Invariant Means, Complementary Averages of Means, and a Characterization of the Beta-Type Means

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Abstract: We prove that whenever the selfmapping $(M_1, \dots, M_p): I^p \rightarrow I^p$, $(p \in \mathbb{N}$ and M_i -s are p -variable means on the interval I) is invariant with respect to some continuous and strictly monotone mean $K: I^p \rightarrow I$ then for every nonempty subset $S \subseteq \{1, \dots, p\}$ there exists a uniquely determined mean $K_S: I^p \rightarrow I$ such that the mean-type mapping $(N_1, \dots, N_p): I^p \rightarrow I^p$ is K -invariant, where $N_i := K_S$ for $i \in S$ and $N_i := M_i$ otherwise. Moreover $\min(M_i: i \in S) \leq K_S \leq \max(M_i: i \in S)$. Later we use this result to: (1) construct a broad family of K -invariant mean-type mappings, (2) solve functional equations of invariant-type, and (3) characterize Beta-type means.

Keywords: invariant means; complementary averages of means; characterizations; beta-type means

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1. Preliminaries

In the whole paper $I \subset \mathbb{R}$ stands for an interval, $p \in \mathbb{N}$, $p > 1$ is fixed, and $\mathbb{N}_p := \{1, \dots, p\}$. A function $M: I^p \rightarrow I$ is called a *mean* on I if

$$\min(x_1, \dots, x_p) \leq M(x_1, \dots, x_p) \leq \max(x_1, \dots, x_p), \quad x_1, \dots, x_p \in I,$$

or, briefly, if

$$\min \mathbf{x} \leq M(\mathbf{x}) \leq \max \mathbf{x}, \quad \mathbf{x} = (x_1, \dots, x_p) \in I^p.$$

The mean M is called *strict*, if for all nonconstant vectors $\mathbf{x}$ these inequalities are sharp; and *symmetric*, if $M(x_{\sigma(1)}, \dots, x_{\sigma(p)}) = M(x_1, \dots, x_p)$ for all $x_1, \dots, x_p \in I$ and all permutations σ of the set $\mathbb{N}_p$. Mean M is *monotone* if it is increasing in each of its variables. It is important to emphasize that every strictly monotone mean is strict.

A mapping $\mathbf{M}: I^p \rightarrow I^p$ is referred to as *mean-type* if there exist some means $M_i: I^p \rightarrow I$, $i \in \mathbb{N}_p$, such that $\mathbf{M} = (M_1, \dots, M_p)$. For a mean-type mapping $\mathbf{M}: I^p \rightarrow I^p$ we denote a projection onto the i -th coordinate by $[\mathbf{M}]_i: I^p \rightarrow I$. In this case we obviously have $[\mathbf{M}]_i = M_i$ for all $i \in \mathbb{N}_p$.

We say that a function $K: I^p \rightarrow \mathbb{R}$ is invariant with respect to $\mathbf{M}$ (briefly $\mathbf{M}$ -invariant), if $K \circ \mathbf{M} = K$.

Theorem 1 (Invariance Principle, [1]). *If $\mathbf{M}: I^p \rightarrow I^p$, $\mathbf{M} = (M_1, \dots, M_p)$ is a continuous mean-type mapping such that*

$$\max \mathbf{M}(\mathbf{x}) - \min \mathbf{M}(\mathbf{x}) < \max(\mathbf{x}) - \min(\mathbf{x}), \quad \mathbf{x} \in I^p \setminus \Delta(I^p),$$

where $\Delta(I^p) := \{ \mathbf{x} = (x_1, \dots, x_p) \in I^p : x_1 = \dots = x_p \}$ then there is a unique $\mathbf{M}$ -invariant mean $K : I^p \rightarrow I$ and the sequence of iterates $(\mathbf{M}^n)_{n \in \mathbb{N}}$ of the mean-type mapping $\mathbf{M}$ converges to $\mathbf{K} := (K, \dots, K)$ pointwise on I^p .

2. A Family of Complementary Means

Let us recall the following result.

Remark 1 (Matkowski [2]). Assume that $K : I^2 \rightarrow I$ is a symmetric mean which is continuous and monotone. Then

- (i) for an arbitrary mean $M_1 : I^2 \rightarrow I$ there is a unique mean $M_2 : I^2 \rightarrow I$ such that K is (M_1, M_2) -invariant,

$$K \circ (M_1, M_2) = K;$$

$\mathcal{K}_{M_1} := M_2$, is referred to as a K -complementary mean for M_1 ; and we have

$$\mathcal{K}_{M_1} = M_2 \iff \mathcal{K}_{M_2} = M_1;$$

- (ii) if $M_1, M_2 : I^2 \rightarrow I$ are means such that K is (M_1, M_2) -invariant, then there exists a unique mean $M : I^2 \rightarrow I$ such that

$$\min(M_1, M_2) \leq M \leq \max(M_1, M_2)$$

and

$$K \circ (M, M) = K;$$

moreover

$$M = K.$$

In this case, $p \geq 3$ the counterpart of part (i) of Remark 1 is false which shows the following

Example 1. Let $I = \mathbb{R}$, and $K = A$ where $A(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3}{3}$. The functions $M_1(x_1, x_2, x_3) = \frac{x_1 + x_2}{2}$, $M_2(x_1, x_2, x_3) = x_2$ are means in $\mathbb{R}$. Then it is easy to see that there is no mean M_3 such that

$$A \circ (M_1, M_2, M_3) = A,$$

but a partial counterpart of part (ii) holds true.

The main result of this section reads as follows.

Theorem 2. Let $M_1, \dots, M_p : I^p \rightarrow I$ be means. Assume that $K : I^p \rightarrow I$ is a continuous and monotone mean which is invariant with respect to the mean type mapping $\mathbf{M} := (M_1, \dots, M_p)$.

Then for every nonempty subset $S \subseteq \mathbb{N}_p$ there exists a unique mean $K_S(\mathbf{M}) : I^p \rightarrow I$ such that K is $\mathbf{K}_S(\mathbf{M})$ -invariant, where $\mathbf{K}_S(\mathbf{M}) : I^p \rightarrow I^p$ is given by

$$[\mathbf{K}_S(\mathbf{M})]_i := \begin{cases} K_S(\mathbf{M}) & \text{for } i \in S, \\ M_i & \text{for } i \in \mathbb{N}_p \setminus S. \end{cases} \quad (1)$$

Moreover, $\min(M_i : i \in S) \leq K_S(\mathbf{M}) \leq \max(M_i : i \in S)$.

Proof. In the case $S = \mathbb{N}_p$ the $\mathbf{K}_S(\mathbf{M})$ -invariance of K implies $K_S(\mathbf{M}) = K$ and the statement is obvious. From now on we assume that $S \neq \mathbb{N}_p$.

Denote briefly $M_\vee := \max\{M_i : i \in S\}$ and $M_\wedge := \min\{M_i : i \in S\}$. Fix $\mathbf{x} \in I^n$ arbitrarily. Define a function $T: I \rightarrow I^p$ by

$$[T(\alpha)]_i := \begin{cases} M_i(\mathbf{x}) & i \in \mathbb{N}_p \setminus S, \\ \alpha & i \in S. \end{cases}$$

and $f: I \rightarrow I$ by $f(\alpha) := K \circ T(\alpha)$. Then, as K is continuous and strictly increasing, so is f . Therefore in view of the equality $K \circ \mathbf{M}(\mathbf{x}) = K(\mathbf{x})$ we obtain $f(M_\vee(\mathbf{x})) \geq K(\mathbf{x})$ and $f(M_\wedge(\mathbf{x})) \leq K(\mathbf{x})$. Thus, there exists unique number $\alpha_{\mathbf{x}} \in [M_\wedge(\mathbf{x}), M_\vee(\mathbf{x})]$ such that $f(\alpha_{\mathbf{x}}) = K(\mathbf{x})$. Now, as $\mathbf{x} \in I^n$ was arbitrary we define $K_S(\mathbf{M})(\mathbf{x}) := \alpha_{\mathbf{x}}$.

Then we have

$$K(\mathbf{x}) = f(\alpha_{\mathbf{x}}) = f(K_S(\mathbf{M})(\mathbf{x})) = (K \circ T \circ (K_S(\mathbf{M}))) (\mathbf{x}) = (K \circ (\mathbf{K}_S(\mathbf{M}))) (\mathbf{x}),$$

which shows that K is $\mathbf{K}_S(\mathbf{M})$ -invariant.

Now we need to show that $K_S(\mathbf{M})$ is uniquely determined. Assume that K is $\mathbf{K}_S(\mathbf{M})$ -invariant and $K_S(\mathbf{M})(\mathbf{x}) \neq \alpha_{\mathbf{x}}$ for some $\mathbf{x} \in I^p$. Then, as f is a monomorphism we obtain

$$K \circ (\mathbf{K}_S(\mathbf{M})) (\mathbf{x}) = (K \circ T \circ (K_S(\mathbf{M}))) (\mathbf{x}) \neq f(\alpha_{\mathbf{x}}) = K(\mathbf{x})$$

contradicting the $\mathbf{K}_S(\mathbf{M})$ -invariance. $\square$

Let us underline that we do not assume that means $M_1, \dots, M_p$ are continuous. This is relatively recent approach to invariant property which was studied by the authors in [3].

The intuition beyond this theorem is the following. Once we have a continuous and monotone mean K such that $\mathbf{M}$ is K -invariant mean we can unite a subfamily $(M_s)_{s \in S}$ into a single mean (denoted by $K_S(\mathbf{M})$) to preserve the K -invariance. In view of Theorem 1, such a mean is unique. In this connection we propose the following

Definition 1. Let $K: I^p \rightarrow I$ be a continuous and monotone mean which is invariant with respect to the mean type mapping $\mathbf{M} := (M_1, \dots, M_p)$.

For each set $S \subseteq \mathbb{N}_p$:

- (i) the mean $K_S(\mathbf{M})$ is called a K -complementary averaging of the means $\{M_i : i \in S\}$ with respect to the invariant mean-type mapping $\mathbf{M} = (M_1, \dots, M_p)$;
- (ii) the mean-type mapping $\mathbf{K}_S(\mathbf{M})$ given by (1) is called a K -complementary averaging of the mean-type mapping $\mathbf{M} = (M_1, \dots, M_p)$ with respect to the means $\{M_i : i \in S\}$.

Moreover, the set

$$\mathfrak{K}(K, \mathbf{M}) := \{\mathbf{K}_S(\mathbf{M}) : S \subseteq \mathbb{N}_p, S \neq \emptyset\}$$

is called the family of all K -complementary averaging of the mean-type mapping $\mathbf{M} = (M_1, \dots, M_p)$.

We can now reapply this result to the complementary of the establish a K -complementary of $\mathbf{K}_S(\mathbf{M})$ for the set $\mathbb{N}_p \setminus S$. More precisely we obtain

Corollary 1. Under the assumptions of Theorem 2 there exists unique mean $K_S^*(\mathbf{M}): I^p \rightarrow I$ such that K is $\mathbf{K}_S^*(\mathbf{M})$ -invariant, where $\mathbf{K}_S^*(\mathbf{M}): I^p \rightarrow I^p$ is given by

$$[\mathbf{K}_S^*(\mathbf{M})]_i := \begin{cases} K_S(\mathbf{M}) & \text{for } i \in S, \\ K_S^*(\mathbf{M}) & \text{for } i \in \mathbb{N}_p \setminus S. \end{cases}$$

Moreover, $\min(M_i : i \in \mathbb{N}_p \setminus S) \leq K_S^*(\mathbf{M}) \leq \max(M_i : i \in \mathbb{N}_p \setminus S)$.

Let us underline that the value $K_S^*(\mathbf{M})$ does **not** depend on $\mathbf{M}$ explicitly. The whole system of dependences is illustrated in Figure 1.

Observe that, as the mean $K_S^*(\mathbf{M})$ is uniquely determined, we obtain

$$K_S^*(\mathbf{M}) \in \mathfrak{K}(K, \mathbf{M}) \iff \text{all means } (M_i : i \in \mathbb{N}_p \setminus S) \text{ are equal to each other.}$$

Figure 1. Map of dependencies between means. Rectangle vertexes are dependent on S . Dotted line means that there could exists more $\mathbf{M}$ -invariant mean satisfying the conditions of Theorem 2.

3. Application in Solving Functional Equations

Theorem 3. Let $\mathbf{M} = (M_1, \dots, M_p)$ be a mean-type mapping such that $M_1, \dots, M_p : (0, \infty)^p \rightarrow (0, \infty)$ are strictly monotonic and homogeneous. Then

- (i) the sequence $(\mathbf{M}^n : n \in \mathbb{N})$ of iterates of $\mathbf{M}$ converge uniformly on compact subsets to a mean-type mapping $\mathbf{K} = (K, \dots, K)$, where K is a unique $\mathbf{M}$ -invariant mean.
- (ii) K is monotone, homogeneous and for every $S \subset \mathbb{N}_p$ the iterates of $\mathbf{K}_S(\mathbf{M})$ converge uniformly on compact subsets to a mean-type mapping $\mathbf{K} = (K, \dots, K)$;
- (iii) a function $F : (0, \infty)^p \rightarrow \mathbb{R}$ is continuous on the diagonal $\Delta((0, \infty)^p) := \{(x_1, \dots, x_p) \in (0, \infty)^p : x_1 = \dots = x_p\}$ and satisfies the functional equation

$$F \circ \mathbf{M} = F \tag{2}$$

- if and only if $F = \varphi \circ K$, where $\varphi : (0, \infty) \rightarrow \mathbb{R}$ is an arbitrary continuous function of a single variable;
- (iv) a function $F : (0, \infty)^p \rightarrow \mathbb{R}$ is continuous on the diagonal $\Delta((0, \infty)^p)$ and satisfies the simultaneous system of functional equations

$$F \circ \mathbf{K}_S(\mathbf{M}) = F, \quad S \subset \mathbb{N}_p; \tag{3}$$

if and only if $F = \varphi \circ K$, where $\varphi : (0, \infty) \rightarrow \mathbb{R}$ is an arbitrary continuous function of a single variable (so (2) and (3) are equivalent)

Proof. The homogeneity and monotonicity of $M_1, \dots, M_p$ imply their continuity (cf. ([4], Theorem 2)), so the invariance principle implies (i).

Now we prove that K is monotone. Indeed, take two vectors $v, w \in (0, \infty)^p$ such that $v_i \leq w_i$ for all $i \in \mathbb{N}_p$ and $v_{i_0} < w_{i_0}$ for certain $i_0 \in \mathbb{N}_p$. Then, as each M_i is monotone, there exists a constant $\theta \in (0, 1)$ such that $M_i(v) \leq \theta M_i(w)$ for all $i \in \mathbb{N}_p$.

Then for all $n \in \mathbb{N}$ and $i \in \mathbb{N}_p$ we have

$$\begin{aligned} [\mathbf{M}^n(v)]_i &= [\mathbf{M}^{n-1}(M_1(v), \dots, M_p(v))]_i \leq [\mathbf{M}^{n-1}(\theta M_1(w), \dots, \theta M_p(w))]_i \\ &= \theta [\mathbf{M}^{n-1}(M_1(w), \dots, M_p(w))]_i = \theta [\mathbf{M}^n(w)]_i. \end{aligned}$$

In a limit case as $n \rightarrow \infty$ in view of the first part of this statement we obtain $K(v) \leq \theta K(w) < K(w)$. Thus, K is monotone, which is (ii).

(iii) Assume first that $F : (0, \infty)^p \rightarrow \mathbb{R}$ that is continuous on the diagonal $\Delta((0, \infty)^p)$ and satisfies Equation (2). Hence, by induction,

$$F = F \circ \mathbf{M}^n, \quad n \in \mathbb{N},$$

By (ii) the sequence $(\mathbf{M}^n : n \in \mathbb{N})$ converges to $\mathbf{K} = (K, \dots, K)$. Since F is continuous on $\Delta((0, \infty)^p)$, we hence get for all $\mathbf{x} \in (0, \infty)^p$,

$$\begin{aligned} F(\mathbf{x}) &= \lim_{n \rightarrow \infty} F(\mathbf{M}^n(\mathbf{x})) = F\left(\lim_{n \rightarrow \infty} \mathbf{M}^n(\mathbf{x})\right) = F(\mathbf{K}(\mathbf{x})) \\ &= F(K(\mathbf{x}), \dots, K(\mathbf{x})). \end{aligned}$$

Setting

$$\varphi(t) := F(t, \dots, t), \quad t \in (0, \infty),$$

we hence get $F(\mathbf{x}) = \varphi(K(\mathbf{x}))$ for all $\mathbf{x} \in (0, \infty)^p$.

To prove the converse implication, take an arbitrary function $\varphi : I \rightarrow \mathbb{R}$ and put $F := \varphi \circ K$. Then, for all $\mathbf{x} \in (0, \infty)^p$, making use of the K -invariance with respect to $\mathbf{M}$, we have

$$F(\mathbf{M}(\mathbf{x})) = (\varphi \circ K)(\mathbf{M}(\mathbf{x})) = \varphi(K(\mathbf{M}(\mathbf{x}))) = \varphi(K(\mathbf{x})) = F(\mathbf{x}),$$

which completes the proof of (iii).

(iv) we omit similar argument. $\square$

Part (ii) of this result gives rise to the following extension.

3.1. General Complementary Process

Once we have a mean-type $\mathbf{M} : I^p \rightarrow I^p$ and a continuous and monotone mean $K : I^p \rightarrow I$ which is $\mathbf{M}$ -invariant let $\mathfrak{K}^+(\mathbf{M}, K)$ be the smallest family of mean-type mappings containing $\mathbf{M}$ which is closed under K -complementary averaging.

More precisely, for every $\mathbf{X} \in \mathfrak{K}^+(\mathbf{M}, K)$ and nonempty subset $S \subseteq \mathbb{N}_p$ we have $\mathbf{K}_S(\mathbf{X}) \in \mathfrak{K}^+(\mathbf{M}, K)$, too. We also define a family of means

$$\mathfrak{K}_0(\mathbf{M}, K) := \{[\mathbf{X}]_i : \mathbf{X} \in \mathfrak{K}^+(\mathbf{M}, K) \text{ and } i \in \mathbb{N}_p\}$$

Obviously using notions from Theorem 2 and Corollary 1 we have

$$\mathfrak{K}^+(\mathbf{M}, K) \supseteq \mathfrak{K}^+(\mathbf{K}_S(\mathbf{M}), K) \supseteq \mathfrak{K}^+(\mathbf{K}_S^*(\mathbf{M}), K)$$

Furthermore, we have the following.

Proposition 1. *Given an interval $I \subset \mathbb{R}$, $p \in \mathbb{N}$, and a mean-type mapping $\mathbf{M} := (M_1, \dots, M_p) : I^p \rightarrow I^p$ which is invariant with respect to some continuous and monotone mean $K : I^p \rightarrow I$. Then*

$$\min(M_1, \dots, M_p) \leq X \leq \max(M_1, \dots, M_p) \quad \text{for all } X \in \mathfrak{K}_0(\mathbf{M}, K).$$

Its inductive proof is obvious in view of Theorem 2 (moreover part).

Now we prove that complementary means preserve symmetry.

Proposition 2. *If a continuous and monotone mean $K: I^p \rightarrow I$ is invariant with respect to a mean-type mapping $\mathbf{M} := (M_1, \dots, M_p): I^p \rightarrow I^p$ such that all M_i -s are symmetric, then K and all means in $\mathfrak{K}_0(\mathbf{M}, K)$ are symmetric.*

Proof. Fix a nonconstant vector $x \in I^p$ and a permutation σ of $\mathbb{N}_p$. First observe that $K(x) = K \circ \mathbf{M}(x) = K \circ \mathbf{M}(x \circ \sigma) = K(x \circ \sigma)$, which implies that K is symmetric.

As the family $\mathfrak{K}_0(\mathbf{M}, K)$ is generating by complementing, we need to show that symmetry is preserved by a single complement. Therefore, it is sufficient to show that the mean $K_S(\mathbf{M})$ defined in Theorem 2 is symmetric. However, using the notions therein, we have

$$K \circ (K_S(\mathbf{M}))(x) = K(x) = K(x \circ \sigma) = K \circ (K_S(\mathbf{M}))(x \circ \sigma).$$

By monotonicity of K , if $K_S(\mathbf{M})(x) < K_S(\mathbf{M})(x \circ \sigma)$ we would have

$$K \circ (K_S(\mathbf{M}))(x) = K(x) < K \circ (K_S(\mathbf{M}))(x \circ \sigma)$$

contradicting the above equality. Similarly we exclude the case $K_S(\mathbf{M})(x) > K_S(\mathbf{M})(x \circ \sigma)$. Therefore $K_S(\mathbf{M})(x) = K_S(\mathbf{M})(x \circ \sigma)$ which, as x and σ were taken arbitrarily, yields the symmetry of $K_S(\mathbf{M})$. $\square$

4. An Applications to Beta-Type Means

Following [5], for a given $k \in \mathbb{N}$ we define a p -variable Beta-type mean $\mathcal{B}_p: \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ by

$$\mathcal{B}_p(x_1, \dots, x_p) := \left(\frac{px_1 \cdots x_p}{x_1 + \cdots + x_p} \right)^{\frac{1}{p-1}}.$$

This is a particular case of so-called biplanar-combinatoric means (Media biplana combinatoria) defined in Gini [6] and Gini-Zappa [7].

In order to formulate the next results, we adapt the notation that A , G and H are arithmetic, geometric and harmonic means of suitable dimension, respectively.

In [8], the invariance $G \circ (A, H) = G$, equivalent to the Pythagorean proportion, has been extended for arbitrary $p \geq 3$. In case $p = 3$ it takes the form $G \circ (A, F, H) = G$, where

$$F(x_1, x_2, x_3) =: \frac{x_2x_3 + x_3x_1 + x_1x_2}{x_1 + x_2 + x_3}, \quad x_1, x_2, x_3 > 0,$$

and $H \leq F \leq A$. Hence, making use of Corollary 1 with $p = 3$, $K = G$, $S = \{1, 2\}$ we obtain the following.

Remark 2. *For all x_1, x_2, x_3 , the following inequality holds*

$$H(x_1, x_2, x_3) \leq \mathcal{B}_3(x_1, x_2, x_3) \leq A(x_1, x_2, x_3),$$

and the inequalities are sharp for nonconstant vectors $x = (x_1, x_2, x_3) \in (0, \infty)^3$.

Passing to the main part of this section, first observe the following easy-to-see lemma.

Lemma 1. *Let $p \in \mathbb{N}$, $p \geq 2$. Then there exists exactly one mean $M: I^p \rightarrow I$ such that $G \circ (A, \underbrace{M, \dots, M}_{(p-1) \text{ times}}) = G$. Furthermore $M = \mathcal{B}_p$.*

Its simple proof which is a straightforward implication of Theorem 2 is omitted. Based on this lemma it is natural to define a mean-type mapping $\mathbf{B}: I^p \rightarrow I^p$ by $[\mathbf{B}]_1 := A$ and $[\mathbf{B}]_i := \mathcal{B}_p$ for all

$i \in \{2, \dots, p\}$. Then we have $G \circ \mathbf{B} = G$, which implies that the geometric mean is the unique $\mathbf{B}$ -invariant mean.

We are now going to establish the set $\mathfrak{K}^+(\mathbf{B}, G)$. It is quite easy to observe that all means in $\mathfrak{K}_0(\mathbf{B}, G)$ are of the form $\mathcal{H}_{p,\alpha}: I^p \rightarrow I$ ($\alpha \in \mathbb{R}$) given by

$$\mathcal{H}_{p,\alpha}(x_1, \dots, x_p) := (x_1 \cdots x_p)^{\frac{1-\alpha}{p}} \left(\frac{x_1 + \cdots + x_p}{p} \right)^\alpha$$

including $\mathcal{B}_p = \mathcal{H}_{p,-\frac{1}{p-1}}$. In the next lemma we show some elementary properties of the family $(\mathcal{H}_{p,\alpha})$.

Lemma 2. *Let $p \in \mathbb{N}$. Then*

1. $\mathcal{H}_{p,\alpha}$ is reflexive for all $\alpha \in \mathbb{R}$, that is $\mathcal{H}_{p,\alpha}(x, \dots, x) = x$ for all $x \in \mathbb{R}_+$,
2. $\mathcal{H}_{p,\alpha}$ is continuous for all $\alpha \in \mathbb{R}$ (as a p -variable function),
3. $\mathcal{H}_{p,\alpha}$ is a strict mean for all $\alpha \in [-\frac{1}{p-1}, 1]$,
4. $\mathcal{H}_{p,\alpha}$ is a symmetric function for all $\alpha \in \mathbb{R}$, that is $\mathcal{H}_{p,\alpha}(x \circ \sigma) = \mathcal{H}_{p,\alpha}(x)$ for all $x \in \mathbb{R}_+^p$ and a permutation σ of $\mathbb{N}_p$,
5. $\mathcal{H}_{p,1}$ and $\mathcal{H}_{p,0}$ are p -variable arithmetic and geometric means, respectively,
6. $\mathcal{H}_{p,\alpha}$ is increasing with respect to α , that is $\mathcal{H}_{p,\alpha}(x) < \mathcal{H}_{p,\beta}(x)$ for every nonconstant vector $x \in \mathbb{R}_+^p$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$.

Proof. By the definition of $\mathcal{H}_{p,\alpha}$ we can easily verify that (1), (2), (4) and (5) holds.

From now on fix a nonconstant vector $x = (x_1, \dots, x_p) \in \mathbb{R}_+^p$. By (4) we may assume without loss of generality that $x_1 \leq \dots \leq x_p$. Denote briefly

$$g := \sqrt[p]{x_1 \cdots x_p} \quad \text{and} \quad a := \frac{x_1 + \cdots + x_p}{p}.$$

By Cauchy inequality we have $x_1 < g < a < x_p$. Moreover, by the definition $\mathcal{H}_{p,\alpha}(x) = a^\alpha g^{1-\alpha}$. Thus, for all $\alpha < \beta$ we have

$$\mathcal{H}_{p,\beta}(x) = a^\beta g^{1-\beta} = a^\alpha g^{1-\alpha} \left(\frac{a}{g} \right)^{\beta-\alpha} > a^\alpha g^{1-\alpha} = \mathcal{H}_{p,\alpha}(x),$$

which completes the proof of (6). The only remaining part to be proved is (3). However, applying (6), it is sufficient to show that

$$\mathcal{H}_{p,1}(x) < \max(x) = x_p \quad \text{and} \quad \mathcal{H}_{p,-\frac{1}{p-1}}(x) > \min(x) = x_1.$$

By (5) we immediately obtain $\mathcal{H}_{p,1}(x) = a < x_p$. For the second part observe that

$$\mathcal{H}_{p,-\frac{1}{p-1}}(x) = \sqrt[p-1]{\frac{g^p}{a}} = \sqrt[p-1]{\frac{x_1 \cdots x_p}{a}} > \sqrt[p-1]{\frac{x_1 \cdots x_p}{x_p}} = \sqrt[p-1]{x_1 \cdots x_{p-1}} \geq x_1,$$

which completes the proof. $\square$

Now we generalize Lemma 1 to the following form

Lemma 3. *Let $p \in \mathbb{N}$, $p \geq 2$ and $\alpha \in \mathbb{R}^p$. Then $G \circ (\mathcal{H}_{p,\alpha_1}, \dots, \mathcal{H}_{p,\alpha_p}) = G$, if and only if $\alpha_1 + \cdots + \alpha_p = 0$.*

Its proof is obvious in view of the identity $G \circ (\mathcal{H}_{p,\alpha_1}, \dots, \mathcal{H}_{p,\alpha_p}) = \mathcal{H}_{p,\frac{1}{p}(\alpha_1 + \cdots + \alpha_p)}$. Having this proved, let us show the next important result.

Theorem 4. Let $p \geq 2$. Then

$$\begin{aligned} \mathfrak{K}^+((A, \underbrace{\mathcal{B}_p, \dots, \mathcal{B}_p}_{(p-1) \text{ times}}, G) \\ \subseteq \{(\mathcal{H}_{p,\alpha_1}, \dots, \mathcal{H}_{p,\alpha_p}) \mid \alpha_1, \dots, \alpha_p \in \mathbb{Q} \cap [-\frac{1}{p-1}, 1] \text{ and } \alpha_1 + \dots + \alpha_p = 0\}. \end{aligned}$$

In particular

$$\mathfrak{K}_0 = ((A, \mathcal{B}_p, \dots, \mathcal{B}_p), G) \subseteq \{\mathcal{H}_{p,\alpha} \mid \alpha \in \mathbb{Q} \cap [-\frac{1}{p-1}, 1]\}.$$

Proof. First observe that in view of Lemma 1 G is $\mathbf{B}$ -invariant, and thus the set $\mathfrak{K}^+(\mathbf{B}, G)$ is well-defined. Now denote briefly

$$\Lambda := \{(\mathcal{H}_{p,\alpha_1}, \dots, \mathcal{H}_{p,\alpha_p}) \mid \alpha_1, \dots, \alpha_p \in \mathbb{Q} \cap [-\frac{1}{p-1}, 1] \text{ and } \alpha_1 + \dots + \alpha_p = 0\}.$$

Obviously $\mathbf{B} \in \Lambda$, so it is sufficient to prove that Λ is closed with respect to G -complementary averaging. To this end take an arbitrary vector $(\alpha_1, \dots, \alpha_p)$ real numbers such that $\mathbf{H} := (\mathcal{H}_{p,\alpha_1}, \dots, \mathcal{H}_{p,\alpha_p}) \in \Lambda$ and a nonempty subset $S \subset \mathbb{N}_p$.

By Theorem 2 there exists exactly one mean $G_S(\mathbf{H}): I^p \rightarrow I$ such that G is $G_S(\mathbf{H})$ -invariant, where $G_S(\mathbf{H}): I^p \rightarrow I^p$ is given by

$$[G_S(\mathbf{H})]_i := \begin{cases} \mathcal{H}_{p,\alpha_i} & \text{for } i \in \mathbb{N}_p \setminus S, \\ G_S(\mathbf{H}) & \text{for } i \in S. \end{cases}$$

On the other hand, in view of Lemma 3 we obtain that G is invariant with respect to the mean-type mapping $\mathbf{H}_0: I^p \rightarrow I^p$ given by

$$[\mathbf{H}_0]_i := \begin{cases} \mathcal{H}_{p,\alpha_i} & \text{for } i \in \mathbb{N}_p \setminus S, \\ \mathcal{H}_{p,\beta} & \text{for } i \in S, \end{cases} \text{ where } \beta = \frac{1}{|S|} \sum_{i \in S} \alpha_i.$$

As G is both $G_S(\mathbf{H})$ -invariant and $\mathbf{H}_0$ -invariant we obtain $G_S(\mathbf{H}) = \mathcal{H}_{p,\beta}$, and consequently $G_S(\mathbf{H}) = \mathbf{H}_0$. Observe that $\mathbf{H}_0 \in \Lambda$ is a straightforward implication of the equality

$$\sum_{i \in \mathbb{N}_p \setminus S} \alpha_i + |S|\beta = \sum_{i \in \mathbb{N}_p} \alpha_i = 0.$$

Now we show that $\beta \in \mathbb{Q} \cap [-\frac{1}{p-1}, 1]$, which would complete the proof. However, this is easy in view of the definition of β and the analogous property $\alpha_i \in \mathbb{Q} \cap [-\frac{1}{p-1}, 1]$, which is valid for all $i \in S$. $\square$

Remark 3. Let us emphasize that inclusions in the above theorem are strict. More precisely, we can prove by simple induction that the denominator of α in irreducible form has no prime divisors greater than p .

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