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Prime Geodesic Theorems for Compact Locally Symmetric Spaces of Real Rank One

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Abstract: Our basic objects will be compact, even-dimensional, locally symmetric Riemannian manifolds with strictly negative sectional curvature. The goal of the present paper is to investigate the prime geodesic theorems that are associated with this class of spaces. First, following classical Randol's appraoch in the compact Riemann surface case, we improve the error term in the corresponding result. Second, we reduce the exponent in the newly acquired remainder by using the Gallagher–Koyama techniques. In particular, we improve DeGeorge's bound $O\left(x^{\eta}\right)$, $2\rho-\frac{\rho}{n}\leq\eta<2\rho$ up to $O\left(x^{2\rho-\frac{\rho}{n}}\left(\log x\right)^{-1}\right)$, and reduce the exponent $2\rho-\frac{\rho}{n}$ replacing it by $2\rho-\rho\frac{4n+1}{4n^2+1}$ outside a set of finite logarithmic measure. As usual, n denotes the dimension of the underlying locally symmetric space, and ρ is the half-sum of the positive roots. The obtained prime geodesic theorem coincides with the best known results proved for compact Riemann surfaces, hyperbolic three-manifolds, and real hyperbolic manifolds with cusps.

Keywords: prime geodesic theorem; Selberg and Ruelle zeta functions; locally symmetric spaces; logarithmic measure

MSC: 11M36; 11F72; 58J50

1. Introduction

Let $Y=\Gamma\setminus G\ /\ K=\Gamma\setminus X$ be a compact, n-dimensional (n even), locally symmetric Riemannian manifold with strictly negative sectional curvature, where G is a connected semisimple Lie group of a real rank one, K is a maximal compact subgroup of G, and Γ is a discrete cocompact torsion-free subgroup of G.

Following [1] (p. 17), we require G to be linear in order to have the possibility of complexification.

We assume that the Riemannian metric over Y induced from the Killing form is normalized, so that the sectional curvature of Y varies between -4 and -1.

The universal covering X of Y is a Riemannian symmetric space of rank one and, hence, is known to be either a real $H\mathbb{R}^k$ or a complex $H\mathbb{C}^m$, or a quaternionic hyperbolic space $H\mathbb{H}^m$, or the hyperbolic Cayley plane $H\mathbb{C}a^2$.

Hence, $n=k$, $2m$, $4m$, and 16, respectively.

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, and $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$. Fix a system of positive roots $\Phi^{+}\left(\mathfrak{g},\mathfrak{a}\right)$, and put $\mathfrak{n}=\sum\limits_{\alpha\in\Phi^{+}\left(\mathfrak{g},\mathfrak{a}\right)}\mathfrak{n}_{\alpha}$ to be the sum of the root spaces. Define $\rho=\frac{1}{2}\sum\limits_{\alpha\in\Phi^{+}\left(\mathfrak{g},\mathfrak{a}\right)}\dim\left(\mathfrak{n}_{\alpha}\right)\alpha$.

Now, if $n=k$, $2m$, $4m$, and 16, then $\rho=\frac{1}{2}\left(k-1\right)$, m, $2m+1$, 11, respectively.

By $\pi_{\Gamma}\left(x\right)$, we denote the number of prime geodesics on Y of length not larger than $\log x$.

As it is known, the prime geodesic theorem gives a growth asymptotic for the function $\pi_\Gamma(x)$. Moreover, the statement regarding the number $\pi_\Gamma(x)$, as $x \to \infty$, $x \notin E$, where E is a set of finite logarithmic measure, is known as the Gallagherian prime geodesic theorem. Usually, the Gallagherian prime geodesic theorem improves the corresponding classical result at the cost of excluding a set of finite logarithmic measure. In this research, we are interested in both kinds of theorems.

In literature, the prime geodesic theorem appear in two forms: refined and non-refined. While, in the refined form, the counting function $\pi_\Gamma(x)$ is represented as a sum of two parts: the explicit part, and some carefully derived remainder, in the non-refined form $\pi_\Gamma(x)$ is given as an asymptotic estimate without the error terms.

DeGeorge [2] obtained the best known estimate of the error term in the prime geodesic theorem in our setting in 1977. Thus, DeGeorge's result is given in the refined form and states that there is a constant η, such that $2\rho - \frac{\rho}{n} \leq \eta < 2\rho$, and (see, Theorem 1 and Remark 2 in [2] (pp. 135–136)):

$$\pi_\Gamma(x) = \int_1^{\log x} \frac{e^{2\rho t}}{t} dt + O(x^\eta) \tag{1}$$

as $x \to \infty$. Clearly, the optimal error term in (1) is $O\left(x^{2\rho - \frac{\rho}{n}}\right)$. Here, as earlier, $n = k, 2m, 4m, 16$, and $\rho = \frac{1}{2}(k-1), m, 2m+1, 11$, respectively. The main purpose of this research is to improve the bound $O(x^\eta)$ in (1) up to $O\left(x^{2\rho - \frac{\rho}{n}} (\log x)^{-1}\right)$ in the classical sense, and then prove that the exponent $2\rho - \frac{\rho}{n}$ of x in the newly acquired bound $O\left(x^{2\rho - \frac{\rho}{n}} (\log x)^{-1}\right)$ can be reduced in the Gallagherian sense and, hence, replaced by $2\rho - \rho\frac{4n+1}{4n^2+1}$ (see, Theorems 2 and 3 below).

To put our research into historical context, let us recall the following related results. Gangolli [3] (see, Theorem 4.4, and page 423), proved the non-refined prime geodesic theorem when Y is compact (also see, [4] (p. 89)):

$$\pi_\Gamma(x) \sim \frac{x^{n-1}}{(n-1)\log x}, \tag{2}$$

where $f(x) \sim g(x)$ means that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$. The relation (2) was also proved by Gangolli-Warner [5] (p. 40, Prop. 5.4) when Y is not necessarily compact but has a finite volume. In (2), $n = k, 2m, 4m, 16$, respectively, as before. It is easy to see that the prime geodesic theorem (1) is a refinement of (2) when Y is compact. The first refinement of the corresponding result (2) of Gangolli–Warner (hence, for Y non-compact), for k-dimensional real hyperbolic manifolds with cusps ($n = k$, $\rho = \frac{1}{2}(k-1)$), was achieved by Park [4] (p. 91, Th. 1.2). It states that:

$$\pi_\Gamma(x) = \sum_{\frac{3}{2}\rho < s_i(j) \leq 2\rho} (-1)^j \operatorname{li}\left(x^{s_i(j)}\right) + O\left(x^{\frac{3}{2}\rho} (\log x)^{-\frac{1}{2}}\right) \tag{3}$$

as $x \to \infty$, where $(s_i(j) - j)(2\rho - j - s_i(j))$ is a small eigenvalue in $\left[0, \frac{3}{4}\rho^2\right]$ of Δ_j on $\pi_{\sigma_j, \lambda_i(j)}$ with $s_i(j) = \rho + \mathrm{i}\lambda_i(j)$ or $s_i(j) = \rho - \mathrm{i}\lambda_i(j)$ in $\left(\frac{3}{2}\rho, 2\rho\right]$, Δ_j is the Laplacian acting on the space of j-forms over Y, and $\pi_{\sigma_j, \lambda_i(j)}$ is the principal series representation. The result (3) was further improved by Avdispahić-Gušić in [6] (p. 367, Th. 1), where the authors derived a variant of (3) with the error term $O\left(x^{\frac{3}{2}\rho} (\log x)^{-1}\right)$. As explained in [7], the correct size of the error term in [4] resp. [6] is $O\left(x^{\frac{4\rho^2+\rho}{2\rho+1}} (\log x)^{-\frac{1}{2}}\right)$ resp. $O\left(x^{\frac{4\rho^2+\rho}{2\rho+1}} (\log x)^{-1}\right)$. The omission was present in [4] and, thus, inherited in [6] because of the missing term $O\left(x^{2\rho-1}h\right)$ obtained during reduction from the level of $k-1$ times integrated Chebyshev function $\psi_{2\rho}(x)$ to $\psi_0(x)$ (see, [4] (p. 101, (3.21)) and [6] (p. 370, (7))). Finally, the bound $O\left(x^{\frac{4\rho^2+\rho}{2\rho+1}} (\log x)^{-1}\right)$ obtained in [7] is additionally improved in [8] in the Gallagherian sense, where the authors proved that the exponent $\frac{4\rho^2+\rho}{2\rho+1}$ can be replaced by $(k-1)\left(1-\frac{2k+1}{4k^2+2}\right)$ outside a set of finite logarithmic measure. The investigations that were conducted in [8] were undoubtedly inspired by the recent research of Koyama [9] in the case of compact hyperbolic surfaces and the generic hyperbolic surfaces of finite volume. The ingredients applied in [9] come from the results of Hejhal [10,11], Iwaniec [12], and Gallagher [13], where the author in [13] (under assuming the Rimemann hypothesis) improved the error term in the prime number theorem from $\psi(x) = x + O\left(x^{\frac{1}{2}}(\log x)^2\right)$ to $\psi(x) = x + O\left(x^{\frac{1}{2}}(\log\log x)^2\right)$ outside a set of finite logarithmic measure with the Chebyshev counting function $\psi(x)$ defined over powers of primes by $\psi(x) = \sum_{p^k \le x} \log p$. Hejhal, in his comprehensive treatise [10,11], studied the Selberg zeta function over a hyperbolic Riemann surface Y ($n = k = 2$, $\rho = \frac{1}{2}(k-1) = \frac{1}{2}$), which is, when Γ is cocompact (Y compact) and cofinite (Y non-compact) discrete subgroup of $G = PSL(2,\mathbb{R})$, respectively. His prime geodesic theorem comes with the error terms and states that (also see [14–16]):

$$\pi_\Gamma(x) = \sum_{\frac{3}{4} < s_i \le 1} \mathrm{li}\left(x^{s_i}\right) + O\left(x^{\frac{3}{4}}(\log x)^{-\frac{1}{2}}\right) \tag{4}$$

as $x \to \infty$, where $\lambda_i = s_i(1-s_i)$ is a small eigenvalue in $\left[0, \frac{3}{16}\right]$ of the Laplacian Δ_0 acting on $L^2(Y)$. The prime geodesic theorem (4) refines the corresponding result (2) for cocompact and cofinite $\Gamma \subseteq PSL(2,\mathbb{R})$ in the same way the prime geodesic theorem (1) refines (2) when Y is compact. The best estimate up to now of the error term in a variant of the prime geodesic theorem (4) for compact Riemann surfaces is $O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$, and it is achieved by Randol [17] (see also, [18]). An important ingredient, which is implicitly applied in [17] and explicitly in [4], is the Ruelle zeta function. The bound $O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$ is also achieved in the case of prime geodesic theorem derived for compact symmetric spaces formed as quotients of the Lie group $SL_4(\mathbb{R})$, which is, when Y is locally symmetric space $\Gamma\backslash G/K$, where $G = SL_4(\mathbb{R})$, K is the maximal compact subgroup of G, and Γ is a discrete cocompact subgroup of G (see, [19]). By Theorem 4.4.1 in [19] (p. 197):

$$\pi_\Gamma(x) = 2\,\mathrm{li}(x) + O\left(x^{\frac{3}{4}}(\log x)^{-1}\right) \tag{5}$$

as $x \to \infty$, where $\pi_\Gamma(x) = \sum_{\substack{[\gamma] \in \mathcal{E}_P^p(\Gamma) \\ e^{l_\gamma} \le x}} \chi_1(\Gamma_\gamma)$ is the first higher Euler characteristic of the centraliser Γ_γ of γ in Γ, l_γ is the length of γ, P is a parabolic subgroup of G, and $\mathcal{E}_P^p(\Gamma) \subset \mathcal{E}_P(\Gamma)$ is the subset of primitive classes, where $\mathcal{E}_P(\Gamma)$ is the set of all conjugacy classes $[\gamma]$ in Γ, such that γ is conjugate in G to an element of A^-B, with A^- the negative Weyl chamber in

$$A = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} : a > 0 \right\}$$

and

$$B = \begin{pmatrix} SO(2) & \\ & SO(2) \end{pmatrix}.$$

Deitmar obtained an analogous result of the result (5) in [20] in the case of complex cubic fields, extending, in that way, the work of Sarnak [21] in the real quadratic case. The research has been extended to a noncompact situation in [22], while the full higher rank case has been explored in [23,24] (see also, [25]). As it is known, the Selberg zeta function for compact or generic hyperbolic surfaces satisfies an analogue of the Riemann hypothesis. This fact raises the expectation that the exponent $\frac{3}{4}$ of x in (4) could be decreased to $\frac{1}{2}$. However, the quantity of zeros of the Selberg zeta function causes major obstacles in achieving such result. Thus, the aforementioned $\frac{3}{4}$ was only successfully reduced in the case of modular surfaces. In particular, Iwaniec [12] obtained $\frac{35}{48}+\varepsilon$, Luo and Sarnak [26] $\frac{7}{10}+\varepsilon$, Cai [27] $\frac{71}{102}+\varepsilon$, and Soundararajan and Young [28] $\frac{25}{36}+\varepsilon$. Finally, if $\Gamma \subset PSL(2,\mathbb{C})$ is a cocompact group or a noncompact cofinite group satisfying the condition $\sum\limits_{\gamma_i>0} \frac{x^{\beta_i-1}}{\gamma_i^2} = O\left(\frac{1}{1+(\log x)^3}\right)$, as $x \to \infty$, where $\beta_i + \mathrm{i}\,\gamma_i$ are poles of the corresponding scattering determinant, then the following prime geodesic theorem for hyperbolic 3-manifolds holds true (see, [7] (p. 691, Th. 1.1)):

$$\pi_\Gamma(x) = \mathrm{li}\left(x^2\right) + \sum_{j=1}^{M} \mathrm{li}\left(x^{s_j}\right) + O\left(x^{\frac{5}{3}}(\log x)^{-1}\right) \tag{6}$$

as $x \to \infty$, where s_1, s_2, ..., s_M are the real zeros of the attached Selberg zeta function lying in the interval $(1,2)$.

The motivation to work within the described setting, i.e., with compact, even-dimensional, locally symmetric Riemannian manifolds of strictly negative sectional curvature, stems from the author's desire to improve the best known error term $O\left(x^{2\rho-\frac{\rho}{n}}\right)$ in the corresponding prime geodesic theorem (1), dating back to 1977 up to $O\left(x^{2\rho-\frac{\rho}{n}}(\log x)^{-1}\right)$, and the wish to further reduce the exponent $2\rho-\frac{\rho}{n}$ of x in the Gallagherian sense, which is, the wish to replace $2\rho-\frac{\rho}{n}$ with better, smaller one $2\rho-\rho\frac{4n+1}{4n^2+1}$ outside a set of finite logarithmic measure. Additionally, the fact that it was an improvement of a more than forty-year-old result was quite motivating for the author.

Regarding the techniques that were applied in the proofs of our results, we want to point out that the proofs of Theorems 1 and 2 are inspired by Randol's 1977 approach in the case of compact Riemann surfaces [17] (pp. 245–246), and that the proof of Theorem 3 relies on the 1980 method developed by Gallagher and applied in the classical case on prime number theorem [13]. Hence, the mentioned techniques are not new, and are already known in literature. However, it must be noted that new techniques are nothey t invented so often in this area of research, and that the ones given above have been exploited many times since 1977 and 1980. In particular, in addition to 1977, Randol's method was successfully applied in 2002 in the proof of prime geodesic theorem for complex cubic fields [20] (p. 165), and then again, in 2006 and 2008 for the same reason, in the case of compact symmetric spaces formed as quotients of the Lie group $SL_4(\mathbb{R})$ (see, [29] (pp. 62–65), [19] (p. 197)). Furthermore, it was applied in 2012 in the proof of prime geodesic theorem for real hyperbolic manifolds with cusps [6] (p. 370) (also see, [22]), etc. On the other hand, besides 1980, Gallagher's technique was re-updated by Koyama in 2016 in the proof of the corresponding Gallagherian-style prime geodesic theorem derived for compact hyperbolic surfaces and generic hyperbolic surfaces of finite volume [9] (p. 78, Th. 2). Thereafter, the technique was re-applied to obtain the following improved results: first, in 2018, in Koyama's own setting [30], then, in 2018, in the case of $PSL(2,\mathbb{Z})$ [31], once again in 2018 in the case of hyperbolic 3-manifolds [7] (p. 691, Th. 1.2), one more time in 2020 in the case of real hyperbolic manifolds with cusps [8] (p. 3021, Th. 2), etc. Summarizing what is said above, we may emphasize that the methods that were applied in this research, although old, are still not obsolete, and represent valuable and unavoidable tool in achieving more refined error terms in prime geodesic theorems for various types of underlying locally symmetric spaces. Accordingly, once again, the techniques are not new, but the results are, and the results are all that we are interested in. Like most of similar pure mathematics researches: [2–17,27,28,30–36], etc., the present research has no direct application. In fact, it is a typical example of research in the field of pure, theoretical mathematics. So, one could hardly expect to obtain some immediate application. Finally, regarding the author's additional motivation to consider this subject, we recall that, in the Concluding Remark of [17] (p. 246), Randol noted that it would be interesting to determine the extent to which his methods are applicable for more general spaces. Accordingly, in the same way Theorem 1 in [6] (see, pages 367 and 371) represents the answer to this query in the case of real hyperbolic manifolds with cusps, now, Theorems 1 and 2 can be interpreted as the answer to the same query in the case at hand.

2. Preliminary Material

We introduce the notation following [1] (see also, [37,38]).

Because $\Gamma \subset G$ is cocompact and torsion-free, there are only two types of conjugacy classes: the class of the identity $1 \in \Gamma$ and classes of hyperbolic elements. Let $C\Gamma$ be the set of all conjugacy classes $[\gamma]$ in Γ. To simplify the notation, we shall write γ for an element of $C\Gamma$, and γ_0 for a primitive element. Thus, if γ and γ_0 occur in the same formula, it is understood that γ_0 will be the primitive element underlying γ.

Denote, by M, the centralizer of $\mathfrak{a}$ in K with the Lie algebra $\mathfrak{m}$.

Let $i^* : R(K) \to R(M)$ be the restriction map that is induced by the embedding $i : M \hookrightarrow K$, where $R(K)$ and $R(M)$ are the representation rings over $\mathbb{Z}$ of K and M, respectively (see, [1] (p. 19)).

Suppose that $\sigma \in \hat{M}$, where $\hat{M}$ is the unitary dual of the Lie group M.

Following [1] (p. 40), we choose a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m}$. Subsequently, $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We choose a positive root system $\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ having the property that, for $\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$, $\alpha_{|\mathfrak{a}} \in \Phi^+(\mathfrak{g}, \mathfrak{a})$ implies $\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. Let $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})} \alpha$, and set $\rho_{\mathfrak{m}} = \delta - \rho$. Define the root vector $H_\alpha \in \mathfrak{a}$ for $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$ by $\lambda(H_\alpha) = \frac{(\lambda, \alpha)}{(\alpha, \alpha)}$ for all $\lambda \in \mathfrak{a}^*$. We also define $\varepsilon_\alpha(\sigma) \in \left\{0, \frac{1}{2}\right\}$ for $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$ by $e^{2\pi \mathrm{i}\, \varepsilon_\alpha(\sigma)} = \sigma(\exp(2\pi \mathrm{i}\, H_\alpha)) \in \{\pm 1\}$.

The root system $\Phi^+(\mathfrak{g}, \mathfrak{a})$ is of the form $\Phi^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}$ or $\Phi^+(\mathfrak{g}, \mathfrak{a}) = \left\{\frac{\alpha}{2}, \alpha\right\}$, where α is the long root (see, [1] (p. 47)). We set $T = |\alpha|$, and define $\epsilon_\sigma \in \left\{0, \frac{1}{2}\right\}$ by $\epsilon_\sigma \equiv \frac{|\rho|}{T} + \varepsilon_\alpha(\sigma) \bmod \mathbb{Z}$. We define the lattice $L(\sigma)$ by $L(\sigma) = T(\epsilon_\sigma + \mathbb{Z})$, and the polynomial $P_\sigma(\lambda)$ by $P_\sigma(\lambda) = \prod_{\beta \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})} \frac{(\lambda + \mu_\sigma + \rho_{\mathfrak{m}}, \beta)}{(\delta, \beta)}$, where μ_σ is the highest weight of the representation σ of M.

Suppose that $\chi \in \hat{\Gamma}$, where $\hat{\Gamma}$ is the unitary dual of Γ.

By Proposition 1.2 in [1] (p. 23), we find an element $\gamma \in R(K)$, such that $i^*(\gamma) = \sigma$ (see also, [1] (p. 27)).

Following [1] (p. 30), we define for $s \in \mathbb{C}$ the multiplicities $m_\chi(s, \gamma, \sigma)$ and $m_d(s, \gamma, \sigma)$ by $m_\chi(s, \gamma, \sigma) = \mathrm{Tr}\, E_{A_{Y,\chi}(\gamma,\sigma)}(\{s\})$ and $m_d(s, \gamma, \sigma) = \mathrm{Tr}\, E_{A_d(\gamma,\sigma)}(\{s\})$, respectively, where $E_A(.)$ is the family of spectral projections of a normal operator A. The multiplicities $m_\chi(s, \gamma, \sigma)$ and $m_d(s, \gamma, \sigma)$ are weighted dimensions of eigenspaces of the operators $A_{Y,\chi}(\gamma, \sigma)$ and $A_d(\gamma, \sigma)$ introduced in [1] (p. 28).

By Definition 1.17 in [1] (p. 49), $\gamma \in R(K)$ is called σ-admissible if $i^*(\gamma) = \sigma$ and $m_d(s, \gamma, \sigma) = P_\sigma(s)$ for all $0 \leq s \in L(\sigma)$. Moreover, by Lemma 1.18 of the same book, there exists a σ-admissible $\gamma \in R(K)$ for every $\sigma \in \hat{M}$.

There are the Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = KAN$.

If $g \in G$ is a hyperbolic element, then g is conjugated to some element $am \in A^+M$ (see, e.g., [3,5]), where $A^+ = \exp(\mathfrak{a}^+)$, and $\mathfrak{a}^+$ is the positive Weyl chamber in $\mathfrak{a}$. Following [1] (p. 59), we put $l(g) = |\log(a)|$.

For finite-dimensional unitary representations σ, χ of M, Γ, and $s \in \mathbb{C}$, $\mathrm{Re}(s) > \rho$ ($\mathrm{Re}(s) > 2\rho$), the Selberg zeta function $Z_{S,\chi}(s, \sigma)$ (the Ruelle zeta function $Z_{R,\chi}(s, \sigma)$) is indroduced by Definition 3.2 (Definition 3.1) in [1]. Subsequently, $Z_{S,\chi}(s, \sigma)$ is meromorphically continued to the whole complex plane with the singularities that were described by Theorem 3.15 of the same book.

For the sake of simplicity, we fix some $\chi \in \hat{\Gamma}$, $\sigma \in \hat{M}$, and reduce the notation by omitting to write them in the sequel unless necessary.

By [1] (p. 99), there are sets $I_p = \left\{(\tau,\lambda) : \tau \in \hat{M},\, \lambda \in \mathbb{R}\right\}$, such that (also see, [39]):

$$Z_R(s) = \prod_{p=0}^{n-1} \prod_{(\tau,\lambda)\in I_p} Z_S(s+\rho-\lambda,\tau)^{(-1)^p},$$

where the shifts λ's are always contained in $[0,2\rho]$.

For $\gamma \in \Gamma$, let $N(\gamma) = e^{l(\gamma)}$ and $\Lambda(\gamma) = \log N(\gamma_0)$.

Finally, we introduce the functions $\psi_j(x)$, $j \in \mathbb{N}$ recursively by $\psi_j(x) = \int_0^x \psi_{j-1}(t)\,dt$, where $\psi_0(x) = \sum_{1\neq[\gamma]\in C\Gamma,\, N(\gamma)\le x} \Lambda(\gamma)$.

3. Results

3.1. Prime Geodesic Theorem

We prove the following theorem:

Theorem 1. *Let Y be as above. Subsequently:*

$$\psi_0(x) = \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha\in S^{\mathbb{R}}_{p,\tau,\lambda} \\ 2\rho-\frac{\rho}{n}<\alpha\le 2\rho}} \alpha^{-1}x^{\alpha} + O\left(x^{2\rho-\frac{\rho}{n}}\right),$$

where $S^{\mathbb{R}}_{p,\tau,\lambda}$ denotes the set of real singularities of $Z_S(s+\rho-\lambda,\tau)$.

Proof. As the starting point, we take the following explicit formula for $\psi_k(x)$, $k \in \mathbb{N}$, $k \ge 2n$:

$$\psi_k(x) = \sum_{\alpha\in S_k} c_k(\alpha), \tag{7}$$

where S_k is the set of poles of $-\frac{Z_R'(s)}{Z_R(s)} \frac{x^{s+k}}{s(s+1)\dots(s+k)}$, and $c_k(\alpha)$ is the residue at α.

Note that Formula (7) is easily obtained, as in the case of compact Riemann surfaces [17] (p. 245) and the compact symmetric spaces formed as quotients of the Lie group $SL_4(\mathbb{R})$ [29] (p. 63).

Fix $k = 2n$.

The relation (7) becomes:

$$\psi_{2n}(x) = \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\alpha\in S_{p,\tau,\lambda}} c_{p,\tau,\lambda}(\alpha), \tag{8}$$

where $S_{p,\tau,\lambda}$ denotes the set of poles of $\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)} \frac{x^{s+2n}}{s(s+1)\dots(s+2n)}$, and $c_{p,\tau,\lambda}(\alpha)$ denotes the residue at α.

For the sake of clarity, we highlight the following facts.

Because n is even, we know now that there is a τ-admissible element in $R(K)$ for each τ occurring in (8). Hence, if $(\tau,\lambda) \in I_p$ for some $p \in \{0,1,\dots,n-1\}$, and $\gamma_{p,\tau,\lambda}$ is τ-admissible, then, by Theorem 3.15 in [1] (p. 113), the singularities of $Z_S(s+\rho-\lambda,\tau)$ are the following ones: at $-\rho+\lambda \pm is$ of order $m\left(s,\gamma_{p,\tau,\lambda},\tau\right)$ if $s \neq 0$ is an eigenvalue of $A_Y\left(\gamma_{p,\tau,\lambda},\tau\right)$, at $-\rho+\lambda$ of order $2m\left(0,\gamma_{p,\tau,\lambda},\tau\right)$ if 0 is an eigenvalue of $A_Y\left(\gamma_{p,\tau,\lambda},\tau\right)$, at $-\rho+\lambda-T(k-\epsilon_\tau)$, $k\in\mathbb{N}$ of order $-2(-1)^{\frac{n}{2}} \frac{\dim(\chi)\operatorname{vol}(Y)}{\operatorname{vol}(X_d)} m_d\left(T(k-\epsilon_\tau),\gamma_{p,\tau,\lambda},\tau\right)$ (in this case, $T(k-\epsilon_\tau)$ is an eigenvalue of $A_d\left(\gamma_{p,\tau,\lambda},\tau\right)$). Here, X_d is a compact dual space of X (see, [1] (p. 18)). If two singularities coincide, their orders add up.

The singularities in the third group are called topological and they are all less than $-\rho + \lambda$. The remaining, spectral singularities, belong to $[-2\rho + \lambda, \lambda] \cup \{-\rho + \lambda + \mathrm{i}\, r \,:\, r \in \mathbb{R} \setminus \{0\}\}$. There may occur an overlap between the topological and the spectral singularities at finitely many points in $[-2\rho + \lambda, -\rho + \lambda)$.

Because $\rho = \frac{1}{2}(k-1), m, 2m+1, 11$ if $n = k, 2m, 4m, 16$, the inequality $-2n < -2\rho$ is always valid.

Bearing in mind these facts, we calculate the residues in (8) in the same way as Hejhal did in [10] (pp. 88–89) for the compact Riemann surfaces. We obtain the following explicit formula:

$$\begin{aligned}\psi_{2n}(x) = {} & \sum_{j=0}^{2n} \alpha_{2n-j} x^{2n-j} \log x + \sum_{j=0}^{2n} \beta_{2n-j} x^{2n-j} + \\ & \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha \in S^{\mathbb{R}}_{p,\tau,\lambda} \\ \alpha \le -2n-1}} \alpha^{-1} (\alpha+1)^{-1} \ldots (\alpha+2n)^{-1} x^{\alpha+2n} + \\ & \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha \in S^{\mathbb{R}}_{p,\tau,\lambda} \\ \alpha > -2n-1}} \alpha^{-1} (\alpha+1)^{-1} \ldots (\alpha+2n)^{-1} x^{\alpha+2n} + \\ & \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\alpha \in S^{-\rho+\lambda}_{p,\tau,\lambda}} \alpha^{-1} (\alpha+1)^{-1} \ldots (\alpha+2n)^{-1} x^{\alpha+2n},\end{aligned} \tag{9}$$

where $S^{\mathbb{R}}_{p,\tau,\lambda}$ is the set of real singularities of $Z_S(s+\rho-\lambda,\tau)$ not containing the integers $0, -1, \ldots, -2n$, $S^{-\rho+\lambda}_{p,\tau,\lambda}$ is the set of non-real singularities of $Z_S(s+\rho-\lambda,\tau)$, and α_j, β_j, $j \in \{0, 1, \ldots, 2n\}$ are some explicitly computable constants.

Consider the sum over $\alpha \le -2n-1$ on the right hand side of (9).

Because $\alpha \le -2n-1$, it follows that each α is of the form $-\rho + \lambda - T(k - \epsilon_\tau)$ for some $k \in \mathbb{N}$. Now, $-\rho + \lambda - T(k-\epsilon_\tau) \le -2n-1$ yields that $k \ge \frac{1}{T}(2n+1-\rho+\lambda) + \epsilon_\tau$. The order of $\alpha = -\rho + \lambda - T(k-\epsilon_\tau)$ is $-2(-1)^{\frac{n}{2}} \frac{\dim(\chi)\operatorname{vol}(Y)}{\operatorname{vol}(X_d)} m_d\left(T(k-\epsilon_\tau), \gamma_{p,\tau,\lambda}, \tau\right)$. Because $\gamma_{p,\tau,\lambda}$ is τ-admissible, it follows that $m_d\left(s, \gamma_{p,\tau,\lambda}, \tau\right) = P_\tau(s)$ for all $0 \le s \in L(\tau)$.

Hence, for $0 < T(k-\epsilon_\tau) \in L(\tau)$, $k \in \mathbb{N}$, we obtain that $m_d\left(T(k-\epsilon_\tau), \gamma_{p,\tau,\lambda}, \tau\right) = P_\tau(T(k-\epsilon_\tau))$.

Consequently:

$$\begin{aligned} & \sum_{\substack{\alpha \in S^{\mathbb{R}}_{p,\tau,\lambda} \\ \alpha \le -2n-1}} \alpha^{-1} (\alpha+1)^{-1} \ldots (\alpha+2n)^{-1} x^{\alpha+2n} = \\ & -2(-1)^{\frac{n}{2}} \frac{\dim(\chi)\operatorname{vol}(Y)}{\operatorname{vol}(X_d)} \sum_{k \ge \frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau} P_\tau(T(k-\epsilon_\tau)) \times \\ & \prod_{j=0}^{2n} \left(-\rho+\lambda-T(k-\epsilon_\tau)+j\right)^{-1} x^{-\rho+\lambda-T(k-\epsilon_\tau)+2n} \\ & = O\left(x^{-1} \sum_{k \ge \frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau} \frac{1}{k^{n+2}}\right) = O\left(x^{-1}\right) \end{aligned}$$

since the polynomial P_τ is known to be of degree $n-1$ (Cf. [36]).

Thus:

$$\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S^{\mathbb{R}}_{p,\tau,\lambda}\\ \alpha\le -2n-1}}\alpha^{-1}(\alpha+1)^{-1}\ldots(\alpha+2n)^{-1}x^{\alpha+2n}=O\left(x^{-1}\right). \tag{10}$$

It is known that $\psi_0(x)\le d^{-2n}\Delta\psi_{2n}(x)$, where the function Δ is defined by:

$$\Delta f(x)=\int_{x}^{x+d}\int_{t_{2n}}^{t_{2n}+d}\ldots\int_{t_2}^{t_2+d}f^{(2n)}(t_1)\,dt_1\ldots dt_{2n}$$

for at least $2n$ times differentiable function f and a constant d.

Notice that we are interested in achieving the bound $d=O(x)$.

For $\alpha\in S^{-\rho+\lambda}_{p,\tau,\lambda}$, it easily follows that (see, e.g., [6] (p. 370, (8))):

$$d^{-2n}\Delta\alpha^{-1}(\alpha+1)^{-1}\ldots(\alpha+2n)^{-1}x^{\alpha+2n}=O\left(\min\left\{|\alpha|^{-1}x^{\rho},d^{-2n}|\alpha|^{-2n-1}x^{\rho+2n}\right\}\right).$$

We obtain (Cf. [17] (p. 246)):

$$\begin{aligned}&\sum_{\alpha\in S^{-\rho+\lambda}_{p,\tau,\lambda}}d^{-2n}\Delta\alpha^{-1}(\alpha+1)^{-1}\ldots(\alpha+2n)^{-1}x^{\alpha+2n}\\ =&O\left(x^{\rho}\int_{|-\rho+\lambda|}^{K}t^{-1}dN(t)\right)+O\left(d^{-2n}x^{\rho+2n}\int_{K}^{+\infty}t^{-2n-1}dN(t)\right)\\ =&O\left(x^{\rho}K^{n-1}\right)+O\left(d^{-2n}x^{\rho+2n}K^{-n-1}\right),\end{aligned} \tag{11}$$

where $N(t)=At^{n}+O\left(t^{n-1}\right)$ is the number of singularities of the Selberg zeta function $Z_S(s,\tau)$ at points $\mathrm{i}\,x$, $0<x<t$, and A is some explicitly known constant (see, [40] (p. 89, Th. 9.1)).

By the mean value theorem:

$$\Delta x^{r}=d^{2n}r(r-1)\ldots(r-(2n-1))\,\tilde{x}^{r-2n}$$

for some $\tilde{x}\in[x,x+2nd]$, so:

$$d^{-2n}\Delta\left(\sum_{j=0}^{2n}\alpha_{2n-j}x^{2n-j}\log x+\sum_{j=0}^{2n}\beta_{2n-j}x^{2n-j}\right)=O(\log x) \tag{12}$$

and:

$$\begin{aligned}&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S^{\mathbb{R}}_{p,\tau,\lambda}\\ \alpha>-2n-1}}d^{-2n}\Delta\alpha^{-1}(\alpha+1)^{-1}\ldots(\alpha+2n)^{-1}x^{\alpha+2n}\\ =&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S^{\mathbb{R}}_{p,\tau,\lambda}\\ 0<\alpha\le 2\rho}}\alpha^{-1}x^{\alpha}+O\left(x^{2\rho-1}d\right).\end{aligned} \tag{13}$$

The relations (9)–(13) give us:

$$\psi_0(x) \leq \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha \in S^{\mathbb{R}}_{p,\tau,\lambda} \\ 0<\alpha\leq 2\rho}} \alpha^{-1} x^{\alpha} + O\left(x^{2\rho-1} d\right) + O\left(x^{\rho} K^{n-1}\right) + \\ O\left(d^{-2n} x^{\rho+2n} K^{-n-1}\right) + O(\log x) + O\left(d^{-2n} x^{-1}\right).$$

The optimal size of the error term is achieved for $d = x^{1-\frac{\rho}{n}}$ and $K = x^{\frac{\rho}{n}}$.
We obtain:

$$\psi_0(x) \leq \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha \in S^{\mathbb{R}}_{p,\tau,\lambda} \\ 2\rho-\frac{\rho}{n}<\alpha\leq 2\rho}} \alpha^{-1} x^{\alpha} + O\left(x^{2\rho-\frac{\rho}{n}}\right).$$

In a similar way:

$$\psi_0(x) \geq \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha \in S^{\mathbb{R}}_{p,\tau,\lambda} \\ 2\rho-\frac{\rho}{n}<\alpha\leq 2\rho}} \alpha^{-1} x^{\alpha} + O\left(x^{2\rho-\frac{\rho}{n}}\right).$$

This completes the proof. □

An immediate consequence of Theorem 1 is the following theorem:

Theorem 2. *(Prime Geodesic Theorem) Let Y be as above. Subsequetly:*

$$\pi_{\Gamma}(x) = \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha \in S^{\mathbb{R}}_{p,\tau,\lambda} \\ 2\rho-\frac{\rho}{n}<\alpha\leq 2\rho}} \mathrm{li}\left(x^{\alpha}\right) + O\left(x^{2\rho-\frac{\rho}{n}} (\log x)^{-1}\right),$$

as $x \to \infty$.

Proof. The derived relation for $\psi_0(x)$ (Theorem 1) yields the assertion of theorem (see, e.g., [4] (p. 102)).
This completes the proof. □

3.2. Gallagherian Prime Geodesic Theorem

Theorem 3. *Let Y be as above. For $\varepsilon > 0$, there exists a set E of finite logarithmic measure, such that:*

$$\pi_{\Gamma}(x) = \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha \in S^{\mathbb{R}}_{p,\tau,\lambda} \\ 2\rho-\rho\frac{4n+1}{4n^2+1}<\alpha\leq 2\rho}} \mathrm{li}\left(x^{\alpha}\right) + O\left(x^{2\rho-\rho\frac{4n+1}{4n^2+1}} (\log x)^{\frac{n-1}{4n^2+1}-1} (\log\log x)^{\frac{n-1}{4n^2+1}+\varepsilon}\right),$$

as $x \to \infty$, $x \notin E$.

Proof. As the starting point, we take the explicit formula for $\psi_{2n}(x)$ given by the relation (9).

Following work of Avdispahić-Šabanac [8] (p. 3022) in the case of real hyperbolic manifolds with cusps, we split the last sum on the right hand side of (9) into three parts:

$$\begin{aligned}&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ |\mathrm{Im}(\alpha)|\leq Y}}+\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ Y<|\mathrm{Im}(\alpha)|\leq W}}+\\&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ |\mathrm{Im}(\alpha)|>W}}.\end{aligned}\tag{14}$$

Define the sets $E_{p,\tau,\lambda}^{j}$, as follows:

$$\begin{aligned}E_{p,\tau,\lambda}^{j}=\Bigg\{x\in\left[e^{j},e^{j+1}\right):&\left|\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ Y<|\mathrm{Im}(\alpha)|\leq W}}\alpha^{-1}(\alpha+1)^{-1}\ldots(\alpha+2n)^{-1}x^{\alpha+2n}\right|\\&>x^{\alpha}(\log x)^{\beta}(\log\log x)^{\beta+\varepsilon}\Bigg\}.\end{aligned}$$

We estimate $\mu^{\times}E_{p,\tau,\lambda}^{j}=\int\limits_{E_{p,\tau,\lambda}^{j}}\frac{dx}{x}$ as follows:

$$\begin{aligned}&\int\limits_{E_{p,\tau,\lambda}^{j}}x^{2\alpha}(\log x)^{2\beta}(\log\log x)^{2\beta+2\varepsilon}\frac{1}{x^{2\alpha}(\log x)^{2\beta}(\log\log x)^{2\beta+2\varepsilon}}\frac{dx}{x}\\
&=O\left(\int\limits_{E_{p,\tau,\lambda}^{j}}\left|\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ Y<|\mathrm{Im}(\alpha)|\leq W}}\alpha^{-1}(\alpha+1)^{-1}\ldots(\alpha+2n)^{-1}x^{\alpha+2n}\right|^{2}\frac{1}{x^{2\alpha}(\log x)^{2\beta}(\log\log x)^{2\beta+2\varepsilon}}\frac{dx}{x}\right)\\
&=O\Bigg(\int\limits_{e^{j}}^{e^{j+1}}x^{2(-\rho+\lambda+2n)}\left|\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ Y<|\mathrm{Im}(\alpha)|\leq W}}\alpha^{-1}(\alpha+1)^{-1}\ldots(\alpha+2n)^{-1}x^{\mathrm{i}\,\mathrm{Im}(\alpha)}\right|^{2}\times\\
&\quad\frac{1}{x^{2\alpha}(\log x)^{2\beta}(\log\log x)^{2\beta+2\varepsilon}}\frac{dx}{x}\Bigg)\\
&=O\left(\frac{e^{2(j+1)(-\rho+\lambda+2n)}}{e^{2j\alpha}j^{2\beta}(\log j)^{2\beta+2\varepsilon}}\int\limits_{e^{j}}^{e^{j+1}}\left|\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ Y<|\mathrm{Im}(\alpha)|\leq W}}\alpha^{-1}(\alpha+1)^{-1}\ldots(\alpha+2n)^{-1}x^{\mathrm{i}\,\mathrm{Im}(\alpha)}\right|^{2}\frac{dx}{x}\right)\end{aligned}$$

$$=O\left(\frac{e^{2(\rho+2n-\alpha)j}}{j^{2\beta}(\log j)^{2\beta+2\varepsilon}}\int\limits_{e^j}^{e^{j+1}}\left|\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ Y<|\mathrm{Im}(\alpha)|\le W}}\alpha^{-1}(\alpha+1)^{-1}\dots(\alpha+2n)^{-1}x^{\mathrm{i}\,\mathrm{Im}(\alpha)}\right|^2\frac{dx}{x}\right).$$

Putting $x=e^{j+2\pi\left(u+\frac{1}{4\pi}\right)}$, we obtain:

$$\mu^{\times}E_{p,\tau,\lambda}^{j}=O\left(\frac{e^{2(\rho+2n-\alpha)j}}{j^{2\beta}(\log j)^{2\beta+2\varepsilon}}\int\limits_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}}\left|\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ Y<|\mathrm{Im}(\alpha)|\le W}}\frac{e^{\mathrm{i}\,\mathrm{Im}(\alpha)\left(j+\frac{1}{2}\right)}}{\alpha(\alpha+1)\dots(\alpha+2n)}e^{2\pi\,\mathrm{i}\,\mathrm{Im}(\alpha)u}\right|^2du\right). \tag{15}$$

Now, we apply the Gallagher lemma (see, [9] (p. 78, Lemma 1)) to the last integral (see also, [13,41], with $\nu=\mathrm{Im}(\alpha)$, $\theta=U=\frac{1}{4\pi}$, and:

$$c(\nu)=\begin{cases} e^{\mathrm{i}\,\nu\left(j+\frac{1}{2}\right)}/\alpha(\alpha+1)\dots(\alpha+2n), & Y<|\nu|\le W\\ 0, & \text{otherwise.}\end{cases}$$

It follows that:

$$\begin{aligned}&\int\limits_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}}\left|\sum_{\substack{\alpha\in S_{p,\tau,\lambda}^{-\rho+\lambda}\\ Y<|\mathrm{Im}(\alpha)|\le W}}\frac{e^{\mathrm{i}\,\mathrm{Im}(\alpha)\left(j+\frac{1}{2}\right)}}{\alpha(\alpha+1)\dots(\alpha+2n)}e^{2\pi\,\mathrm{i}\,\mathrm{Im}(\alpha)u}\right|^2du\\ &\le\left(\frac{\frac{1}{4}}{\sin\frac{1}{4}}\right)^2\int\limits_{-\infty}^{+\infty}\left(\sum_{\substack{t\le|\mathrm{Im}(\alpha)|\le t+1\\ Y<|\mathrm{Im}(\alpha)|\le W}}\frac{1}{|\alpha||\alpha+1|\dots|\alpha+2n|}\right)^2dt.\end{aligned} \tag{16}$$

Since $N(t)=At^n+O\left(t^{n-1}\right)$, the number of $|\mathrm{Im}(\alpha)|$ participants in the last sum is $O\left(t^{n-1}\right)$. Moreover, $\frac{1}{\alpha(\alpha+1)\dots(\alpha+2n)}=O\left(t^{-2n-1}\right)$.
Hence:

$$\int\limits_{-\infty}^{+\infty}\left(\sum_{\substack{t\le|\mathrm{Im}(\alpha)|\le t+1\\ Y<|\mathrm{Im}(\alpha)|\le W}}\frac{1}{|\alpha||\alpha+1|\dots|\alpha+2n|}\right)^2dt=O\left(\int\limits_{Y-1}^{W+1}\frac{1}{t^{2n+4}}dt\right)=O\left(\frac{1}{Y^{2n+3}}\right). \tag{17}$$

Combining the relations (15)–(17), we conclude that:

$$\mu^{\times}E_{p,\tau,\lambda}^{j}=O\left(\frac{e^{2(\rho+2n-\alpha)j}}{Y^{2n+3}j^{2\beta}(\log j)^{2\beta+2\varepsilon}}\right).$$

Taking:

$$Y\sim e^{\frac{1}{2n+3}(2\rho+4n-2\alpha)j}j^{\frac{1-2\beta}{2n+3}}(\log j)^{\frac{1-2\beta}{2n+3}},$$

we get $\mu^{\times}E_{p,\tau,\lambda}^{j}=O\left(\frac{1}{j(\log j)^{1+2\varepsilon}}\right)$, and so $\mu^{\times}\bigcup\limits_{p}\bigcup\limits_{(\tau,\lambda)\in I_p}E_{p,\tau,\lambda}^{j}=O\left(\frac{1}{j(\log j)^{1+2\varepsilon}}\right)$.

Hence, the set $E = \bigcup_j \bigcup_p \bigcup_{(\tau,\lambda)\in I_p} E^j_{p,\tau,\lambda}$ has a finite logarithmic measure.

Now, for $x \notin E$, i.e., for x outside a set of finite logarithmic measure, the definition of $E^j_{p,\tau,\lambda}$ yields that the second sum in (14) is estimated by:

$$O\left(x^{\alpha}\left(\log x\right)^{\beta}\left(\log\log x\right)^{\beta+\varepsilon}\right).$$

Consequently, for $x \notin E$:

$$\begin{aligned} &d^{-2n}\Delta\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S^{-\rho+\lambda}_{p,\tau,\lambda}\\ Y<|\mathrm{Im}(\alpha)|\le W}}\alpha^{-1}\left(\alpha+1\right)^{-1}\ldots\left(\alpha+2n\right)^{-1}x^{\alpha+2n}\\ &=O\left(\frac{x^{\alpha}\left(\log x\right)^{\beta}\left(\log\log x\right)^{\beta+\varepsilon}}{d^{2n}}\right). \end{aligned} \tag{18}$$

Consider the first sum in (14).
Reasoning in the same way as in the derivation of (11), we conclude that:

$$d^{-2n}\Delta\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S^{-\rho+\lambda}_{p,\tau,\lambda}\\ |\mathrm{Im}(\alpha)|\le Y}}\alpha^{-1}\left(\alpha+1\right)^{-1}\ldots\left(\alpha+2n\right)^{-1}x^{\alpha+2n}=O\left(x^{\rho}Y^{n-1}\right). \tag{19}$$

Similarly, for the third sum in (14), we have:

$$d^{-2n}\Delta\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S^{-\rho+\lambda}_{p,\tau,\lambda}\\ |\mathrm{Im}(\alpha)|> W}}\alpha^{-1}\left(\alpha+1\right)^{-1}\ldots\left(\alpha+2n\right)^{-1}x^{\alpha+2n}=O\left(d^{-2n}\frac{x^{\rho+2n}}{W^{n+1}}\right). \tag{20}$$

Now, the relations (9), (10), (12)–(14), and (18)–(20) give us for $x \notin E$:

$$\begin{aligned} \psi_0\left(x\right)\le&\sum_{p=0}^{n-1}(-1)^{p+1}\sum_{(\tau,\lambda)\in I_p}\sum_{\substack{\alpha\in S^{\mathbb{R}}_{p,\tau,\lambda}\\ 0<\alpha\le 2\rho}}\alpha^{-1}x^{\alpha}+O\left(x^{2\rho-1}d\right)+O\left(x^{\rho}Y^{n-1}\right)+\\ &O\left(\frac{x^{\alpha}\left(\log x\right)^{\beta}\left(\log\log x\right)^{\beta+\varepsilon}}{d^{2n}}\right)+O\left(d^{-2n}\frac{x^{\rho+2n}}{W^{n+1}}\right)+O\left(\log x\right)+O\left(d^{-2n}x^{-1}\right). \end{aligned} \tag{21}$$

Clearly, $x^{2\rho-1}d = x^{\rho}Y^{n-1}$ if:

$$d = x^{1-\rho}Y^{n-1}. \tag{22}$$

Additionally, $x^{2\rho-1}d \le \frac{x^{\alpha}(\log x)^{\beta}(\log\log x)^{\beta+\varepsilon}}{d^{2n}}$ if:

$$d = x^{\frac{\alpha-2\rho+1}{2n+1}}\left(\log x\right)^{\frac{\beta}{2n+1}}\left(\log\log x\right)^{\frac{\beta}{2n+1}}. \tag{23}$$

By our selection of Y:

$$Y^{n-1}\sim x^{(n-1)\frac{2\rho+4n-2\alpha}{2n+3}}\left(\log x\right)^{(n-1)\frac{1-2\beta}{2n+3}}\left(\log\log x\right)^{(n-1)\frac{1-2\beta}{2n+3}}. \tag{24}$$

Combining the relations (22) and (24), and comparing the exponents of x and $\log x$ with the corresponding exponents in (23), we arrive at:

$$\frac{\alpha-2\rho+1}{2n+1} = 1-\rho+(n-1)\,\frac{2\rho+4n-2\alpha}{2n+3},$$
$$\frac{\beta}{2n+1} = (n-1)\,\frac{1-2\beta}{2n+3}.$$

Thus, $\alpha = \frac{8n^3+2n+\rho-6n\rho}{4n^2+1}$, $\beta = \frac{2n^2-n-1}{4n^2+1}$.

Substituting the obtained d and Y into (21), we end up with:

$$\psi_0(x) \le \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha\in S^{\mathbb{R}}_{p,\tau,\lambda} \\ 2\rho-\rho\frac{4n+1}{4n^2+1}<\alpha\le 2\rho}} \alpha^{-1}x^{\alpha} + O\left(x^{2\rho-\rho\frac{4n+1}{4n^2+1}}\,(\log x)^{\frac{n-1}{4n^2+1}}\,(\log\log x)^{\frac{n-1}{4n^2+1}+\varepsilon}\right)$$

as $x\to\infty$, $x\notin E$.

Notice that $d = x^{1-\rho\frac{4n+1}{4n^2+1}}\,(\log x)^{\frac{n-1}{4n^2+1}}\,(\log\log x)^{\frac{n-1}{4n^2+1}} = O(x)$, as required above.

Similarly:

$$\psi_0(x) \ge \sum_{p=0}^{n-1} (-1)^{p+1} \sum_{(\tau,\lambda)\in I_p} \sum_{\substack{\alpha\in S^{\mathbb{R}}_{p,\tau,\lambda} \\ 2\rho-\rho\frac{4n+1}{4n^2+1}<\alpha\le 2\rho}} \alpha^{-1}x^{\alpha} + O\left(x^{2\rho-\rho\frac{4n+1}{4n^2+1}}\,(\log x)^{\frac{n-1}{4n^2+1}}\,(\log\log x)^{\frac{n-1}{4n^2+1}+\varepsilon}\right)$$

as $x\to\infty$, $x\notin E$.

Now, the assertion of theorem follows the same argumentation as in the proof of Theorem 2.

This completes the proof. □

4. Discussion

The bound $O\left(x^{2\rho-\frac{\rho}{n}}\,(\log x)^{-1}\right)$ from Theorem 2 obviously improves DeGeorge's $O(x^{\eta})$ for $2\rho-\frac{\rho}{n}\le\eta<2\rho$. If $n=2$, $\rho=\frac{1}{2}$, the corresponding estimate coincides with the best known result $O\left(x^{\frac{3}{4}}\,(\log x)^{-1}\right)$ in the Riemann surfaces case [17] (p. 245, Th. 2). The obtained error term also fully agrees with $O\left(x^{\frac{5}{3}}\,(\log x)^{-1}\right)$ in (6), as derived for hyperbolic 3-manifolds [7] (p. 691, Th. 1.1), where $n=3$, $\rho=1$, as well as with $O\left(x^{\frac{4\rho^2+\rho}{2\rho+1}}\,(\log x)^{-1}\right)$ for real hyperbolic manifolds with cusps [7] (p. 692, Th. 2.1) when $n=k$ and $\rho=\frac{1}{2}(k-1)$.

The inequality $2\rho-\rho\frac{4n+1}{4n^2+1}\le 2\rho-\frac{\rho}{n}$ is always valid, since the corresponding equivalent inequality $n\ge 1$ is clearly true. The result $O\left(x^{2\rho-\rho\frac{4n+1}{4n^2+1}}\,(\log x)^{\frac{n-1}{4n^2+1}-1}\,(\log\log x)^{\frac{n-1}{4n^2+1}+\varepsilon}\right)$ from Theorem 3 thus improves our $O\left(x^{2\rho-\frac{\rho}{n}}\,(\log x)^{-1}\right)$ outside a set of finite logarithmic measure. It is evident that the bound $O\left(x^{2\rho-\frac{\rho}{n}}\,(\log x)^{-1}\right)$ is not the optimal one, so the search for such a bound still remains open.

Note that Avdispahić-Gušić [42] (p. 311, Th. 9) proved to be a variant of Theorem 2. The omission mentioned in the introduction is also present in their work, since the additional term $O\left(x^{2\rho-1}h\right)$ in reduction from $\psi_{2n}(x)$ to $\psi_0(x)$ is missing [42] (p. 316, (32)). The correct form of the prime geodesic theorem [42] (p. 317, (39)) resp. [42] (p. 311, Th. 9) is given by Theorem 1 resp. Theorem 2.

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