

Article

Repdigits as Product of Fibonacci and Tribonacci Numbers

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Abstract: In this paper, we study the problem of the explicit intersection of two sequences. More specifically, we find all repdigits (i.e., numbers with only one repeated digit in its decimal expansion) which can be written as the product of a Fibonacci by a Tribonacci number (both with the same indexes). To work on this problem, our approach is to combine lower bounds from the Baker's theory with reduction methods (based on the theory of continued fractions) due to Dujella and Pethö.

Keywords: k -generalized Fibonacci numbers; linear forms in logarithms; reduction method

MSC: 11A63; 11B37; 11B39; 11J86

1. Introduction

Before starting with the main problem of this paper, we recall some nomenclature and symbols for the convenience of the reader:

The Fibonacci sequence $(F_n)_n$ is defined by the recurrence

$$F_{n+1} = F_n + F_{n-1}, \quad (1)$$

with initial values $F_0 = 0$ and $F_1 = 1$.

The Tribonacci numbers $(T_n)_n$ are defined by the third-order recurrence

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}, \quad (2)$$

with initial values $T_0 = 0$ and $T_1 = T_2 = 1$.

A repdigit (short for “repeated digit”) is also a number of the form

$$a \left(\frac{10^\ell - 1}{9} \right), \quad (3)$$

where $\ell \geq 1$ and $a \in [1, 9]$ (here, for integers $a < b$, we set $[a, b] = \{a, a + 1, \dots, b\}$), that is, a number with only one distinct digit in its decimal expansion.

The main subject of this work is the Diophantine equations. It is almost unnecessary to stress that these objects play an important role in the Number Theory—for example, the equations $x^2 + y^2 = z^2$ (Pythagoras equation), $x^2 - Dy^2 = c$ (Pell equation), and $x^n + y^n = z^n$ (Fermat equation) intrigued several mathematicians at different times. It is also important to notice that their studies contributed strongly to the advance of mathematics. There are many articles that address Diophantine equations concerning Fibonacci and Lucas numbers (see, e.g., [1–8]). The linear forms in logarithms, which were

probably firstly used for solving Diophantine equations in Dujella and Jadrijević [9], have proved to be a very effective tool for finding solutions to all these equations.

Recently, many authors have been interested in solving Diophantine equations involving repdigits (their sums, products concatenations, etc.) and some special types of linear recurrences (usually their product, sums, etc.), where we refer the reader to [10–18] and references therein.

We point out that Luca [19] and Marques [20] proved that the largest repdigits in the Fibonacci and Tribonacci sequence are $F_{10} = 55$ and $T_8 = 44$, respectively.

The aim of this paper is to continue the study of Diophantine problems involving recurrence sequences and repdigits. More precisely, we search for repdigits which are the product of Fibonacci and Tribonacci numbers with the same index. Our main result is the following:

Theorem 1. *The only solutions of the Diophantine equation*

$$F_n T_n = a \left(\frac{10^\ell - 1}{9} \right), \quad (4)$$

in positive integers n, a and ℓ , with $a \in [1, 9]$, are

$$(n, \ell, a) \in \{(1, 1, 1), (2, 1, 1), (3, 1, 4)\}. \quad (5)$$

2. Auxiliary Results

We recall a well-known non-recursive formula for generating Fibonacci numbers. *Binet's formula* asserts that:

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}, \quad (6)$$

where $\phi = (1 + \sqrt{5})/2$ (the *golden number*). With this formula, we can deduce that:

$$\phi^{n-2} \leq F_n \leq \phi^{n-1}, \text{ for all } n \geq 1. \quad (7)$$

It is also possible to infer that

$$F_n = \frac{\phi^n}{\sqrt{5}} + \nu, \quad (8)$$

where $|\nu| \leq 1/\sqrt{5}$.

Spickerman [21] (in 1982) found the following “Binet-like” formula for Tribonacci numbers (see also [22] (pp. 527–536) for some properties of this sequence):

$$T_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1}, \text{ for all } n \geq 1, \quad (9)$$

where α, β, γ are the roots of $x^3 - x^2 - x - 1 = 0$. More precisely, we have

$$\alpha = \frac{1}{3} + \frac{1}{3}(19 - 3\sqrt{33})^{1/3} + \frac{1}{3}(19 - 3\sqrt{33})^{1/3}, \quad (10)$$

$$\beta = \frac{1}{3} - \frac{1}{6}(1 + i\sqrt{3})(19 - 3\sqrt{33})^{1/3} - \frac{1}{6}(1 - i\sqrt{3})(19 + 3\sqrt{33})^{1/3}, \quad (11)$$

$$\gamma = \frac{1}{3} - \frac{1}{6}(1 - i\sqrt{3})(19 - 3\sqrt{33})^{1/3} - \frac{1}{6}(1 + i\sqrt{3})(19 + 3\sqrt{33})^{1/3}. \quad (12)$$

Another very useful formula provided by Spickermann is

$$T_n = \left\lfloor \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)} \alpha^n \right\rfloor, \quad (13)$$

where, as usual, $\lfloor x \rfloor$ denotes the nearest integer to x (the so-called *Nint* function). In particular, the formula

$$T_n = \alpha' \alpha^n + \eta, \quad (14)$$

holds, where $|\eta| < 1/2$ and $\alpha' := \alpha/(\alpha - \beta)(\alpha - \gamma)$. Moreover, since $\alpha^{-2} < \alpha' = 0.33622 \dots < \alpha$, the previous identity implies that

$$\alpha^{n-3} < T_n < \alpha^{n+2}, \text{ for all } n \geq 1. \quad (15)$$

To prove Theorem 1, we will use Baker's theory. Specifically, we shall use a lower bound for a linear form in three logarithms:

Lemma 1. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ be algebraic numbers and let b_1, b_2, b_3 be non-zero integer numbers. Define

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3. \quad (16)$$

Let $D = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}]$ (degree of field extension) and let A_1, A_2, A_3 be real numbers, such that

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}, \text{ for } j \in \{1, 2, 3\}. \quad (17)$$

Take

$$B \geq \max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\}. \quad (18)$$

If $\Lambda \neq 0$, then

$$\log |\Lambda| \geq -C_1 D^2 A_1 A_2 A_3 \log(1.5eDB \log(eD)), \quad (19)$$

where

$$C_1 = 6750000 \cdot e^4 (20.2 + \log(3^{5.5} D^2 \log(eD))). \quad (20)$$

The proof of this result can be found in [23].

In the previous statement, the *logarithmic height* of a t -degree algebraic number α is defined by

$$h(\alpha) = \frac{1}{t} \left(\log |a| + \sum_{j=1}^t \log \max\{1, |\alpha^{(j)}|\} \right), \quad (21)$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$), and $(\alpha^{(j)})_{1 \leq j \leq t}$ are the algebraic conjugates of α . The next lemma provides some useful properties of this function (we refer to [24] for the proof of the following facts):

Lemma 2. We have

- i. $h(xy) \leq h(x) + h(y)$;
- ii. $h(xy) \leq h(x) + h(y) + \log 2$;
- iii. $h(\alpha^r) = |r| \cdot h(\alpha)$, for all $r \in \mathbb{Q}$.

Our last tool is a reduction method provided by a variant of the well-known Baker-Davenport lemma, proved by Dujella and Pethő. For $x \in \mathbb{R}$, set $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\} = |x - \lfloor x \rfloor|$ for the distance from x to the nearest integer. We refer the reader to Lemma 5 in [25] for the proof of the following lemma.

Lemma 3. For a positive integer M , let p/q be a convergent of the continued fraction of $\gamma \notin \mathbb{Q}$, such that $q > 6M$, and let μ, A , and B be real numbers, with $A > 0$ and $B > 1$. If the number $\epsilon = \|\mu q\| - M\|\gamma q\|$ is positive, then there is no solution to the Diophantine inequality

$$0 < m\gamma - n + \mu < A \cdot B^{-m} \quad (22)$$

in integers $m, n > 0$ with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m < M. \quad (23)$$

Now, we are ready to prove the main theorem.

3. The Proof of Theorem 1

3.1. Finding an Upper Bound for n and ℓ

By using Equations (8) and (14) in Equation (4), we have

$$\left(\frac{\phi^n}{\sqrt{5}} + \nu\right)(\alpha'\alpha^n + \eta) = a\left(\frac{10^\ell - 1}{9}\right). \quad (24)$$

After some manipulations, we arrive at

$$\left|\frac{\alpha'(\alpha\phi)^n}{\sqrt{5}} - a\frac{10^\ell}{9}\right| < 1.6\alpha^n, \quad (25)$$

where we used that $|\nu| \leq 1/\sqrt{5}$, $|\eta| < 1/2$ and $\phi < 1.7 < \alpha$. On dividing through by $\alpha'(\alpha\phi)^n/\sqrt{5}$, we get

$$\left|1 - \frac{a\sqrt{5}10^\ell}{9\alpha'}\right| < \frac{12}{\phi^n}, \quad (26)$$

where we used that $\alpha' > 0.3$. Define

$$\Lambda = \ell \log 10 - n \log(\phi\alpha) + \log \theta_a, \quad (27)$$

where $\theta_a := a\sqrt{5}/9\alpha'$ (for $a \in [1, 9]$). Then, Equation (26) can be rewritten as

$$|e^\Lambda - 1| < \frac{12}{\phi^n}. \quad (28)$$

First, we claim that $\Lambda \neq 0$. To obtain a contradiction, suppose that $\Lambda = 0$, and thus, $10^\ell \theta_a = (\phi\alpha)^n$, and so $\phi^{2n} \in \mathbb{Q}(\alpha)$. Since $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$, then ϕ^{2n} is either a rational or a 3-degree algebraic number. However, ϕ is a quadratic algebraic number, and since $\mathbb{Q}(\phi^n) \subseteq \mathbb{Q}(\phi)$, then the degree of ϕ^{2n} is either 1 or 2. So, we conclude that $\phi^{2n} \in \mathbb{Q}$, which is an absurd, since (by the Binomial theorem) $\phi^{2n} = A_n + B_n\sqrt{5}$, for some positive rational numbers A_n and B_n . Therefore, we have that $\Lambda \neq 0$.

If $\Lambda > 0$, then $\Lambda < e^\Lambda - 1 < 12\phi^{-n}$ (see Equation (28)). If $\Lambda < 0$, then $1 - e^{-|\Lambda|} = |e^\Lambda - 1| < 12\phi^{-n}$. Thus, we get

$$|\Lambda| < e^{|\Lambda|} - 1 < \frac{\phi^{-n}}{1 - \phi^{-n}} < \phi^{-n+1}. \quad (29)$$

Hence, we have $|\Lambda| < 12\phi^{-n+1}$. Therefore,

$$\log |\Lambda| < -(n-1) \log \phi + \log 12. \quad (30)$$

Now, we are in the position to apply Lemma 1. For that, take

$$\alpha_1 = 10, \alpha_2 = \phi\alpha, \alpha_3 = \theta_a, b_1 = \ell, b_2 = -n, b_3 = 1. \quad (31)$$

Since $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\alpha, \phi)$, then $D \leq 6$, and so $C_1 < 1.4 \cdot 10^{10}$.

By using the properties of the logarithm height, we deduce that $h(\alpha_1) = h(10) = \log 10$ and $h(\alpha_2) \leq h(\phi) + h(\alpha) = (\log \phi)/2 + (\log \alpha)/3 < 5(\log 2)/6$. Additionally,

$$h(\alpha_3) = h(\theta_a) \leq h(a) + h(\sqrt{5}) + h(9) + h(\alpha') < 8.6, \quad (32)$$

where we used the definition of α' together with the fact that $h(\alpha) = h(\beta) = h(\gamma)$ (since they are algebraic conjugates). Thus, we can take

$$A_1 = 14, A_2 = 4 \text{ and } A_3 = 52. \quad (33)$$

If $\ell \geq 4$, we have

$$\max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\} = \max\{\ell, 2n/7\}. \quad (34)$$

Now, we combine the bounds in Equations (7) and (15) to derive

- $\phi^{n-2}\alpha^{n-3} < F_n T_n = a(10^\ell - 1)/9 < 10^\ell \implies n < 4.8\ell + 2$
- $\phi^{n-1}\alpha^{n+2} > F_n T_n = a(10^\ell - 1)/9 > 10^{\ell-1} \implies 4.7\ell - 6 < n$.

In conclusion, we have

$$4.7\ell - 6 < n < 4.8\ell + 2. \quad (35)$$

However, $4.7\ell - 6 \geq \ell$ for $\ell > 1$, and so, we can choose $B := n$. Hence, by Lemma 1, we get

$$\log |\Lambda| > -1.5 \times 10^{15} \log(69n). \quad (36)$$

By combining the estimates Equations (30) and (36), we obtain

$$n < 2.9 \times 10^{15} \log(69n). \quad (37)$$

From this inequality, we deduce that $n < 1.3 \times 10^{17}$, and by the estimate $4.7\ell - 6 < n$, we infer that $\ell < 3 \times 10^{16}$.

3.2. Reducing the Bound

Now, we need to reduce the upper bound for n and ℓ . For that, we may suppose, with no loss of generality, that $\Lambda > 0$ (the other case is simply a mimic, considering that $0 < \Lambda' = -\Lambda$).

Since $0 < \Lambda < 12\phi^{-n+1}$, we obtain

$$0 < \ell \log 10 - n \log(\phi\alpha) + \log \theta_a < 1536 \cdot 12^{-\ell}. \quad (38)$$

On dividing through by $\log(\phi\alpha)$, we have

$$0 < \ell\gamma' - n + \mu_a < 1448 \cdot 12^{-\ell}, \quad (39)$$

with $\gamma' := \log 10 / \log(\phi\alpha)$ and $\mu_a := \log \theta_a / \log(\phi\alpha)$.

Clearly, γ' is an irrational number (because α and ϕ are multiplicatively independent). Therefore, we shall denote p_n/q_n as the n -th convergent of the (infinite) continued fraction of γ' .

To reduce our bound on ℓ , we shall apply Lemma 3. For that, we choose $M = 3 \times 10^{16}$, and so

$$\frac{p_{40}}{q_{40}} = \frac{1014377782556875091}{480446934735032799} \quad (40)$$

is enough of an approximant of γ' to fulfill the hypotheses of that lemma. Indeed, $q_{40} = 480,446,934,735,032,799 > 6M$. Additionally, by defining

$$\epsilon_a := \|\mu_a q_{40}\| - M\|\gamma' q_{40}\| \quad (41)$$

for $a \in [1, 9]$, we have that $\min_{a \in [1, 9]} \epsilon_a = \epsilon_9 = 0.0089973 \dots$ (here, we used *Mathematica* software (Wolfram Mathematica version 12, Wolfram Research of Champaign, Illinois, USA), see Appendix A).

Hence, the hypotheses of Lemma 3 are fulfilled for the choice of $A = 1448$ and $B = 12$. Thus, there is no solution of the Diophantine inequality in Equation (39) for ℓ belonging to the range

$$\left[\left\lfloor \frac{\log(Aq_{40}/\epsilon_a)}{\log B} \right\rfloor + 1, M \right] = [22, 3 \times 10^{16}]. \quad (42)$$

Since $\ell < M$, then $\ell \leq 21$ and so $n \leq 103$. Thus, we prepare a *Mathematica* routine which shows that the solutions of $F_n T_n = a(10^\ell - 1)/9$, in the range $n \in [1, 103]$, $\ell \in [1, 22]$ and $a \in [1, 9]$, are

$$(n, \ell, a) \in \{(1, 1, 1), (2, 1, 1), (3, 1, 4)\}. \quad (43)$$

This completes the proof. $\square$

4. Conclusions

In this paper, we solved the Diophantine equation $F_n T_n = a(10^\ell - 1)/9$, where $(F_n)_n$ and $(T_n)_n$ are the Fibonacci and Tribonacci sequences, respectively, in positive integers n, ℓ and a , with $a \in [1, 9]$. In other words, we found all repdigits (i.e., positive integers with only one distinct digit in its decimal expansion) which can be written as a product of a Fibonacci number and a Tribonacci number (both with the same index). In particular, we proved that the only repdigits with the desired property are the trivial ones, that is, those with only one digit ($\ell = 1$). To prove this result, we combined the theory of lower bounds for linear forms in the logarithm of algebraic numbers (from Baker's theory) with reduction methods from Diophantine approximation (based on the theory of continued fractions) due to Dujella and Pethö.

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Appendix A

In this section, we shall provide the main routines and commands (in *Mathematica* software) which were used during the preparation of this work.

Appendix A.1. Tribonacci Sequence, Polynomial, and Roots

The n -th Tribonacci number:

```
t[n_] := SeriesCoefficient[Series[x/(1 - Sum[x^j, {j, 1, 3}]), {x, 0, 5000}], n]
```

The characteristic polynomial of $(T_n)_n$:

$f[x_] := x^3 - x^2 - x - 1$

The roots α , β , and γ :

$\alpha := x /. \text{NSolve}[f[x], x, 140][[3]]$

$\beta := x /. \text{NSolve}[f[x], x, 140][[2]]$

$\gamma := x /. \text{NSolve}[f[x], x, 140][[1]]$

Appendix A.2. The Constants

The constants α' , θ_a , γ' , and μ_a :

$\alpha' := \alpha / ((\alpha - \beta) * (\alpha - \gamma))$

$\theta[a_] := a * \text{Sqrt}[5] / (9 * \alpha')$

$\gamma' := \text{Log}[10] / \text{Log}[\alpha * \text{GoldenRatio}]$

$\mu[a_] := \text{Log}[\theta[a]] / \text{Log}[\alpha * \text{GoldenRatio}]$

Appendix A.3. Functions and Routines

The n -th denominator of the continued fraction expansion of x :

$\text{DeFrac}[x_, n_] := \text{Last}[\text{Denominator}[\text{Convergents}[x, n]]]$

The distance to the nearest integer:

$\text{Near}[x_] := \text{Min}[\text{Abs}[x - \text{Floor}[x]], \text{Abs}[\text{Ceiling}[x] - x]]$

The numbers ϵ_a , for $a \in [1, 9]$

$e[a_] :=$

$\text{N}[\text{Near}[\mu[a] * \text{DeFrac}[\gamma', 40]] -$

$3 * 10^{(16)} * \text{Near}[\gamma' * \text{DeFrac}[\gamma', 40]], 5]$

The routine for searching solutions of the main equation in the obtained range:

$\text{Catch}[\text{Do}[\text{If}[\text{Fibonacci}[n] * t[n] == a * (10^1 - 1) / 9, \text{Print}[\{n, l, a\}]], \{n, 1, 103\}, \{l, 1, 22\}, \{a, 1, 9\}]]$

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