\left(\frac{1}{p}-\frac{1}{\nu}\right)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V(x)\,|v_n|^p\;dx\right) \\
&\quad - \lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} a(x)|v_n|^r\,dx - \frac{1}{\nu}\int_{\mathbb{R}^N}\left(\varrho\,|v_n|^p + \eta(x)\right)\,dx - \mathcal{K}_0 \\
&\ge \frac{1}{2}\left(\frac{1}{p}-\frac{1}{\nu}\right)\|v_n\|^p_{X_s(\mathbb{R}^N)} - \lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|v_n\|^r_{L^p(\mathbb{R}^N)} \\
&\quad - \frac{\varrho}{\nu}\|v_n\|^p_{L^p(\mathbb{R}^N)} - \frac{1}{\nu}\|\eta\|_{L^1(\mathbb{R}^N)} - \mathcal{K}_0,
\end{aligned}$$

which implies

$$1 \le \frac{\varrho}{\nu\frac{C_9}{2}\left(\frac{1}{p}-\frac{1}{\nu}\right)}\limsup_{n\to\infty}\|w_n\|^p_{L^p(\mathbb{R}^N)} = \frac{\varrho}{\nu\frac{C_9}{2}\left(\frac{1}{p}-\frac{1}{\nu}\right)}\|w\|^p_{L^p(\mathbb{R}^N)}. \tag{10}$$

Hence, it follows from (10) that $w \neq 0$. Thus, we can conclude a contradiction. Therefore, $\{v_n\}$ is bounded in $X_s(\mathbb{R}^N)$. This completes the proof. □

Lemma 8. ([36,47]) *Let X be a Banach space and x_0 be a fixed point of X. Suppose that $f : E \to \mathbb{R}\cup\{+\infty\}$ is a lower semi-continuous function, not identically $+\infty$, bounded from below. Then, for every $\varepsilon > 0$ and $y \in X$, such that*

$$f(y) < \inf_X f + \varepsilon,$$

and every $\lambda > 0$, there exists some point $z \in X$, such that

$$f(z) \le f(y), \quad \|z - x_0\|_X \le (1 + \|y\|_X)(e^\lambda - 1),$$

and

$$f(x) \geq f(z) - \frac{\varepsilon}{\lambda(1+\|z\|_X)}\|x-z\|_X \quad \textit{for all} \quad x \in X.$$

Theorem 1. *Assume that* (V), (A) *and* (G1)–(G5) *hold. Then, there is a constant* $\lambda^* > 0$*, such that for any* $\lambda \in (0, \lambda^*)$*, problem* (1) *possesses at least two different nontrivial solutions in* $X_s(\mathbb{R}^N)$*.*

Proof. In accordance with Lemmas 5 and 7, there is a positive number λ^*, such that $\mathcal{I}_\lambda$ has the mountain pass geometry and (C)-condition for any $\lambda \in (0, \lambda^*)$. By invoking the mountain pass theorem, we assert that there exists a critical point $v_0 \in X_s(\mathbb{R}^N)$ of $\mathcal{I}_\lambda$ with $\mathcal{I}_\lambda(v_0) = \overline{\ell} > 0 = \mathcal{I}_\lambda(0)$. Hence, we know that v_0 is a nontrivial weak solution of the problem (1). According to Lemma 5, for a fixed $\lambda \in (0, \lambda^*)$, we can choose $R > 0$ and $0 < \delta < 1$, such that $\mathcal{I}_\lambda(v) \geq R$ if $\|v\|_{X_s(\mathbb{R}^N)} = \delta$. Let us denote $\ell := \inf_{z \in \overline{B}_\delta} \mathcal{I}_\lambda(v)$, where $B_\delta := \{v \in X_s(\mathbb{R}^N) : \|v\|_{X_s(\mathbb{R}^N)} < \delta\}$ with a boundary ∂B_δ. Then, taking (3) and Lemma 5 (3) into account, we have $-\infty < \ell < 0$. Putting $0 < \epsilon < \inf_{z \in \partial B_\delta} \mathcal{I}_\lambda(v) - \ell$, owing to Lemma 8, we can choose $v_\epsilon \in \overline{B}_\delta$, such that

$$\begin{cases} \mathcal{I}_\lambda(v_\epsilon) \leq \ell + \epsilon \\ \mathcal{I}_\lambda(v_\epsilon) < \mathcal{I}_\lambda(v) + \frac{\epsilon}{1+\|v_\epsilon\|_{X_s(\mathbb{R}^N)}}\|v - v_\epsilon\|_{X_s(\mathbb{R}^N)}, \quad \text{for all} \quad v \in \overline{B}_\delta \quad v \neq v_\epsilon. \end{cases} \tag{11}$$

Then, it holds that $v_\epsilon \in B_\delta$ since $\mathcal{I}_\lambda(v_\epsilon) \leq \ell + \epsilon < \inf_{v \in \partial B_\delta} \mathcal{I}_\lambda(v)$. From these facts, we have that v_ϵ is a local minimum of $\widetilde{I}_\lambda(v) = \mathcal{I}_\lambda(v) + \frac{\epsilon}{1+\|v_\epsilon\|_{X_s(\mathbb{R}^N)}}\|v - v_\epsilon\|_{X_s(\mathbb{R}^N)}$. Now, by taking $v = v_\epsilon + tw$ for $w \in B_1$ with $t > 0$ that is small enough, from (11), we deduce

$$0 \leq \frac{\widetilde{I}_\lambda(v_\epsilon + tw) - \widetilde{I}_\lambda(v_\epsilon)}{t} = \frac{\mathcal{I}_\lambda(v_\epsilon + tw) - \mathcal{I}_\lambda(v_\epsilon)}{t} + \frac{\epsilon}{1+\|v_\epsilon\|_{X_s(\mathbb{R}^N)}}\|w\|_{X_s(\mathbb{R}^N)}.$$

Therefore, letting $t \to 0+$, we get

$$\langle \mathcal{I}'_\lambda(v_\epsilon), w \rangle + \frac{\epsilon}{1+\|v_\epsilon\|_{X_s(\mathbb{R}^N)}}\|w\|_{X_s(\mathbb{R}^N)} \geq 0.$$

Substituting $-w$ for w in the argument above, we derive

$$-\langle \mathcal{I}'_\lambda(v_\epsilon), w \rangle + \frac{\epsilon}{1+\|v_\epsilon\|_{X_s(\mathbb{R}^N)}}\|w\|_{X_s(\mathbb{R}^N)} \geq 0.$$

Thus, one has

$$(1+\|v_\epsilon\|_{X_s(\mathbb{R}^N)})\left|\langle \mathcal{I}'_\lambda(v_\epsilon), w \rangle\right| \leq \epsilon\|w\|_{X_s(\mathbb{R}^N)}$$

for any $w \in \overline{B}_1$. Hence, we get

$$(1+\|v_\epsilon\|_{X_s(\mathbb{R}^N)})\|\mathcal{I}'_\lambda(v_\epsilon)\|_{X_s(\mathbb{R}^N)^*} \leq \epsilon. \tag{12}$$

Combining (11) with (12), we can choose a sequence $\{v_n\} \subset B_\delta$, such that

$$\begin{cases} \mathcal{I}_\lambda(v_n) \to \ell \quad \text{as} \quad n \to \infty \\ (1+\|v_n\|_{X_s(\mathbb{R}^N)})\|\mathcal{I}'_\lambda(v_n)\|_{X_s(\mathbb{R}^N)^*} \to 0 \quad \text{as} \quad n \to \infty. \end{cases} \tag{13}$$

Thus, $\{v_n\}$ is a bounded Cerami sequence in $X_s(\mathbb{R}^N)$. Due to Lemma 7, $\{v_n\}$ admits a subsequence $\{v_{n_k}\}$, such that $v_{n_k} \to v_1$ in $X_s(\mathbb{R}^N)$ as $k \to \infty$. With the help of this and (13), we obtain that $\mathcal{I}_\lambda(v_1) = \ell$ and

$\mathcal{I}_\lambda'(v_1) = 0$. Hence, v_1 is a nontrivial solution of the given problem with $\mathcal{I}_\lambda(v_1) < 0$, which is different from v_0. This finishes the proof. □

Next, employing the fountain theorem in [49] (Theorem 3.6), we demonstrate the existence of a sequence of nontrivial weak solutions to problem (1). Let X be a separable and reflexive Banach space. It is well-known that there are $\{e_n\}$ in X and $\{f_n^*\}$ in X^*, such that

$$X = \overline{\operatorname{span}\{e_n : n = 1, 2, \cdots\}}, \quad X^* = \overline{\operatorname{span}\{f_n^* : n = 1, 2, \cdots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $X_n = \operatorname{span}\{e_n\}$, $Y_k = \bigoplus_{n=1}^k X_n$, and $Z_k = \overline{\bigoplus_{n=k}^\infty X_n}$. Then, we recall the fountain theorem.

Lemma 9. ([49]) *Let $\mathcal{X}$ be a real reflexive Banach space, $\mathcal{E} \in C^1(\mathcal{X}, \mathbb{R})$ satisfies the $(C)_c$-condition for any $c > 0$, and $\mathcal{E}$ is even. If, for each sufficiently large $k \in \mathbb{N}$, there exist $\rho_k > \delta_k > 0$, such that the following conditions hold:*

(1) $b_k := \inf\{\mathcal{E}(v) : v \in Z_k, \|v\|_{\mathcal{X}} = \delta_k\} \to \infty$ *as* $k \to \infty$;
(2) $a_k := \max\{\mathcal{E}(v) : v \in Y_k, \|v\|_{\mathcal{X}} = \rho_k\} \leq 0$,

then, the functional $\mathcal{E}$ has an unbounded sequence of critical values—that is, there exists a sequence $\{v_n\} \subset \mathcal{X}$, such that $\mathcal{E}'(v_n) = 0$ and $\mathcal{E}(v_n) \to \infty$ as $n \to \infty$.

Using Lemma 9, the existence of infinitely many nontrivial weak solutions for problem (1) is stated as follows:

Theorem 2. *Let $0 < s < 1 < p < +\infty$ and $ps < N$. Assume that* (V), (A) *and* (G1)–(G4) *hold. If $g(x, -t) = -g(x, t)$ satisfies for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, then, problem* (1) *possesses a sequence of nontrivial weak solutions $\{v_n\}$ in $X_s(\mathbb{R}^N)$, such that $\mathcal{I}_\lambda(v_n) \to \infty$ as $n \to \infty$ for any $\lambda > 0$.*

Proof. Obviously, $\mathcal{I}_\lambda$ is an even functional, and fulfils the $(C)_c$-condition for any $c \in \mathbb{R}$ by Lemma 7. Note that $X_s(\mathbb{R}^N)$ is a separable and reflexive Banach space. Thanks to Lemma 9, it suffices to prove that there exist $\rho_k > \delta_k > 0$, such that

(1) $b_k := \inf\{\mathcal{I}_\lambda(v) : v \in Z_k, \|v\|_{X_s(\mathbb{R}^N)} = \delta_k\} \to \infty$ as $k \to \infty$;
(2) $a_k := \max\{\mathcal{I}_\lambda(v) : v \in Y_k, \|v\|_{X_s(\mathbb{R}^N)} = \rho_k\} \leq 0$,

for a sufficiently large k. We denote

$$\vartheta_k := \sup_{v \in Z_k, \|v\|_{X_s(\mathbb{R}^N)} = 1} \Big(\int_{\mathbb{R}^N} |v(x)|^q \, dx \Big), \quad p < q < p_s^*.$$

Then, we know $\vartheta_k \to 0$ as $k \to \infty$. Indeed, suppose on the contrary that there is a positive constant ε_0 and the sequence $\{v_k\}$ in Z_k, such that

$$\|v_k\|_{X_s(\mathbb{R}^N)} = 1, \ \int_{\mathbb{R}^N} |v_k(x)|^q \, dx \geq \varepsilon_0,$$

for all $k \geq k_0$. Since the sequence $\{v_k\}$ is bounded in $X_s(\mathbb{R}^N)$, there exists an element v in $X_s(\mathbb{R}^N)$, such that $v_k \rightharpoonup v$ in $X_s(\mathbb{R}^N)$ as $k \to \infty$, and

$$\langle f_j^*, v \rangle = \lim_{k\to\infty} \langle f_j^*, v_k \rangle = 0$$

for $j = 1, 2, \cdots$. Hence, we get $v = 0$. However, we know

$$\varepsilon_0 \leq \lim_{k\to\infty} \int_{\mathbb{R}^N} |v_k(x)|^q \, dx = \int_{\mathbb{R}^N} |v(x)|^q \, dx = 0,$$

that it is a contradiction.

For any $v \in Z_k$, it follows from

$$\begin{aligned}
\mathcal{I}_\lambda(v) &= \frac{1}{p}(|v|^p_{W^{s,p}(\mathbb{R}^N)} + \|v\|^p_{L^p(V,\mathbb{R}^N)}) - \lambda \int_{\mathbb{R}^N} a(x)\,|v(x)|^r \, dx - \int_{\mathbb{R}^N} G(x,v)\,dx \\
&\geq \frac{1}{p}(|v|^p_{W^{s,p}(\mathbb{R}^N)} + \|v\|^p_{L^p(V,\mathbb{R}^N)}) - \lambda \int_{\mathbb{R}^N} a(x)\,|v(x)|^r \, dx - \int_{\mathbb{R}^N} \frac{b(x)}{q}\,|v(x)|^q \, dx \\
&\geq \frac{1}{p}(|v|^p_{W^{s,p}(\mathbb{R}^N)} + \|v\|^p_{L^p(V,\mathbb{R}^N)}) - \lambda \|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)} \|v\|^r_{L^p(\mathbb{R}^N)} - \frac{1}{q}\|b\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |v(x)|^q \, dx \\
&\geq \frac{1}{p}(|v|^p_{W^{s,p}(\mathbb{R}^N)} + \|v\|^p_{L^p(V,\mathbb{R}^N)}) - \lambda C_5 \|v\|^r_{X_s(\mathbb{R}^N)} - \frac{C_6}{q}\|v\|^q_{L^q(\mathbb{R}^N)} \\
&\geq \frac{1}{p}\|v\|^p_{X_s(\mathbb{R}^N)} - \lambda C_5 \|v\|^r_{X_s(\mathbb{R}^N)} - \frac{C_6}{q}\vartheta_k^q \|v\|^q_{X_s(\mathbb{R}^N)},
\end{aligned}$$

where C_5 and C_6 are positive constants. Choosing $\delta_k = (C_6 \vartheta_k^q)^{1/(p-q)}$, we assert $\delta_k \to \infty$ as $k \to \infty$, since $p < q$ and $\vartheta_k \to 0$ as $k \to \infty$. Hence, if $v \in Z_k$ and $\|v\|_{X_s(\mathbb{R}^N)} = \delta_k$, then, we deduce that

$$\mathcal{I}_\lambda(v) \geq \left(\frac{1}{p} - \frac{1}{q}\right)\delta_k^p - \lambda C_5 \delta_k^r \to \infty \quad \text{as} \quad k \to \infty,$$

which implies (1).

Next, suppose that condition (2) is not satisfied for some k. Then, there exists a sequence $\{v_n\}$ in Y_k, such that

$$\|v_n\|_{X_s(\mathbb{R}^N)} > 1 \quad \text{and} \quad \|v_n\|_{X_s(\mathbb{R}^N)} \to \infty \text{ as } n \to \infty \quad \text{and} \quad \mathcal{I}_\lambda(v_n) \geq 0. \tag{14}$$

Let $w_n = v_n / \|v_n\|_{X_s(\mathbb{R}^N)}$. Then, it is obvious that $\|w_n\|_{X_s(\mathbb{R}^N)} = 1$. Since $\dim Y_k < \infty$, there exists $w \in Y_k \setminus \{0\}$, such that up to a subsequence,

$$\|w_n - w\|_{X_s(\mathbb{R}^N)} \to 0 \quad \text{and} \quad w_n(x) \to w(x)$$

for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. For $x \in \Omega := \{x \in \mathbb{R}^N : w(x) \neq 0\}$, we obtain $|v_n(x)| \to \infty$ as $n \to \infty$. Hence, we deduce from the assumption (G3) that

$$\lim_{n\to\infty} \frac{G(x,v_n)}{\|v_n\|^p_{X_s(\mathbb{R}^N)}} \geq \lim_{n\to\infty} \frac{G(x,v_n)}{|v_n(x)|^p}\,|w_n(x)|^p = \infty. \tag{15}$$

As shown in the proof of Lemma 7, we can choose $\mathcal{C}_1 \in \mathbb{R}$, such that

$$\frac{G(x,v_n) - \mathcal{C}_1}{\|v_n\|^p_{X_s(\mathbb{R}^N)}} \geq 0 \tag{16}$$

for $x \in \Omega$. Considering (15) and (16) and the Fatou lemma, we assert by a similar argument to (9) that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \frac{G(x,v_n)}{\|v_n\|^p_{X_s(\mathbb{R}^N)}}\,dx &\geq \liminf_{n\to\infty}\int_\Omega \frac{G(x,v_n)-\mathcal{C}_1}{\|v_n\|^p_{X_s(\mathbb{R}^N)}}\,dx\\ &\geq \int_\Omega \liminf_{n\to\infty}\frac{G(x,v_n)}{\|v_n\|^p_{X_s(\mathbb{R}^N)}}\,dx = \infty. \end{aligned} \tag{17}$$

Consequently, using the relation (17), we have

$$\begin{aligned}\mathcal{I}_\lambda(v_n) &\leq \frac{1}{p}\|v_n\|^p_{X_s(\mathbb{R}^N)} - \lambda\int_{\mathbb{R}^N} a(x)\,|v_n(x)|^r\,dx - \int_\Omega G(x,v_n)\,dx\\ &\leq \frac{1}{p}\|v_n\|^p_{X_s(\mathbb{R}^N)}\left(1 - p\int_\Omega \frac{G(x,v_n)}{\|v_n\|^p_{X_s(\mathbb{R}^N)}}\,dx\right) \to -\infty\end{aligned}$$

as $n \to \infty$, which yields a contradiction to (14). The proof is complete. □

3. Conclusions

In this paper, we used the variational methods to get the existence of nontrivial distinct solutions to problem (1) for the case of a combined effect of concave-convex-type nonlinearities. As far as we are aware, the present paper is the first attempt to study the multiplicity of nontrivial weak solutions to Schrödinger-type problems with the concave-convex nonlinearity in these circumstances. Additionally, we address to the readers several comments and perspectives.

I. We point out that with a similar analysis, our main consequences continue to hold when $(-\Delta)^s_p v$ in (1) is changed into any non-local integro-differential operator $\mathcal{L}_K$ in (2), where $K : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ is a kernel function satisfying properties that

(K1) $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^p, 1\}$;
(K2) there exists $\theta > 0$, such that $K(x) \geq \theta|x|^{-(N+ps)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
(K3) $K(x) = K(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

II. A new research direction which has a strong relationship with several related applications is the study of Kirchhoff-type equations

$$\mathcal{M}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}dx\,dy\Big)(-\Delta)^s_p v + V(x)|v|^{p-2}v = \lambda a(x)|v|^{r-2}v + g(x,v) \quad \text{in } \mathbb{R}^N,$$

where $\mathcal{M} \in C(\mathbb{R}^+_0, \mathbb{R}^+)$ is a Kirchhoff-type function and $\mathcal{M} : \mathbb{R}^+_0 \to \mathbb{R}^+$ satisfies the following conditions:

(M1) $\mathcal{M} \in C(\mathbb{R}^+_0, \mathbb{R}^+)$ fulfils $\inf_{t\in\mathbb{R}^+_0}\mathcal{M}(t) \geq m_0 > 0$, where m_0 is a constant.
(M2) There is a positive constant $\theta \in [1, \frac{N}{N-ps})$, such that $\theta\mathfrak{M}(t) \geq \mathcal{M}(t)t$ for any $t \geq 0$, where $\mathfrak{M}(t) := \int_0^t \mathcal{M}(\tau)d\tau$.

Funding: This research received no external funding.

Acknowledgments: The author gratefully thanks to the reviewers for the constructive comments and recommendations which definitely help to improve the readability and quality of the paper.

Conflicts of Interest: The author declares no conflict of interest.

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