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Accelerated Life Tests under Pareto-IV Lifetime Distribution: Real Data Application and Simulation Study

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Abstract: In this article, a progressive-stress accelerated life test (ALT) that is based on progressive type-II censoring is studied. The cumulative exposure model is used when the lifetime of test units follows Pareto-IV distribution. Different estimates as the maximum likelihood estimates (MLEs) and Bayes estimates (BEs) for the model parameters are discussed. Bayesian estimates are derived while using the Tierney and Kadane (TK) approximation method and the importance sampling method. The asymptotic and bootstrap confidence intervals (CIs) of the parameters are constructed. A real data set is analyzed in order to clarify the methods proposed through this paper. Two types of the progressive-stress tests, the simple ramp-stress test and multiple ramp-stress test, are compared through the simulation study. Finally, some interesting conclusions are drawn.

Keywords: progressive-stress; progressive type-II censoring; maximum likelihood estimation; bayes estimation; Tierney and Kadane approximation; simulation study

1. Introduction

In most of the classical life testing and reliability experiments, collecting enough number of failure times is not easy, especially when the products are highly reliable with long lifetimes. Under normal conditions the lifetime experiments with a restricted testing time may produce a very few failures, one of the most famous procedures used to accelerate the occurrence of failure is the ALT, in which products are tested at high levels of stress (e.g., humidity, temperature, pressure, voltage, or vibration). The failure time data from ALTs are then analyzed in order to estimate the life characteristics of the products under normal conditions. There are different ways in which the ALT can work, for example, constant, step, and progressive stress ALT. Nelson discussed these different types [1].

In constant-stress ALT, the test is performed completely under a constant level of stress. Several authors have discussed the constant-stress models; see references [2–9] for more details. The second way to apply accelerated stress is step-stress. In this kind of stress, we raise the stress level step by step at pre-specified times to get early failures. Many authors have discussed the step-stress ALTs; see for example Bai et al. [10], Gouno et al. [11], Miller and Nelson [12], Balakrishnan et al. [13], and Mohie El-Din et al. [14–17]. On the other hand, in progressive-stress ALT, all of the test units are exposed to stress, which is continuously at an increasing rate with time. For more reading about progressive-stress ALT, see Yin and Sheng [18], Mohie El-Din et al. [19], Bai et al. [20], as well as Ronghua and Heliang [21] and Abdel-Hamid and AL-Hussaini [22] who also studied the progressive-stress ALT based on progressive censoring in case of Weibull distribution in [23]. Bayesian prediction intervals under progressively type-II censoring for the half-logistic distribution under progressive-stress model have been discussed by AL-Hussaini et al. [24]. The Bayesian estimates of exponentiated exponential distribution based on type-II progressive hybrid censoring have been discussed by Abdel-Hamid and Abushul [25], depending on the inverse power law and the cumulative exposure model. Abd El-Raheem [26] considered the problem of optimal design of progressive-stress ALT for generalized half normal distribution.

Because of the high cost and long time that the life testing experiments consume, the experimenter needs to end the experiment before observing all failures, so the censoring techniques of data are widely used in order to decrease the test time and cost. Type-I and Type-II conventional censoring are the most two used censoring schemes for life testing. Recently, the most popular censoring type is the progressive type-II censoring scheme (CS). It can be illustrated, as follows: suppose n identical units or devices are put on a life time test with $m \leq n$ is a pre-specified number of failures. At the occurrence of the first failure $t_{1:m:n}$, R_1 surviving units are removed from the test. When the second failure $t_{2:m:n}$ occurs, R_2 surviving units are also removed from the test and so on until the m-th failure is observed, where the test is terminated and the remaining surviving $R_m = n - m - (R_1 + ... + R_{m-1})$ units are removed. For extra studies regarding progressive censoring, see [27,28].

The contribution in this paper is studying statistical inference of progressive-stress ALT for test units whose lifetime follow Pareto-IV distribution in the presence of progressively type-II censored data.

The paper is organized, as follows: in Section 2, the ALT model and test assumptions are presented. In Section 3, the maximum likelihood estimates (MLEs) of the parameters under progressive-stress model are obtained. Using Tierney and Kadane (TK) Approximation and importance sampling method, the Bayes estimates (BEs) for model parameters are discussed in Section 4. The asymptotic and bootstrap confidence intervals for the parameters of the model are constructed in Section 5. In Section 6, a real data set is analyzed in order to illustrate the proposed methods. Section 7 includes the simulation study. Finally, the paper is concluded in Section 8.

2. ALT Model and Test Assumptions

2.1. Pareto-IV Distribution

Arnold presents the hierarchy of Pareto-IV distribution [29]. It was formed by adding additional parameters (shape, scale, location, and inequality) to the conventional Pareto I distribution. The cumulative distribution function (CDF) of Pareto-IV$(\mu, \gamma, \theta, \alpha)$ distribution is given by

$$F(t) = 1 - \left[1 + \left(\frac{t-\mu}{\theta}\right)^{1/\gamma}\right]^{-\alpha}, \quad \gamma, \alpha, \theta > 0,\ t > \mu,\ -\infty < \mu < \infty. \tag{1}$$

With a view to present Pareto-IV distribution as a regular family, we consider parameter μ is known and equal to 0, Serfling [30]. Thus, the corresponding probability density function (PDF) of Pareto-IV $(0, \gamma, \theta, \alpha)$ is given by

$$f(t) = \frac{\alpha \left(\frac{t}{\theta}\right)^{\frac{1}{\gamma}-1}}{\gamma\theta \left(1 + \left(\frac{t}{\theta}\right)^{\frac{1}{\gamma}}\right)^{\alpha+1}}, \quad t > 0,\ \gamma, \alpha, \theta > 0. \tag{2}$$

2.2. Test Assumptions

In this subsection, we list the assumptions that are used through this article in the context of progressive-stress ALT:

1. The lifetime of a unit follows Pareto-IV $(0, \gamma, \theta, \alpha)$.
2. The progressive-stress at level i, $S_i(t)$ is directly proportional to the time with fixed rate ν_i, i.e., $S_i(t) = \nu_i t,\ 0 < \nu_1 < \nu_2 < ... < \nu_k$.
3. The inverse power law is used to model the relationship between the life characteristic θ_i and the stress loading $S_i(t)$, as follows
$$\theta_i(t) = \frac{1}{a[s_i(t)]^b}, \tag{3}$$
where the parameters a and b are unknown and positive. For more information about this acceleration model, see, e.g., Nelson ([1], [Ch. 2]).
4. The cumulative exposure model is used in order to describe the effect of stress on failure time from one level to another, see, e.g., Nelson [1].
5. From the life-stress relationship in (3) the parameter θ_i can be expressed as $\theta_i = \theta_1 \psi_i^b$, where $\psi_i = \frac{\nu_1}{\nu_i}$.

Suppose that the test is started with n units, $S_0 < S_1(t) < ... < S_k(t)$ be the stress levels used during the test and S_0 be the use-stress. Furthermore, suppose n_i units are tested under each progressive-stress level $Si(t)$, $i = 1, 2, ..., k$. The progressive censoring scheme is carried out as follows: When the time of the first failure $t_{i1:m_i:n_i}$ is reached, R_{i1} units are randomly removed from the remaining $n_i - 1$ surviving units. When the time of the second failure $t_{i2:m_i:n_i}$ is reached, R_{i2} units from the remaining $n_i - 2 - R_{i1}$ surviving units are randomly withdrawn from the test. The test is terminated at level i when the m_i-th failure occurs $t_{im_i:m_i:n_i}$, at this time all remaining $R_{im_i} = n_i - m_i - \sum_{j=1}^{m_i-1} R_{ij}$ items are withdrawn. It is clear that the complete samples and type-II censored samples are special cases of this scheme. The observed life times while using the progressive-stress $S_i(t)$ are $t_{i1:m_i:n_i} < t_{i2:m_i:n_i} < ... < t_{im_i:m_i:n_i}$, $i = 1, 2, ..., k$.

Based on the assumption of the of cumulative exposure model, the CDF for the lifetimes that result from progressive-stress $S_i(t)$ is

$$G_i(t) = F_i(\triangle(t)),\ i = 1, 2, ..., k, \tag{4}$$

where $\triangle(t) = \int_0^t \frac{du}{\theta_i(u)} = \frac{t}{\theta_1 \psi_i^b (b+1)}$, and $F_i(.)$ is defined in (2) with scale parameter equals 1. Therefore,

$$G_i(t) = 1 - \left[1 + \left(\frac{t}{\sigma i}\right)^{1/\gamma}\right]^{-\alpha},\quad t > 0,\ \gamma, \alpha, \sigma i > 0, \tag{5}$$

where $\sigma i = \theta_1 \psi_i^b (b+1)$. The PDF of (5) is given by

$$g_i(t) = \frac{\alpha \left(\frac{t}{\sigma i}\right)^{\frac{1}{\gamma}-1}}{\gamma \sigma i \left(1 + \left(\frac{t}{\sigma i}\right)^{\frac{1}{\gamma}}\right)^{\alpha+1}},\quad t > 0,\ \gamma, \alpha, \sigma i, \theta > 0. \tag{6}$$

3. Maximum Likelihood Estimation

In this section, the MLEs of the parameters γ, α, $\theta 1$, and b are obtained. For simplification, let $t_{ij:m_i:n_i} = t_{ij}$, which represent the progressive censoring lifetimes under progressive-stress level $S_i(t)$, $i = 1, 2, ..., k$ and $j = 1, 2, ..., m_i$. The likelihood function of the proposed model is derived while using the CDF in (5) and the corresponding PDF in (6), as follows:

$$L(\gamma, \alpha, \theta_1, b) = \prod_{i=1}^{k} C_i \prod_{j=1}^{m_i} g_i(t_{ij}) \left[1 - G_i(t_{ij})\right]^{R_{ij}}, \tag{7}$$

where $C_i = n_i\,(n_i - 1 - R_{i1})\,(n_i - 2 - R_{i1} - R_{i2}) \ldots \left(n_i - m_i + 1 - \sum_{j=1}^{m_i-1} R_{ij}\right)$.

From (5) and (6) in (7), we obtain

$$L(\gamma,\alpha,\theta_1,b) = \prod_{i=1}^{k} C_i \prod_{j=1}^{m_i} \frac{\alpha\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}-1}}{\gamma\sigma i\left(1+\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}\right)^{\alpha(R_{ij}+1)+1}}. \tag{8}$$

Therefore, the log-likelihood function is given by

$$\begin{aligned}\ell(\gamma,\alpha,\theta_1,b) = &\sum_{i=1}^{k}\log C_i + (\log\alpha - \log\sigma_i - \log\gamma)\sum_{i=1}^{k} m_i + \left(\frac{1}{\gamma}-1\right)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\log\left(\frac{t_{ij}}{\sigma i}\right)\\ &-\sum_{i=1}^{k}\sum_{j=1}^{m_i}(\alpha(R_{ij}+1)+1)\log\left(\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}+1\right),\end{aligned} \tag{9}$$

where the likelihood equations of α, γ, and θ_1 and b are, respectively

$$\frac{\partial\ell}{\partial\alpha} = \frac{\sum_{i=1}^{k} m_i}{\alpha} - \sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[(R_{ij}+1)\log\left(\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}+1\right)\right], \tag{10}$$

$$\frac{\partial\ell}{\partial\gamma} = \frac{-\sum_{i=1}^{k} m_i}{\gamma} + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{-1}{\gamma^2}\log\left(\frac{t_{ij}}{\sigma i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{(\alpha(R_{ij}+1)+1)\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}\log\left(\frac{t_{ij}}{\sigma i}\right)}{\gamma^2\left(\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}+1\right)}, \tag{11}$$

$$\frac{\partial\ell}{\partial\theta_1} = \frac{-\sum_{i=1}^{k} m_i}{\theta_1} - \left(\frac{1}{\gamma}-1\right)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{1}{\theta_1} + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{(\alpha(R_{ij}+1)+1)\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}}{\gamma\theta_1\left(\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}+1\right)}, \tag{12}$$

$$\begin{aligned}\frac{\partial\ell}{\partial b} = &\sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[\frac{-1}{(b+1)} - \log\psi\right] + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\left(\frac{1}{\gamma}-1\right)\left[\frac{-1}{(b+1)} - \log\psi\right]\\ &+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{(\alpha(R_{ij}+1)+1)\left[(b+1)\log\psi_i+1\right]\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}}{\gamma(b+1)\left(\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}+1\right)},\end{aligned} \tag{13}$$

The MLEs of the parameters are obtained by solving the system of four nonlinear Equations (10)–(13) in four unknowns γ, α, θ_1, and b. This can be solved while using the Newton–Raphson iteration method.

4. Bayesian Estimation

In this section, based on square error (SE), linear exponential (LINEX), and the general entropy (GE) loss functions, the BEs for the parameters α, γ, θ_1, and b based on progressive type-II censoring are discussed. The prior distributions of the parameters α, γ, θ_1, and b are assumed to be gamma priors. Thus,

$$\pi_1(\alpha) \propto \alpha^{\mu_1-1}e^{\frac{-\alpha}{\lambda_1}}, \quad \alpha > 0,\ \mu_1,\lambda_1 > 0,$$

$$\pi_2(\theta_1) \propto \theta_1^{\mu_2-1}e^{\frac{-\theta_1}{\lambda_2}}, \quad \theta_1 > 0,\ \mu_2,\lambda_2 > 0,$$

$$\pi_3(b) \propto b^{\mu_3-1}e^{\frac{-b}{\lambda_3}}, \quad b > 0,\ \mu_3,\lambda_3 > 0,$$

and

$$\pi_4(\gamma) \propto \gamma^{\mu_4-1} e^{\frac{-\gamma}{\lambda_4}}, \quad \gamma > 0,\ \mu_4, \lambda_4 > 0.$$

Assume that the parameters of the model are independent, then the joint PDF of prior is given by

$$\pi(\gamma, \alpha, \theta_1, b) \propto b^{\mu_3-1} \theta_1^{\mu_2-1} \alpha^{\mu_1-1} \gamma^{\mu_4-1} e^{-(\frac{\alpha}{\lambda_1}+\frac{\gamma}{\lambda_4}+\frac{\theta_1}{\lambda_2}+\frac{b}{\lambda_3})}. \tag{14}$$

The posterior density function of the parameters γ, α, θ_1 and b can be obtained from (8) and (14), as follows:

$$\begin{aligned} \pi^*(\Theta|t) = \pi^*(\gamma, \alpha, \theta_1, b|t) &\propto L(\gamma, \alpha, \theta_1, b)\ \pi(\gamma, \alpha, \theta_1, b) \\ &\propto b^{\mu_3-1} \theta_1^{\mu_2-1} \alpha^{\mu_1-1} \gamma^{\mu_4-1} e^{-(\frac{\alpha}{\lambda_1}+\frac{\gamma}{\lambda_4}+\frac{\theta_1}{\lambda_2}+\frac{b}{\lambda_3})} \\ &\prod_{i=1}^{k} C_i \prod_{j=1}^{m_i} \frac{\alpha \left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}-1}}{\gamma \sigma i \left(1+\left(\frac{t_{ij}}{\sigma i}\right)^{\frac{1}{\gamma}}\right)^{\alpha(R_{ij}+1)+1}}. \end{aligned} \tag{15}$$

The BE of the function of parameters $U = U(\Theta)$, $\Theta = (\gamma, \alpha, \theta_1, b)$ under the SE loss function is given by

$$\hat{U}_{SE} = \int_{\Theta} U\ \pi^*(\Theta|t)\ d\Theta, \tag{16}$$

Depending on the LINEX loss function, the BE of $U = U(\Theta)$ is given by

$$\hat{U}_{LINEX} = -\frac{1}{c} \log \left[\int_{\Theta} e^{-cU}\ \pi^*(\Theta|t)\ d\Theta \right], \tag{17}$$

where $c \neq 0$ is the shape parameter of the LINEX loss function.

Based on GE loss function, the BE of U is given by

$$\hat{U}_{GE} = \left(\int_{\Theta} U^{-q} \pi^*(\Theta|t) d\Theta \right)^{-1/q}, \tag{18}$$

where $q \neq 0$

It is obvious that the integrals in Equations (16)–(18) are complicated. Consequently, the important sampling method and TK method are applied to obtain an approximation for these integrals.

4.1. Important Sampling Technique

In this part, we use the important sampling method for estimating the unknown parameters. The posterior distribution presented in (15) can be rewritten as

$$\varpi^*(\gamma, \alpha, \theta_1, b) \propto\ G_1(\gamma) \times G_2(b) \times G_3(\theta_1|\gamma) \times G_4(\alpha|\gamma, \theta_1, b) \times \Delta(\gamma, \alpha, \theta_1, b), \tag{19}$$

where

$$G_1(\gamma) = GammaDistribution[\mu_4 - \sum_{i=1}^{k} m_i, \lambda_4], \tag{20}$$

$$G_2(b) = GammaDistribution\left[\mu_3, \left(\frac{1}{\lambda_3} + \sum_{i=1}^{k} m_i Log[\psi_i]\right)^{-1}\right], \tag{21}$$

$$G_3(\theta_1|\gamma) = GammaDistribution\left[\mu_2 - \frac{1}{\gamma}\sum_{i=1}^{k} m_i, \lambda_2\right], \tag{22}$$

and

$$G_4(\alpha|\gamma,\theta_1,b) = GammaDistribution\left[\mu_1 + \sum_{i=1}^{k} m_i, \left(\frac{1}{\lambda_1} + \sum_{i=1}^{k}\sum_{j=1}^{m_i}(R_{ij}+1)Log\left[1+(\frac{t_{ij}}{\psi_i^b(b+1)\theta_1})^{1/\gamma}\right]\right)^{-1}\right], \tag{23}$$

Additionally,

$$\begin{aligned}\Delta\left(\gamma,\alpha,\theta_1,b\right) =& (b+1)^{-(\frac{1}{\gamma}\sum_{i=1}^{k} m_i)} \exp\left\{\sum_{i=1}^{k} m_i(\frac{1}{\gamma}-1)(Log[t_{ij}] - bLog[\psi_i])\right\}\times \\ & \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{m_i} Log\left[1+\left(\frac{t_{ij}}{\psi_i^b(b+1)\theta_1}\right)^{1/\gamma}\right]\right\}.\end{aligned} \tag{24}$$

Algorithm 1 is used to find the BEs of γ, α, θ_1, b.

Algorithm 1 Importance Sampling

1. Generate γ from $G_1(\gamma)$.
2. Generate b from $G_2(b)$.
3. Generate θ_1 from $G_3(\theta_1|\gamma)$.
4. Generate α from $G_4(\alpha|\gamma,\theta_1,b)$.
5. Repeat steps 1 to 4, N times to get $(\alpha_1,\theta_{11},\gamma_1,b_1)$, ..., $(\alpha_N,\theta_{1N},\gamma_N,b_N)$.
6. The BEs estimate of $u(\gamma,\alpha,\theta_1,b)$ under the SE loss function is

$$\tilde{u}_{SE} = \frac{\sum_{i=1}^{N} u(\gamma,\alpha,\theta_1,b)\ \Delta(\gamma,\alpha,\theta_1,b)}{\sum_{i=1}^{N}\Delta(\gamma,\alpha,\theta_1,b)}, \tag{25}$$

7. The BEs estimate of $u(\gamma,\alpha,\theta_1,b)$ under LINEX loss function is

$$\tilde{u}_{LINEX} = \frac{-1}{c}\log\left[\frac{\sum_{i=1}^{N}\exp\{-c\ u(\gamma,\alpha,\theta_1,b)\}\ \Delta(\gamma,\alpha,\theta_1,b)}{\sum_{i=1}^{N}\Delta(\gamma,\alpha,\theta_1,b)}\right], \tag{26}$$

8. The BEs estimate of $u(\gamma,\alpha,\theta_1,b)$ under GE loss function is

$$\tilde{u}_{GE} = \left[\frac{\sum_{i=1}^{N}(u(\gamma,\alpha,\theta_1,b))^{-q}\ \Delta(\gamma,\alpha,\theta_1,b)}{\sum_{i=1}^{N}\Delta(\gamma,\alpha,\theta_1,b)}\right]^{-1/q}. \tag{27}$$

4.2. Tk-Approximation for Bayesian Estimates

There are a lot of approximation methods to obtain an approximate value for the integration that depend on the posterior distribution as Monte Carlo methods (Kloek and Van Dijk [31] and Zellner and Rossi [32]) and Lindley method [33], which are computationally intensive. For extra studies where the MCMC and Lindley method are used, see, e.g., [34–37] and [38]. Tierney and Kadane [39] implemented an easily computable integration approximation method that was based on the posterior distribution as the mean and variance of a non-negative parameter or a smooth parameter function that is non-zero within the space of the parameter. The advantage of the Tierney and Kadane (TK) method over Lindley approximation method is that the second method requires the evaluation of the third derivatives of the posterior density or the likelihood function, which can be tiresome and requires great computational precision.

In this subsection, we use the TK approximation method to compute the Bayesian estimates of the parameters that are based on the SE loss function. TK-method is one of the methods to find the approximate value of the ratio of two integrals as given in Equation (16), which can be rewritten, as follows:

$$\hat{U}(\Theta) = \frac{\int\limits_{\Theta} U(\Theta)\, e^{[l(\Theta|t)+\rho(\Theta|t)]}\, d\Theta}{\int\limits_{\Theta} e^{[l(\Theta|t)+\rho(\Theta|t)]}\, d\Theta}, \tag{28}$$

where $U(\Theta)$ is any function of parameters $(\alpha, \gamma, \theta_1, b)$, $l(\Theta|t)$ is defined in (9), and $\rho(\Theta|t)$ is the logarithm joint prior distribution that is given by

$$\begin{aligned} \rho(\alpha, \gamma, \theta_1, b|t) &= (\mu_1 - 1)ln(\alpha) + (\mu_2 - 1)ln(\theta_1) + (\mu_3 - 1)ln(b) + (\mu_4 - 1)ln(\gamma) \\ &- \left(\frac{\alpha}{\lambda_1} + \frac{\theta_1}{\lambda_2} + \frac{b}{\lambda_3} + \frac{\gamma}{\lambda_4}\right). \end{aligned} \tag{29}$$

To obtain an explicit expression for $\hat{U}_{SE}(\Theta)$ using TK approximation, we consider the functions, defined by

$$\delta(\Theta) = \frac{l(\Theta|t) + \rho(\Theta|t)}{n} \tag{30}$$

$$\delta^*(\Theta) = \delta(\Theta) + \frac{lnU(\Theta)}{n} \tag{31}$$

Now, assume that the following groups of values $(\hat{\alpha}_\delta, \hat{\gamma}_\delta, \hat{\theta}_{1\delta}, \hat{b}_\delta)$ and $(\hat{\alpha}_{\delta^*}, \hat{\gamma}_{\delta^*}, \hat{\theta}_{1\delta^*}, \hat{b}_{\delta^*})$ maximize the functions $\delta(\alpha, \gamma, \theta_1, b)$ and $\delta^*(\alpha, \gamma, \theta_1, b)$, respectively.

Hence, we approximate $\hat{U}(\Theta)$, as follows:

$$\hat{U}_{TK}(\Theta) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}} \exp\left[n\left\{\delta^*(\hat{\alpha}_{\delta^*}, \hat{\gamma}_{\delta^*}, \hat{\theta}_{1\delta^*}, \hat{b}_{\delta^*}) - \delta(\hat{\alpha}_\delta, \hat{\gamma}_\delta, \hat{\theta}_{1\delta}, \hat{b}_\delta)\right\}\right], \tag{32}$$

where $|\Sigma|$ and $|\Sigma^*|$ denote the determinants of negative inverse hessian of $\delta(\alpha, \gamma, \theta_1, b)$ and $\delta^*(\alpha, \gamma, \theta_1, b)$, respectively. Moreover, $|\Sigma|$ and $|\Sigma^*|$ are given by

$$|\Sigma| = \left|\begin{pmatrix} \frac{\partial^2\delta}{\partial\alpha^2} & \frac{\partial^2\delta}{\partial\alpha\partial\gamma} & \frac{\partial^2\delta}{\partial\alpha\partial\theta_1} & \frac{\partial^2\delta}{\partial\alpha\partial b} \\ \frac{\partial^2\delta}{\partial\gamma\partial\alpha} & \frac{\partial^2\delta}{\partial\gamma^2} & \frac{\partial^2\delta}{\partial\gamma\partial\theta_1} & \frac{\partial^2\delta}{\partial\gamma\partial b} \\ \frac{\partial^2\delta}{\partial\theta_1\partial\alpha} & \frac{\partial^2\delta}{\partial\theta_1\partial\gamma} & \frac{\partial^2\delta}{\partial\theta_1^2} & \frac{\partial^2\delta}{\partial\theta_1\partial b} \\ \frac{\partial^2\delta}{\partial b\partial\alpha} & \frac{\partial^2\delta}{\partial b\partial\gamma} & \frac{\partial^2\delta}{\partial b\partial\theta_1} & \frac{\partial^2\delta}{\partial b^2} \end{pmatrix}\right|^{-1} \quad \& \quad |\Sigma^*| = \left|\begin{pmatrix} \frac{\partial^2\delta^*}{\partial\alpha^2} & \frac{\partial^2\delta^*}{\partial\alpha\partial\gamma} & \frac{\partial^2\delta^*}{\partial\alpha\partial\theta_1} & \frac{\partial^2\delta^*}{\partial\alpha\partial b} \\ \frac{\partial^2\delta^*}{\partial\gamma\partial\alpha} & \frac{\partial^2\delta^*}{\partial\gamma^2} & \frac{\partial^2\delta^*}{\partial\gamma\partial\theta_1} & \frac{\partial^2\delta^*}{\partial\gamma\partial b} \\ \frac{\partial^2\delta^*}{\partial\theta_1\partial\alpha} & \frac{\partial^2\delta^*}{\partial\theta_1\partial\gamma} & \frac{\partial^2\delta^*}{\partial\theta_1^2} & \frac{\partial^2\delta^*}{\partial\theta_1\partial b} \\ \frac{\partial^2\delta^*}{\partial b\partial\alpha} & \frac{\partial^2\delta^*}{\partial b\partial\gamma} & \frac{\partial^2\delta^*}{\partial b\partial\theta_1} & \frac{\partial^2\delta^*}{\partial b^2} \end{pmatrix}\right|^{-1}.$$

5. Interval Estimation

In this part of the paper, the approximate and Bootstrap CIs of the parameters γ, α, θ_1, and b are obtained.

5.1. Approximate Confidence Intervals

Based on the asymptotic distribution of the MLEs of the unknown parameters γ, α, θ_1, and b, the normal approximation CIs of the parameters are obtained. The asymptotic distribution of the MLEs of $(\gamma, \alpha, \theta_1, b)$ is given by Miller [40]

$$\left((\hat{\gamma} - \gamma), (\hat{\theta}_1 - \theta_1), (\hat{\alpha} - \alpha), (\hat{b} - b)\right) \sim \mathbf{N}(0, \eta),$$

where $\eta = (\kappa_{ij}),\ i,\ j = 1,2,3,4$ is the variance covariance matrix of the unknown parameters. The 100 $(1-\zeta)\%$ two sided CIs for a general parameter ϑ_i is given by

$$(\hat{\vartheta}_{iL},\ \hat{\vartheta}_{iU}) = \hat{\vartheta}_i \pm Z_{\zeta/2}\sqrt{\kappa_{ii}}, \quad i = 1,2,3,4, \tag{33}$$

where $\vartheta_1 = \gamma,\ \vartheta_2 = \theta_1,\ \vartheta_3 = \alpha,\ \vartheta_4 = b$, and Z_q is the $100q$-th percentile of a standard normal distribution.

5.2. Bootstrap Confidence Intervals

The percentile bootstrap (Boot-p) [41] CIs of the unknown parameters γ, α, θ_1, and b are obtained. The following steps are used in order to obtain the bootstrap bounds for our distribution parameters. Algorithm 2 is used to find Bootstrap CIs.

Algorithm 2 Bootstrap CIs

1. Find MLE estimates of the parameters α, γ, θ_1, and b by solving Equations (10)–(13).
2. Use the estimates in step 1 in order to generate a bootstrap sample $\underline{X}^*$ with the same values of the censoring scheme using the algorithm made by [42]. Let us say that the estimates are $\widehat{\alpha}^*$, $\widehat{\gamma}^*$, $\widehat{\theta 1_1}^*$ and $\widehat{b}^*$.
3. Repeat steps 1 and 2, 1000 times.
4. Arrange the estimates ascendingly $(\phi_t^{(1)}, \phi_t^{(2)}, \cdots, \phi_t^{(N)}),\ t = 1,2,3,4$, where $\phi_1 = \widehat{\alpha}^*,\ \phi_2 = \widehat{\gamma}^*,\ \phi_3 = \widehat{\theta 1_1}^*,\ \phi_4 = \widehat{b}^*$. Let $G(x) = P(\phi_t \le x)$ be the CDF of ϕ_t.
5. Define $\phi_{tboot} = G^{-1}(x)$ for given x. Accordingly, the upper and lower bounds are given by:

$$\left[\phi_{tboot}\left(\frac{\beta}{2}\right), \phi_{tboot}\left(1-\frac{\beta}{2}\right)\right]. \tag{34}$$

6. Application

In this section, we illustrate the proposed methods by analyzing reliability experiment carried out previously by Zhu [43]. The data in Table 1 were observed from ramp-voltage experiment of miniature light bulbs. In this ramp-voltage experiment, 62 and 61 light bulbs were experimented under ramp-rate 2.01 V/h, and 2.015 V/h, respectively, with experimental design stress is 2 V.

To determine whether the real data fit the Pareto-IV distribution, we apply Kolmogorov–Smirnov (K-S) test. Using the MLEs of parameters which are presented in Table 3, the values of K-S statistic and corresponding p-values are calculated and shown in Table 2. From results presented in Table 2, we conclude that Pareto-IV distribution is a good fit to the given data according to the p-values.

Table 3 includes the MLES and BEs of the model parameters as well as the approximate (App) and Boot CIs for the data set in Table 1. The BEs are obtained while using both TK method and importance sampling method under SE, LINEX, and GE loss functions with $c = -2$ and $q = 2$. Furthermore, the values of hyper-parameters used for obtaining BEs are $\mu_1 = 131.8$, $\mu_2 = 3717.6$, $\mu_3 = 1.3$, $\mu_4 = 2606.2$, $\lambda_1 = 0.0027$, $\lambda_2 = 0.00016$, $\lambda_3 = 0.00027$, and $\lambda_4 = 0.00061$.

Table 1. Failure times of miniature light bulbs from ramp-voltage experiment.

Stress Volt per Hour 2.01 V/h						Stress Volt per Hour 2.015 V/h					
Number	**t**	**Number**	**t**	**Number**	**t**	**Number**	**t**	**Number**	**t**	**Number**	**t**
1	13.57	22	72.33	43	42.06	1	19.3	22	49.65	43	31.00
2	19.92	23	72.60	44	47.88	2	23.28	23	51.42	44	34.81
3	23.3	24	75.43	45	54.21	3	23.50	24	51.27	45	36.03
4	27.81	25	75.85	46	54.55	4	26.50	25	53.25	46	43.08
5	31.16	26	76.20	47	55.85	5	27.42	26	54.25	47	45.63
6	31.56	27	77.78	48	56.43	6	28.32	27	55.47	48	46.03
7	34.00	28	79.13	49	58.86	7	28.62	28	56.83	49	46.33
8	46.26	29	80.65	50	60.60	8	30.62	29	56.17	50	49.62
9	46.41	30	82.65	51	62.48	9	34.42	30	8.85	51	49.86
10	50.60	31	90.33	52	62.81	10	35.30	31	11.31	52	50.66
11	56.76	32	14.51	53	63.41	11	35.48	32	11.83	53	50.93
12	56.85	33	15.61	54	63.76	12	38.30	33	14.50	54	51.03
13	60.13	34	15.85	55	64.18	13	40.52	34	14.83	55	51.73
14	65.00	35	17.73	56	66.15	14	43.83	35	17.73	56	51.95
15	65.86	36	19.65	57	66.41	15	43.00	36	19.35	57	52.36
16	66.20	37	21.05	58	69.91	16	43.00	37	25.50	58	54.78
17	66.40	38	21.20	59	71.73	17	43.12	38	26.15	59	55.58
18	66.80	39	24.21	60	72.46	18	44.43	39	27.45	60	55.83
19	66.93	40	24.85	61	73.78	19	45.32	40	27.61	61	57.13
20	68.25	41	31.18	62	78.91	20	47.58	41	28.05		
21	70.23	42	35.08			21	47.65	42	30.96		

Table 2. K–S statistic and corresponding p-values for each stress level.

Stress	2.01 V/h	2.015 V/h
Value of statistic	0.2258	0.125
p-value	0.0846	0.7865

Table 3. Maximum likelihood estimates (MLEs), Tierney and Kadane (TK) estimates, importance sampling Bayes estimates (BEs) under square error (SE), linear exponential (LINEX), and general entropy (GE) loss functions, and length of 95% CIs.

Θ	MLE	TK	SE	LINEX	GE	App CI Length	Boot CI length
$\hat{\alpha}$	6.0973	6.09726	6.0975	6.0975	6.09623	14.0879	22.0187
$\hat{\theta}_1$	36.5901	36.5901	36.5818	36.5814	36.584	48.9879	41.0014
$\hat{\gamma}$	0.3631	0.36312	0.3598	0.3598	0.3595	0.11123	0.1551
$\hat{b}$	1.6143	1.61439	1.6727	1.6724	1.670	3.5197	1.63704

7. Simulation Studies

The performance of the suggested estimators is studied while using Monte Carlo simulations. The suggested estimators are compared in terms of their relative absolute biases (RABs) and mean square errors (MSEs) for different sample sizes (n_i, m_i, $i = 1, 2, ..., k$), and censoring schemes (R_{ij}, $j = 1, 2, ..., m_i$). Furthermore, the simulation study is carried out for two designs of the progressive-stress ALT, the first one is the simple ramp-stress with two stress levels ($k = 2$) and their ramp values are $\nu_1 = 10$ and $\nu_2 = 16$. The second one is multiple ramp-stress test contains four stress levels ($k = 4$) with ramp values $\nu_1 = 10$, $\nu_2 = 16$, $\nu_3 = 20$, and $\nu_4 = 25$. Tables 4–7 contain the results that were obtained from the simulation study. Tables 4 and 5 give the values of the MSEs and RABs of MLEs and BEs using the TK method and importance sampling method under SE, LINEX, and GE loss functions with $c = 2$ and $q = 2$. Tables 6 and 7 introduce the length of CIs and their coverage probabilities. In both Tables 4 and 5, each parameter faces to two horizontal lines, the first containing values of MSEs and the second containing values of RABs. The results of simulation studies are obtained using two censoring schemes (CS 1 and CS 2), which are defined as

$$\text{CS 1:} \qquad R_{ij} = \begin{cases} n_i - m_i, \ j = 1, \\ 0, \ other\ wise. \end{cases}$$

$$\text{CS 2:} \qquad \text{If } m_i \text{ is even,} \quad R_{ij} = \begin{cases} n_i - m_i, \ j = \frac{m_i}{2}, \\ 0, \ other\ wise. \end{cases}$$

$$\text{If } m_i \text{ is odd,} \quad R_{ij} = \begin{cases} n_i - m_i, \ j = \frac{m_i+1}{2}, \\ 0, \ other\ wise. \end{cases}$$

The simulation study is carried out while using Algorithm 3.

Algorithm 3 Steps of the simulation study

---

1. Assign the values of $k, \alpha, \theta_1, \gamma, b, n_i, m_i$, and ν_i, $i = 1, 2, ..., k$.
2. Generate according to the number of levels in the test, random samples of size m_i from Uniform$(0, 1)$ distribution, $(U_{i1}, U_{i2}, ..., U_{im_i})$, $i = 1, 2, ..., k$.
3. Assign the values of CS, R_{ij}, $i = 1, 2, ..., k$, and $j = 1, 2, ..., m_i$ such that $\sum_{j=1}^{m_i} R_{ij} = n_i - m_i$.
4. Set $E_{ij} = U_{ij}^{1/(j+\sum_{d=m_i-j+1}^{m_i} R_{id})}$, $j = 1, 2, ..., m_i$, and $i = 1, 2, ..., k$.
5. Obtain $(U^*_{i1}, U^*_{i2}, ..., U^*_{im_i})$, where $U^*_{ij} = 1 - \prod_{d=m_i-j+1}^{m_i} E_{id}$, $j = 1, 2, ..., m_i$, $i = 1, 2, ..., k$.
6. Generate random samples $(t_{i1}, t_{i2}, ..., t_{im_i})$, $i = 1, 2, ..., k$, from (5) as follows:
$$t_{ij} = \left((\theta_1)(b+1)(\frac{\nu_1}{\nu_i})^b \left(-1 + (1 - U^*_{ij})^{\frac{-1}{\alpha}} \right)^{\gamma} \right), \quad j = 1, 2, ..., m_i, \ i = 1, 2, ..., k.$$
7. Use the samples generated in step 6 to obtain the MLEs of the parameters by finding a solution for the non-linear system in ((10)–(13)).
8. Use the samples generated in step 6 to obtain TK Bayes estimates of the parameters from Equation (32).
9. Obtain the BEs using importance sampling method under SE, LINEX, and GE loss functions, according to Algorithm 1.
10. Find the of 95% normal approximate CIs of the parameters from Equation (33).
11. Find the of 95% bootstrap CIs of the parameters, using Algorithm 2.
12. Repeat the above steps from step ((2)–(11)), 1000 times.
13. Find the mean values of the RABs, MSEs, and length of CIs for the model parameters.

Table 4. Mean square errors (MSEs) and relative absolute biases (RABs) for MLEs and BEs of α, θ_1, γ, and b with true values ($\gamma = 1.4$, $\alpha = 0.5$, $\theta_1 = 0.7$ and $b = 1.2$), values of hyper-parameters ($\mu_1 = 196000$, $\mu_2 = 2500$, $\mu_3 = 490$, $\mu_4 = 1440$, $\lambda_1 = 7.1428 \times 10^{-6}$, $\lambda_2 = 0.0002$, $\lambda_3 = 0.0014$, and $\lambda_4 = 0.0008$), $k = 2$, $\nu_1 = 10$ and $\nu_2 = 16$.

n_i	m_i	CS	θ	ML	TK	Important Sampling		
						SE	LINEX	GE
$n_i = \begin{cases} 8 & i=1 \\ 9 & i=2 \end{cases}$	$m_i = \begin{cases} 4 & i=1 \\ 5 & i=2 \end{cases}$	1	α	0.200304	4.37×10^{-7}	0.000112	0.000111	0.000111
				0.232240	0.001010	0.016650	0.016650	0.016650
			θ_1	0.461802	4.27×10^{-6}	0.000985	0.000985	0.000985
				0.343000	0.002450	0.036280	0.036280	0.036280
			β	0.119454	7.40×10^{-8}	9.17×10^{-6}	9.00×10^{-6}	9.00×10^{-6}
				0.070850	0.000020	0.001840	0.001840	0.001840
			b	0.623000	6.83×10^{-6}	0.000588	0.000588	0.000588
				0.295570	0.000210	0.000610	0.000610	0.000610
		2	α	0.058976	1.17×10^{-7}	0.000135	0.000135	0.000135
				0.312900	0.000560	0.018370	0.018370	0.018370
			θ_1	0.162161	3.07×10^{-6}	0.000631	0.000631	0.000631
				0.251000	0.001730	0.029570	0.028600	0.029570
			β	0.230800	4.20×10^{-9}	0.000014	0.000014	0.000014
				0.135000	0.000010	0.002470	0.002340	0.002340
			b	0.210573	6.17×10^{-6}	0.000722	0.000622	0.000733
				0.465570	0.000590	0.017810	0.020540	0.020440
$n_i = \begin{cases} 8 & i=1 \\ 9 & i=2 \end{cases}$	$m_i = \begin{cases} 8 & i=1 \\ 9 & i=2 \end{cases}$		α	0.100304	3.97×10^{-7}	0.000100	0.000001	0.000002
				0.113224	0.001000	0.001870	0.001880	0.001940
			θ_1	0.261802	3.22×10^{-6}	0.000096	0.000116	0.000140
				0.013320	0.002100	0.013430	0.014810	0.016420
			β	0.011945	5.20×10^{-8}	0.000053	0.000890	0.000001
				0.010850	0.000010	0.000170	0.000176	0.000170
			b	0.023400	5.68×10^{-6}	0.000039	0.000029	0.000027
				0.115570	0.000580	0.004730	0.003950	0.003760
$n_i = \begin{cases} 14 & i=1 \\ 15 & i=2 \end{cases}$	$m_i = \begin{cases} 9 & i=1 \\ 9 & i=2 \end{cases}$	1	α	0.100400	8.67×10^{-7}	1.73×10^{-6}	2.00×10^{-6}	2.00×10^{-6}
				0.122400	0.001490	0.002110	0.002410	0.002130
			θ_1	0.218020	6.99×10^{-6}	0.000332	0.000366	0.000411
				0.243000	0.003030	0.025670	0.027030	0.028660
			β	0.114000	1.59×10^{-8}	1.06×10^{-6}	2.00×10^{-6}	2.00×10^{-6}
				0.010850	0.000020	0.000180	0.000180	0.000180
			b	0.323000	8.56×10^{-6}	0.000111	0.000092	0.000088
				0.195570	0.000650	0.008300	0.007480	0.007280
		2	α	0.189759	8.68×10^{-7}	1.73×10^{-6}	2.00×10^{-6}	3.00×10^{-6}
				0.212900	0.001490	0.002120	0.002150	0.018370
			θ_1	0.142000	6.30×10^{-6}	0.000322	0.000357	0.000401
				0.151000	0.002870	0.025310	0.026660	0.028290
			β	0.211000	1.81×10^{-9}	1.10×10^{-7}	1.14×10^{-7}	1.27×10^{-7}
				0.123000	0.000030	0.000190	0.000180	0.000180
			b	0.110573	8.22×10^{-6}	0.000124	0.000104	0.000100
				0.345700	0.000650	0.008840	0.008020	0.007820
$n_i = \begin{cases} 14 & i=1 \\ 15 & i=2 \end{cases}$	$m_i = \begin{cases} 14 & i=1 \\ 15 & i=2 \end{cases}$		α	0.090304	1.78×10^{-7}	1.99×10^{-6}	2.00×10^{-6}	2.00×10^{-6}
				0.102400	0.002080	0.002180	0.002190	0.002260
			θ_1	0.121802	9.63×10^{-6}	0.001540	0.000116	0.001740
				0.010200	0.003440	0.056040	0.057390	0.059110
			β	0.011945	5.20×10^{-9}	1.30×10^{-7}	19.0×10^{-7}	1.00×10^{-7}
				0.010050	0.000030	0.000110	0.000116	0.000110
			b	0.013400	8.90×10^{-6}	0.000615	0.000565	0.000554
				0.105570	0.000638	0.002730	0.003450	0.003260
$n_i = \begin{cases} 50 & i=1 \\ 30 & i=2 \end{cases}$	$m_i = \begin{cases} 35 & i=1 \\ 10 & i=2 \end{cases}$	1	α	0.070030	7.71×10^{-8}	0.000016	0.000021	0.000117
				0.132240	0.000120	0.001430	0.002210	0.004970
			θ_1	0.036180	2.33×10^{-6}	0.000648	0.000648	0.000648
				0.093320	0.000990	0.012800	0.012800	0.012800
			β	0.011945	3.67×10^{-9}	6.72×10^{-6}	7.00×10^{-6}	7.00×10^{-6}
				0.070850	0.000040	0.001600	0.001600	0.001600
			b	0.074542	2.16×10^{-6}	0.000588	0.000587	0.000588
				0.195570	0.001050	0.015910	0.015910	0.015910

Table 4. *Cont.*

n_i	m_i	CS	θ	ML	TK	Important Sampling SE	LINEX	GE
		2	α	0.058976	1.17×10^{-7}	0.000048	5.00×10^{-6}	0.000034
				0.112900	0.000170	0.002690	0.001020	0.002310
			θ_1	0.142161	6.52×10^{-6}	0.000632	0.000632	0.000632
				0.230510	0.001680	0.014820	0.014820	0.014820
			β	0.045211	2.35×10^{-9}	1.47×10^{-6}	1.00×10^{-6}	1.00×10^{-6}
				0.130850	0.000030	0.000820	0.000820	0.000820
			b	0.110573	2.46×10^{-6}	0.000456	0.000456	0.000456
				0.265570	0.001150	0.017810	0.017800	0.017800
$n_i=\begin{cases}50 & i=1\\30 & i=2\end{cases}$	$m_i=\begin{cases}50 & i=1\\30 & i=2\end{cases}$		α	0.064000	4.71×10^{-8}	0.000014	0.000011	0.000127
				0.122240	0.000110	0.001230	0.001210	0.001970
			θ_1	0.026180	1.33×10^{-6}	0.000448	0.000448	0.000548
				0.083320	0.000880	0.011800	0.011100	0.011800
			β	0.011045	2.10×10^{-9}	4.21×10^{-6}	6.00×10^{-6}	6.00×10^{-6}
				0.050850	0.000030	0.001200	0.001300	0.001200
			b	0.056542	1.16×10^{-6}	0.000239	0.000249	0.000234
				0.185570	0.001010	0.014910	0.014210	0.014211
$n_i=\begin{cases}65 & i=1\\45 & i=2\end{cases}$	$m_i=\begin{cases}45 & i=1\\20 & i=2\end{cases}$	1	α	0.059339	1.37×10^{-7}	1.51×10^{-6}	2.00×10^{-6}	8.00×0^{-6}
				0.096560	0.000170	0.000570	0.000550	0.000560
			θ_1	0.081640	0.000012	0.000594	0.000630	0.000631
				0.133570	0.002160	0.015360	0.015910	0.015920
			β	0.017258	8.08×10^{-9}	1.19×10^{-6}	1.00×10^{-6}	1.00×10^{-6}
				0.068490	0.000050	0.000650	0.000650	0.000650
			b	0.051445	5.78×10^{-6}	0.000316	0.000291	0.000286
				0.151740	0.001810	0.014380	0.013770	0.013630
		2	α	0.161424	5.76×10^{-7}	1.15×10^{-6}	1.00×10^{-6}	1.00×0^{-6}
				0.184700	0.000320	0.000520	0.000340	0.000470
			θ_1	0.020008	0.000026	0.000484	0.000529	0.000530
				0.093340	0.002740	0.014620	0.015290	0.015300
			β	0.017739	1.12×10^{-9}	1.73×10^{-6}	1.00×10^{-9}	1.00×10^{-9}
				0.094870	0.000070	0.000250	0.000250	0.000260
			b	0.146713	4.06×10^{-6}	0.000316	0.000291	0.000286
				0.151740	0.001810	0.014380	0.013770	0.013630
$n_i=\begin{cases}65 & i=1\\45 & i=2\end{cases}$	$m_i=\begin{cases}65 & i=1\\45 & i=2\end{cases}$		α	0.110898	2.87×10^{-7}	2.55×10^{-7}	2.10×10^{-7}	1.10×10^{-7}
				0.146290	0.000200	0.000220	0.000210	0.000260
			θ_1	0.129907	0.000012	0.001092	0.001156	0.001158
				0.223740	0.001910	0.022010	0.022650	0.022660
			β	0.007118	7.38×10^{-9}	4.77×10^{-6}	4.00×10^{-7}	4.00×10^{-7}
				0.050270	0.000050	0.000480	0.000480	0.000480
			b	0.057922	2.83×10^{-6}	0.001645	0.001561	0.001543
				0.164010	0.001260	0.033740	0.032860	0.032680
$n_i=\begin{cases}100 & i=1\\60 & i=2\end{cases}$	$m_i=\begin{cases}60 & i=1\\40 & i=2\end{cases}$	1	α	0.136049	$2.36\times10-7$	7.49×10^{-6}	1.00×10^{-6}	1.00×0^{-6}
				0.134770	0.000190	0.000350	0.000360	0.000360
			θ_1	0.131050	0.000014	0.000956	0.001015	0.001017
				0.196560	0.002230	0.020580	0.021230	0.021240
			β	0.017954	8.40×10^{-9}	6.49×10^{-6}	1.00×10^{-6}	1.00×10^{-6}
				0.085750	0.000050	0.000570	0.000570	0.000570
			b	0.052372	4.14×10^{-6}	0.001495	0.001417	0.001401
				0.161230	0.001490	0.032170	0.031320	0.031140
		2	α	0.122402	$2.45\times10-7$	6.95×10^{-7}	1.00×10^{-6}	1.00×0^{-6}
				0.159660	0.000210	0.000300	0.000280	0.000300
				0.198720	0.002290	0.022440	0.023080	0.023100
			θ_1	0.006794	1.45×10^{-8}	5.40×10^{-7}	1.00×10^{-6}	1.00×10^{-6}
			β	0.051130	0.000080	0.000520	0.000530	0.000530
			b	0.052372	3.84×10^{-6}	0.001258	0.001185	0.001170
				0.176690	0.001380	0.029540	0.028670	0.028480
$n_i=\begin{cases}100 & i=1\\60 & i=2\end{cases}$	$m_i=\begin{cases}100 & i=1\\60 & i=2\end{cases}$		α	0.085274	3.51×10^{-7}	4.06×10^{-7}	1.00×10^{-7}	1.10×10^{-7}
				0.117070	0.000260	0.000220	0.000180	0.000170
			θ_1	0.099751	0.000020	0.002664	0.002763	0.002767
				0.183710	0.002390	0.034390	0.035030	0.035050
			β	0.008279	1.46×10^{-9}	1.27×10^{-6}	1.00×10^{-7}	1.00×10^{-7}
				0.051540	0.000070	0.000780	0.000790	0.000790
			b	0.052874	4.58×10^{-6}	0.003914	0.003775	0.003749
				0.171990	0.001280	0.052100	0.051170	0.050990

Table 5. Mean square errors (MSEs) and RABs for MLEs and BEs of α, θ_1, γ, and b with true values ($\gamma = 1.4$, $\alpha = 0.5$, $\theta_1 = 0.7$ and $b = 1.2$), values of hyper-parameters ($\mu_1 = 196000$, $\mu_2 = 2500$, $\mu_3 = 490$, $\mu_4 = 1440$, $\lambda_1 = 7.1428 \times 10^{-6}$, $\lambda_2 = 0.0002$, $\lambda_3 = 0.0014$, and $\lambda_4 = 0.0008$), $k = 4$, $\nu_1 = 10$, $\nu_2 = 16$ $\nu_3 = 20$, $\nu_4 = 25$.

n_i	m_i	CS	θ	ML	TK	Important Sampling		
						SE	LINEX	GE
$n_i = \begin{cases} 5 & i=1 \\ 5 & i=2 \\ 5 & i=3 \\ 5 & i=4 \end{cases}$	$m_i = \begin{cases} 3 & i=1 \\ 3 & i=2 \\ 3 & i=3 \\ 3 & i=4 \end{cases}$	1	α	0.204326	2.32×10^{-6}	1.6208×10^{-4}	2×10^{-4}	2×10^{-4}
				0.211159	0.002440	0.002260	0.002300	0.002370
			θ_1	0.021981	3.49×10^{-6}	0.000368	0.000406	0.000454
				0.210009	0.003310	0.029160	0.031830	0.030040
			β	0.067425	3.60×10^{-9}	1.32×10^{-6}	1.21×10^{-7}	1.23×10^{-7}
				0.185115	0.000050	0.000371	0.000381	0.000371
			b	0.061369	9.44×10^{-7}	0.002951	0.002187	0.003085
				0.174800	0.000801	0.041678	0.042197	0.031158
		2	α	0.000921	3.32×10^{-6}	5.28×10^{-6}	1.00×10^{-6}	1.00×10^{-6}
				0.072126	0.002911	0.002600	0.002630	0.002680
			θ_1	0.000847	4.32×10^{-6}	0.000644	0.000671	0.000625
				0.051600	0.002990	0.044340	0.044260	0.045440
			β	0.028234	1.40×10^{-8}	1.91×10^{-4}	1.21×10^{-4}	1.31×10^{-4}
				0.210180	0.000021	0.000230	0.000229	0.000229
			b	0.000072	6.21×10^{-4}	0.002472	0.002558	0.002156
				0.007158	0.000740	0.035921	0.034931	0.034928
$n_i = \begin{cases} 5 & i=1 \\ 5 & i=2 \\ 5 & i=3 \\ 5 & i=4 \end{cases}$	$m_i = \begin{cases} 5 & i=1 \\ 5 & i=2 \\ 5 & i=3 \\ 5 & i=4 \end{cases}$		α	0.008912	1.51×10^{-6}	7.57×10^{-5}	1.42×10^{-5}	1.00×10^{-5}
				0.217193	0.004312	0.004167	0.004274	0.004219
			θ_1	0.039508	8.85×10^{-6}	0.003178	0.031151	0.003125
				0.321788	0.003123	0.075780	0.065910	0.067004
			β	0.039245	4.92×10^{-10}	5.69×10^{-6}	1.00×10^{-7}	1.00×10^{-7}
				0.212110	0.000051	0.000510	0.000521	0.000540
			b	0.032629	8.32×10^{-5}	0.006486	0.006405	0.005342
				0.132073	0.000770	0.008691	0.007860	0.007660
$n_i = \begin{cases} 7 & i=1 \\ 7 & i=2 \\ 7 & i=3 \\ 7 & i=4 \end{cases}$	$m_i = \begin{cases} 5 & i=1 \\ 5 & i=2 \\ 5 & i=3 \\ 5 & i=4 \end{cases}$	1	α	0.194326	2.10×10^{-6}	1.55×10^{-5}	2.10×10^{-4}	2.12×10^{-4}
				0.191115	0.002310	0.002212	0.002200	0.002170
			θ_1	0.021871	3.20×10^{-6}	0.000378	0.000426	0.000424
				0.220009	0.003411	0.039160	0.032830	0.031040
			β	0.064425	3.20×10^{-9}	1.34×10^{-6}	1.31×10^{-7}	1.00×10^{-7}
				0.215115	0.000051	0.000471	0.000481	0.000337
			b	0.052369	8.20×10^{-7}	0.002951	0.002217	0.002108
				0.194800	0.000701	0.032678	0.034197	0.034158
		2	α	0.000641	3.00×10^{-6}	5.00×10^{-6}	1.10×10^{-6}	1.20×10^{-6}
				0.052146	0.002311	0.002300	0.002330	0.002580
			θ_1	0.000544	4.00×10^{-6}	0.000724	0.000641	0.000715
				0.041600	0.002980	0.044350	0.044160	0.045340
			β	0.029134	1.00×10^{-7}	1.42×10^{-5}	1.21×10^{-5}	1.31×10^{-5}
				0.210280	0.000022	0.000211	0.000239	0.000229
			b	0.000092	6.21×10^{-6}	0.002522	0.002518	0.002157
				0.007218	0.000141	0.021592	0.033231	0.034328
$n_i = \begin{cases} 7 & i=1 \\ 7 & i=2 \\ 7 & i=3 \\ 7 & i=4 \end{cases}$	$m_i = \begin{cases} 7 & i=1 \\ 7 & i=2 \\ 7 & i=3 \\ 7 & i=4 \end{cases}$		α	0.007112	1.00×10^{-6}	7.00×10^{-6}	1.31×10^{-5}	1.00×10^{-5}
				0.197123	0.003312	0.003167	0.003274	0.004319
			θ_1	0.031508	8.85×10^{-6}	0.003078	0.032151	0.003325
				0.221788	0.002123	0.065780	0.015910	0.061004
			β	0.031245	4.00×10^{-9}	5.00×10^{-6}	1.00×10^{-7}	1.00×10^{-7}
				0.182110	0.000041	0.000410	0.000412	0.000414
			b	0.022628	8.00×10^{-5}	0.004486	0.004405	0.004342
				0.131207	0.000670	0.007691	0.006861	0.005661
$n_i = \begin{cases} 25 & i=1 \\ 20 & i=2 \\ 20 & i=3 \\ 20 & i=4 \end{cases}$	$m_i = \begin{cases} 10 & i=1 \\ 10 & i=2 \\ 10 & i=3 \\ 10 & i=4 \end{cases}$	1	α	0.010433	1.62×10^{-6}	1.62×10^{-6}	2.00×10^{-6}	2.00×10^{-6}
				0.171590	0.002440	0.002260	0.002300	0.002370
			θ_1	0.016198	3.49×10^{-6}	0.000368	0.000406	0.000454
				0.167090	0.002310	0.026860	0.028300	0.030040
			β	0.057426	3.60×10^{-9}	1.27×10^{-6}	1.00×10^{-7}	1.00×10^{-7}
				0.145150	0.000040	0.000350	0.000350	0.000350
			b	0.043697	9.44×10^{-7}	0.001951	0.001868	0.001850
				0.144480	0.000790	0.036780	0.035970	0.035810
		2	α	0.000979	2.11×10^{-6}	4.87×10^{-6}	1.00×10^{-6}	1.00×10^{-6}
				0.062600	0.002910	0.001400	0.001630	0.002080
			θ_1	0.000848	3.98×10^{-6}	0.000545	0.000575	0.000615
				0.041600	0.002850	0.033340	0.034260	0.035440
			β	0.028306	1.40×10^{-10}	1.75×10^{-6}	1.00×10^{-7}	1.00×10^{-7}
				0.120180	0.000010	0.000300	0.000290	0.000290
			b	0.000062	5.85×10^{-7}	0.001647	0.001580	0.001566
				0.006580	0.000640	0.033820	0.033130	0.032980

Table 5. *Cont.*

n_i	m_i	CS	θ	ML	TK	Important Sampling SE	LINEX	GE
$n_i = \begin{cases} 25 & i=1 \\ 20 & i=2 \\ 20 & i=3 \\ 20 & i=4 \end{cases}$	$m_i = \begin{cases} 25 & i=1 \\ 20 & i=2 \\ 20 & i=3 \\ 20 & i=4 \end{cases}$		α	0.008938	1.52×10^{-6}	7.57×10^{-7}	1.00×10^{-6}	1.00×10^{-6}
				0.171930	0.002200	0.001670	0.001740	0.001880
			θ_1	0.029508	8.85×10^{-6}	0.002078	0.002151	0.002249
				0.217880	0.003830	0.064780	0.065910	0.067400
			β	0.039245	4.92×10^{-10}	5.68×10^{-6}	1.00×10^{-7}	1.00×10^{-7}
				0.135110	0.000040	0.000400	0.000400	0.000400
			b	0.017628	8.96×10^{-7}	0.005188	0.005049	0.005024
				0.107340	0.000780	0.059090	0.058290	0.058140
$n_i = \begin{cases} 40 & i=1 \\ 30 & i=2 \\ 30 & i=3 \\ 30 & i=4 \end{cases}$	$m_i = \begin{cases} 25 & i=1 \\ 20 & i=2 \\ 20 & i=3 \\ 20 & i=4 \end{cases}$	1	α	0.003908	2.01×10^{-6}	0.000011	0.000011	0.000011
				0.120590	0.002550	0.005610	0.005530	0.005460
			θ_1	0.022139	0.000016	0.001389	0.001456	0.001548
				0.202780	0.004530	0.046120	0.047610	0.049550
			β	0.028294	1.16×10^{-8}	3.19×10^{-6}	3.00×10^{-7}	3.00×10^{-7}
				0.089840	0.000060	0.001010	0.001000	0.001000
			b	0.056264	8.13×10^{-7}	0.005337	0.005218	0.005197
				0.170840	0.000730	0.059590	0.058890	0.058760
		2	α	0.002635	3.33×10^{-6}	4.01×10^{-6}	4.01×10^{-6}	5.00×10^{-6}
				0.080740	0.003140	0.003050	0.003160	0.003390
			θ_1	0.016658	9.89×10^{-6}	0.002737	0.002806	0.002901
				0.130320	0.003810	0.074270	0.075210	0.076520
			β	0.031085	4.62×10^{-9}	6.60×10^{-7}	1.00×10^{-7}	1.00×10^{-7}
				0.119090	0.000050	0.000490	0.000490	0.000490
			b	0.028543	9.39×10^{-7}	0.004388	0.004264	0.004242
				0.130920	0.000730	0.055060	0.054280	0.054130
$n_i = \begin{cases} 40 & i=1 \\ 30 & i=2 \\ 30 & i=3 \\ 30 & i=4 \end{cases}$	$m_i = \begin{cases} 40 & i=1 \\ 30 & i=2 \\ 30 & i=3 \\ 30 & i=4 \end{cases}$		α	0.003945	1.82×10^{-6}	9.92×10^{-7}	1.00×10^{-6}	1.00×10^{-6}
				0.125620	0.002700	0.001990	0.002060	0.002200
			θ_1	0.051662	0.000043	0.008010	0.008105	0.008248
				0.324700	0.009350	0.127860	0.128610	0.129740
			β	0.002229	1.01×10^{-9}	1.05×10^{-7}	1.00×10^{-7}	1.00×10^{-7}
				0.033720	0.000001	0.000730	0.000740	0.000740
			b	0.045313	5.02×10^{-7}	0.010600	0.010347	0.010308
				0.177390	0.000590	0.085800	0.084770	0.084610
$n_i = \begin{cases} 80 & i=1 \\ 60 & i=2 \\ 60 & i=3 \\ 60 & i=4 \end{cases}$	$m_i = \begin{cases} 60 & i=1 \\ 45 & i=2 \\ 45 & i=3 \\ 45 & i=4 \end{cases}$	1	α	0.005745	5.87×10^{-6}	8.29×10^{-6}	8.00×10^{-6}	8.00×10^{-6}
				0.124970	0.003280	0.004920	0.004860	0.004740
			θ_1	0.009045	0.000024	0.017979	0.018045	0.018163
				0.122810	0.005420	0.190050	0.190400	0.191030
			β	0.045596	6.58×10^{-9}	5.43×10^{-6}	5.00×10^{-6}	5.00×10^{-6}
				0.122530	0.000050	0.001340	0.001340	0.001340
			b	0.047806	2.16×10^{-6}	0.019736	0.019666	0.019657
				0.117930	0.001140	0.114960	0.114740	0.114710
		2	α	0.002655	8.71×10^{-6}	0.000023	0.000024	0.000025
				0.093450	0.004660	0.007180	0.007280	0.007470
			θ_1	0.018653	0.000039	0.022114	0.022194	0.022340
				0.170830	0.006340	0.211600	0.211990	0.212710
			β	0.006572	7.98×10^{-9}	1.88×10^{-6}	2.00×10^{-6}	2.00×10^{-6}
				0.047190	0.000060	0.000690	0.000680	0.000680
			b	0.041606	2.21×10^{-6}	0.016220	0.016126	0.016114
				0.151820	0.001130	0.105700	0.105400	0.105360
$n_i = \begin{cases} 80 & i=1 \\ 60 & i=2 \\ 60 & i=3 \\ 60 & i=4 \end{cases}$	$m_i = \begin{cases} 80 & i=1 \\ 60 & i=2 \\ 60 & i=3 \\ 60 & i=4 \end{cases}$		α	0.002446	2.01×10^{-6}	0.000062	0.000062	0.000062
				0.092370	0.002430	0.012940	0.012900	0.012840
			θ_1	0.020805	5.39×10^{-6}	0.019193	0.019251	0.019352
				0.157070	0.003020	0.194610	0.194930	0.195470
			β	0.028698	9.49×10^{-9}	2.28×10^{-6}	2.00×10^{-6}	2.00×10^{-6}
				0.098050	0.000050	0.000900	0.000910	0.000910
			b	0.054623	1.94×10^{-6}	0.021122	0.020898	0.020873
				0.073830	0.001140	0.166880	0.166260	0.166220

Table 6. Lengths and coverage probabilities of 95% approximate and bootstrap CIs of α, θ_1, γ and b with true values ($\gamma = 1.4$, $\alpha = 0.5$, $\theta_1 = 0.7$ and $b = 1.2$), $k = 2$, $\nu_1 = 10$, and $\nu_2 = 16$.

n_i	m_i	CS	θ	Approximate CI		Bootstrap CI	
				Length	Coverage Probability	Length	Coverage Probability
$n_i = \begin{cases} 8 & i=1 \\ 9 & i=2 \end{cases}$	$m_i = \begin{cases} 4 & i=1 \\ 5 & i=2 \end{cases}$	1	α	3.895	0.827	1.12	0.700
			θ_1	1.436	0.780	0.903	0.875
			β	1.469	0.870	0.679	0.756
			b	2.480	0.780	0.869	0.788
		2	α	3.528	0.820	0.982	0.661
			θ_1	1.427	0.840	0.998	0.712
			β	1.403	0.891	0.678	0.870
			b	2.600	0.890	0.760	0.670
$n_i = \begin{cases} 8 & i=1 \\ 9 & i=2 \end{cases}$	$m_i = \begin{cases} 8 & i=1 \\ 9 & i=2 \end{cases}$		α	3.822	0.891	1.111	0.890
			θ_1	1.120	0.880	0.991	0.955
			β	1.269	0.890	0.659	0.796
			b	2.100	0.878	0.819	0.912
$n_i = \begin{cases} 14 & i=1 \\ 15 & i=2 \end{cases}$	$m_i = \begin{cases} 9 & i=1 \\ 9 & i=2 \end{cases}$	1	α	2.891	0.910	1.101	0.841
			θ_1	1.236	0.880	0.802	0.885
			β	1.1698	0.890	0.590	0.816
			b	2.280	0.940	0.789	0.818
		2	α	2.800	0.860	0.821	0.861
			θ_1	1.427	0.840	0.998	0.712
			β	1.410	0.811	0.547	0.810
			b	2.100	0.880	0.560	0.801
$n_i = \begin{cases} 14 & i=1 \\ 15 & i=2 \end{cases}$	$m_i = \begin{cases} 14 & i=1 \\ 15 & i=2 \end{cases}$		α	2.910	0.901	0.990	0.910
			θ_1	0.960	0.890	0.810	0.960
			β	1.026	0.930	0.830	0.830
			b	1.980	0.780	0.719	0.921
$n_i = \begin{cases} 50 & i=1 \\ 35 & i=2 \end{cases}$	$m_i = \begin{cases} 20 & i=1 \\ 10 & i=2 \end{cases}$	1	α	2.940	0.929	0.942	1
			θ_1	1.378	0.880	0.883	0.975
			β	1.369	0.950	0.579	0.990
			b	1.969	0.900	0.690	0.980
		2	α	2.245	0.920	0.892	1
			θ_1	1.2175	0.900	0.856	0.900
			β	1.209	0.950	0.489	0.990
			b	2.208	1	0.68	0.98
$n_i = \begin{cases} 50 & i=1 \\ 35 & i=2 \end{cases}$	$m_i = \begin{cases} 50 & i=1 \\ 35 & i=2 \end{cases}$		α	2.045	0.990	0.792	1
			θ_1	1.110	0.990	0.756	0.980
			β	1.009	0.890	0.970	0.990
			b	2.008	0.880	0.610	0.990
$n_i = \begin{cases} 65 & i=1 \\ 45 & i=2 \end{cases}$	$m_i = \begin{cases} 45 & i=1 \\ 20 & i=2 \end{cases}$	1	α	1.627	1	0.952	1
			θ_1	1.126	1	0.842	1
			β	0.955	1	0.470	1
			b	2.175	1	0.676	1
		2	α	2.625	1	1.043	1
			θ_1	1.420	1	0.796	1
			β	1.597	1	0.484	1
			b	1.731	1	0.651	1
$n_i = \begin{cases} 65 & i=1 \\ 45 & i=2 \end{cases}$	$m_i = \begin{cases} 65 & i=1 \\ 45 & i=2 \end{cases}$		α	0.772	0.900	0.913	1
			θ_1	0.681	1	0.839	1
			β	0.766	1	0.462	1
			b	1.829	1	0.674	1
$n_i = \begin{cases} 100 & i=1 \\ 60 & i=2 \end{cases}$	$m_i = \begin{cases} 60 & i=1 \\ 40 & i=2 \end{cases}$	1	α	0.874	0.900	1.030	1
			θ_1	0.663	1	0.865	1
			β	0.721	1	0.485	1
			b	1.825	1	0.67491	1
		2	α	2.003	1	0.926	1
			θ_1	1.153	1	0.846	1
			β	0.895	1	0.438	1
			b	2.122	1	0.690	1
$n_i = \begin{cases} 100 & i=1 \\ 60 & i=2 \end{cases}$	$m_i = \begin{cases} 100 & i=1 \\ 60 & i=2 \end{cases}$		α	0.635	0.900	0.950	1
			θ_1	0.572	0.900	0.842	1
			β	0.636	1	0.3878	1
			b	1.402	1	0.684	1

Table 7. Lengths and coverage probabilities of 95% approximate and bootstrap CIs of α, θ_1, γ and b with true values ($\gamma = 1.4$, $\alpha = 0.5$, $\theta_1 = 0.7$ and $b = 1.2$), $k = 4$, $\nu_1 = 10$, $\nu_2 = 16$, $\nu_3 = 20$ and $\nu_4 = 25$.

n_i	m_i	CS	θ	Approximate CI		Bootstrap CI	
				Length	Coverage Probability	Length	Coverage Probability
$n_i =$ 5 $i=1$	$m_i =$ 3 $i=1$	1	α	0.632	0.900	0.2681	0.920
5 $i=2$	3 $i=2$		θ_1	2.849	0.911	0.416	0.913
5 $i=3$	3 $i=3$		β	1.261	0.921	0.801	0.711
5 $i=4$	3 $i=4$		b	4.891	0.7140	0.713	0.816
		2	α	0.718	0.808	0.292	0.841
			θ_1	1.927	0.910	0.312	0.970
			β	1.328	0.900	0.617	0.910
			b	3.922	0.820	0.718	0.990
$n_i =$ 5 $i=1$	$m_i =$ 5 $i=1$		α	0.612	0.951	0.240	0.912
5 $i=2$	5 $i=2$		θ_1	2.517	0.720	0.389	0.912
5 $i=3$	5 $i=3$		β	0.921	1	0.611	1
5 $i=4$	5 $i=4$		b	3.219	1	0.718	1
$n_i =$ 7 $i=1$	$m_i =$ 5 $i=1$	1	α	0.532	0.910	0.241	0.970
7 $i=2$	5 $i=2$		θ_1	2.749	0.920	0.316	0.930
7 $i=3$	5 $i=3$		β	1.211	0.981	0.6201	0.911
7 $i=4$	5 $i=4$		b	4.291	0.8140	0.813	0.816
		2	α	0.718	0.808	0.292	0.841
			θ_1	1.927	0.910	0.312	0.970
			β	1.328	0.900	0.617	0.910
			b	3.922	0.820	0.718	0.990
$n_i =$ 7 $i=1$	$m_i =$ 7 $i=1$		α	0.412	0.961	0.210	0.922
7 $i=2$	7 $i=2$		θ_1	2.417	0.820	0.319	0.912
7 $i=3$	7 $i=3$		β	0.911	1	0.611	1
7 $i=4$	7 $i=4$		b	3.119	1	0.618	1
$n_i =$ 25 $i=1$	$m_i =$ 10 $i=1$	1	α	0.722	0.950	0.242	0.980
20 $i=2$	10 $i=2$		θ_1	2.489	1	0.391	1
20 $i=3$	10 $i=3$		β	1.091	1	0.708	1
20 $i=4$	10 $i=4$		b	4.854	0.940	0.683	0.990
		2	α	0.742	0.88	0.272	0.99
			θ_1	1.726	0.940	0.362	0.970
			β	1.188	0.900	0.567	0.940
			b	3.872	0.920	0.658	0.990
$n_i =$ 25 $i=1$	$m_i =$ 25 $i=1$		α	0.777	1	0.280	1
20 $i=2$	20 $i=2$		θ_1	2.216	0.610	0.381	1
20 $i=3$	20 $i=3$		β	0.991	1	0.581	1
20 $i=4$	20 $i=4$		b	3.309	1	0.658	1
$n_i =$ 40 $i=1$	$m_i =$ 25 $i=1$	1	α	0.778	0.860	0.280	0.960
30 $i=2$	20 $i=2$		θ_1	2.232	0.820	0.397	0.983
30 $i=3$	20 $i=3$		β	0.893	0.880	0.586	0.990
30 $i=4$	20 $i=4$		b	3.286	0.670	0.674	0.987
		2	α	0.764	0.880	0.238	0.970
			θ_1	2.122	0.920	0.392	0.900
			β	0.853	0.838	0.581	0.929
			b	3.218	0.880	0.674	0.987
$n_i =$ 40 $i=1$	$m_i =$ 40 $i=1$		α	0.778	1	0.280	1
30 $i=2$	30 $i=2$		θ_1	2.216	1	0.381	1
30 $i=3$	30 $i=3$		β	0.919	1	0.581	1
30 $i=4$	30 $i=4$		b	2.900	1	0.658	1
$n_i =$ 80 $i=1$	$m_i =$ 60 $i=1$		α	0.410	0.910	0.238	0.975
60 $i=2$	45 $i=2$	1	θ_1	1.181	1	0.394	1
60 $i=3$	45 $i=3$		β	0.648	1	0.441	1
60 $i=4$	45 $i=4$		b	2.437	1	0.666	1
			α	0.393	1	0.200	1
		2	θ_1	1.146	1	0.398	1
			β	0.588	0.800	0.444	1
			b	2.052	1	0.690	1
$n_i =$ 80 $i=1$	$m_i =$ 80 $i=1$		α	0.382	1	0.200	1
60 $i=2$	60 $i=2$		θ_1	1.210	1	0.364	1
60 $i=3$	60 $i=3$		β	0.615	1	0.416	1
60 $i=4$	60 $i=4$		b	2.054	1	0.655	1

From the results in Tables 4–7, we concluded that

1. The BEs of the model parameters give more accurate results than MLEs according to MSEs and RABs.
2. The TK method gives more accurate results than the importance sampling method in most cases.
3. With respect to the importance sampling method, in most of the iteration LINEX loss function had the best values when compared to SE and GE loss functions.
4. The length of bootstrap CIs is smaller than the corresponding approximate CIs.
5. The coverage probability of bootstrap CIs of the four parameters is greater than the corresponding probability of approximate CIs.

8. Conclusions

This study addressed statistical inference that was based on progressively type-II censored data from Pareto-IV distribution under progressive-stress accelerated life test. The statistical inference that was covered in this paper includes: estimating the unknown parameters of Pareto-IV distribution by several methods, including the classical method and Bayesian methods. In this context, two methods were used to find Bayes estimates, namely the TK approximation method and importance sampling method. A comprehensive simulation study was conducted to compare different estimation methods; from them, we concluded that Bayes' estimation methods are more accurate than the classical method. Furthermore, TK method is more accurate than importance sampling method according to MSEs and RABs. Regarding to the interval estimation, the normal approximate and bootstrap confidence intervals were obtained. We have deduced from the simulation study that the length of bootstrap CIs is smaller than approximate CIs. Moreover, the coverage probability of bootstrap CIs is greater than the corresponding approximate CIs. We did not rely only on simulation studies, but also analyzed real data from ramp-voltage experiment of miniature light bulbs in order to clarify the different estimation methods.

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Abbreviations

The following abbreviations are used in this manuscript:

ALT	Accelerated life test
MLE	Maximum likelihood estimation
BE	Bayesian estimation
TK	Tierney and Kadane
CI	Confidence interval
Pareto-IV	Pareto distribution of type IV
CS	Censoring scheme
CDF	Cumulative distribution function
PDF	Probability density function
SE	Squared error
LINEX	Linear exponential
GE	Generalized entropy
K-S	Kolmogorov-Smirnov
MSE	Mean square error
RAB	Relative absolute bias

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