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Article

Martingale Convergence Theorem for the Conditional Intuitionistic Fuzzy Probability

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Abstract: For the first time, the concept of conditional probability on intuitionistic fuzzy sets was introduced by K. Lendelová. She defined the conditional intuitionistic fuzzy probability using a separating intuitionistic fuzzy probability. Later in 2009, V. Valenčáková generalized this result and defined the conditional probability for the MV-algebra of inuitionistic fuzzy sets using the state and probability on this MV-algebra. She also proved the properties of conditional intuitionistic fuzzy probability on this MV-algebra. B. Riečan formulated the notion of conditional probability for intuitionistic fuzzy sets using an intuitionistic fuzzy state. We use this definition in our paper. Since the convergence theorems play an important role in classical theory of probability and statistics, we study the martingale convergence theorem for the conditional intuitionistic fuzzy probability. The aim of this contribution is to formulate a version of the martingale convergence theorem for a conditional intuitionistic fuzzy probability induced by an intuitionistic fuzzy state **m**. We work in the family of intuitionistic fuzzy sets introduced by K. T. Atanassov as an extension of fuzzy sets introduced by L. Zadeh. We proved the properties of the conditional intuitionistic fuzzy probability.

Keywords: intuitionistic fuzzy event; intuitionistic fuzzy observable; intuitionistic fuzzy state; product; conditional intuitionistic fuzzy probability; martingale convergence theorem

MSC: 03B52; 60A86; 60G48

1. Introduction

The notion of intuitionistic fuzzy sets was introduced by K. T. Atanassov in [1,2]. In this paper we work with the family of intuitionistic fuzzy events given by

$$\mathcal{F} = \{(\mu_A, \nu_A)\,; \mu_A + \nu_A \leq 1_\Omega\},$$

where μ_A, ν_A are $\mathcal{S}$-measurable functions, $\mu_A, \nu_A : \Omega \to [0,1]$.

In [3] K. Lendelová introduced the conditional intuitionistic fuzzy probability $\mathbf{p}\big((a_1, a_2) \mid y\big)$ as a couple of two Borel measurable functions $\mathbf{p}^\flat\big((a_1, a_2) \mid y\big)$, $\mathbf{p}^\sharp\big((a_1, a_2) \mid y\big) : R \to R$ such that

$$\left[\int\limits_B \mathbf{p}^\flat(a_1|y^\flat)\, \mathrm{d}\mathcal{P}^\flat, \int\limits_B \mathbf{p}^\sharp(1-a_2|y^\sharp)\, \mathrm{d}\mathcal{P}^\sharp\right] = \mathcal{P}\big((a_1, a_2) \cdot y(B)\big)$$

for each $B \in \mathcal{B}(R)$, where $\mathcal{P}$ is a separating intuitionistic fuzzy probability given by $\mathcal{P}\big((a_1, a_2)\big) = \big[\mathcal{P}^\flat(a_1), 1 - \mathcal{P}^\sharp(a_2)\big]$, the functions $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{T} \to [0,1]$ are probabilities, $\mathcal{T}$ is Lukasiewicz tribe and $a_1, a_2 \in \mathcal{T}$ with $a_1 + a_2 \leq 1$.

Later in [4] V. Valenčáková defined a conditional probability $p(A \mid y)$ on a family $\mathcal{M} = \{(\mu_A, \nu_A)\,; \mu_A, \nu_A : \Omega \to [0,1], \mu_A, \nu_A$ are $\mathcal{S}$-measurable$\}$ using an MV-state $m : \mathcal{M} \to [0,1]$ as a Borel measurable function such that

$$\int_C p(A \mid y)\, dm_y = m(A \cdot y(C))$$

for each $C \in \mathcal{B}(R)$. Here, $A \in \mathcal{M}$ and $y : \mathcal{B}(R) \to \mathcal{M}$ are MV-observable. The algebraic system $(\mathcal{M}, 0_{\mathcal{M}}, 1_{\mathcal{M}}, \neg, \oplus, \odot, \cdot)$ is an MV-algebra with product, $1_{\mathcal{M}} = (1_\Omega, 0_\Omega)$, $0_{\mathcal{M}} = (0_\Omega, 1_\Omega)$, $\neg(\mu_A, \nu_A) = (1_\Omega - \mu_A, 1_\Omega - \nu_A)$, $(\mu_A, \nu_A) \oplus (\mu_B, \nu_B) = ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega))$, $(\mu_A, \nu_A) \odot (\mu_B, \nu_B) = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))$, $(\mu_A, \nu_A) \cdot (\mu_B, \nu_B) = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B)$. Here, the corresponding ℓ-group is $(\mathcal{M}, +, \leq)$ with the neutral element $\mathbf{0} = (0_\Omega, 1_\Omega)$, $(\mu_A, \nu_A) + (\mu_B, \nu_B) = (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega)$, $(\mu_A, \nu_A) \leq (\mu_A, \nu_A) \iff \mu_A \leq \mu_B, \nu_A \geq \nu_B$ and with the lattice operations $(\mu_A, \nu_A) \vee (\mu_B, \nu_B) = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$, $(\mu_A, \nu_A) \wedge (\mu_B, \nu_B) = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$. Since $\mathcal{F} \subset \mathcal{M}$ and by [5] to each intuitionistic fuzzy state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one MV-state $m : \mathcal{M} \to [0,1]$ such that $m \mid \mathcal{F} = \mathbf{m}$, V. Valenčáková in [4] defined a conditional intuitionistic fuzzy probability of an intuitionistic fuzzy event $\mathbf{A} \in \mathcal{F}$ wit respect to an intuitionistic fuzzy observable $x : \mathcal{B}(R) \to \mathcal{F}$ with help of a conditional probability defined on $\mathcal{M}$. She proved the properties of a conditional probability on $\mathcal{M}$, too.

In [6] B. Riečan introduced the conditional intuitionistic fuzzy probability $\mathbf{p}(\mathbf{A} \mid x)$ as a Borel measurable function f (i.e., $B \in \mathcal{B}(R) \Longrightarrow f^{-1}(B) \in \mathcal{B}(R)$) such that

$$\int_B \mathbf{p}(\mathbf{A} \mid x)\, d\mathbf{m}_x = \mathbf{m}(\mathbf{A} \cdot x(B))$$

for each $B \in \mathcal{B}(R)$, where $\mathbf{m} : \mathcal{F} \to [0,1]$ is the intuitionistic fuzzy state, $\mathbf{A} \in \mathcal{F}$ is an intuitionistic fuzzy event and $x : \mathcal{B}(R) \to \mathcal{F}$ is an intuitionistic fuzzy observable.

The convergence theorems play an important role in the theory of probability and statistics and in its application (see [7–9]). In [10–12] the authors studied the martingale measures in connection with fuzzy approach in financial area. They used a geometric Levy process, the Esscher transformed martingale measures and the minimal L^p equivalent martingale measure on the fuzzy numbers for an option pricing. A practical use of results is a good motivation for studying a theory of martingales. In this paper, we formulate the modification of the martingale convergence theorem for the conditional intuitionistic fuzzy probability using the intuitionistic fuzzy state $\mathbf{m}$. As a method, we use a transformation of an intuitionistic probability space to the Kolmogorov probability space.

The paper is organized as follows: Section 2 includes the basic notions from intuitionistic fuzzy probability theory as an intuitionistic fuzzy event, an intuitionistic fuzzy state, an intuitionistic fuzzy observable and a joint intuitionistic fuzzy observable. In Section 3 we present a definition of a conditional intuitionistic fuzzy probability using an intuitionistic fuzzy state and we prove its properties. In Section 3, we formulate a martingale convergence theorem for a conditional intuitionistic fuzzy probability. Last section contains concluding remarks and a future research.

We note that in the whole text we use a notation IF as an abbreviation for intuitionistic fuzzy.

2. Basic Notions of the Intuitionistic Fuzzy Probability Theory

In this section we recall the definitions of basic notions connected with IF-probability theory (see [13–15]).

Definition 1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.*

Definition 2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. By an IF-event we mean an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ is denoted by $\mathcal{F}$, $\mu_A : \Omega \longrightarrow [0,1]$ is called *the membership function* and $\nu_A : \Omega \longrightarrow [0,1]$ is called *the non-membership function*.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Lukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\begin{aligned} \mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)) \end{aligned}$$

and the partial ordering is given by

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

In the IF-probability theory (see [6]) we use the notion of *state* instead of the notion of probability.

Definition 3. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;
(ii) *if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;*
(iii) *if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e., $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.*

One of the most useful results in the IF-state theory is the following representation theorem ([16]):

Theorem 1. *To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that*

$$\mathbf{m}(\mathbf{A}) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega \nu_A dP\right)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

Proof. In [16] *Theorem*. □

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a,b) = \{x \in R : a \leq x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called *the σ-algebra of Borel sets*. Its elements are called Borel sets.

Definition 4. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;
(ii) *if $A \cap B = \emptyset$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;*
(iii) *if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.*

If we denote $x(A) = \left(x^\flat(A), 1_\Omega - x^\sharp(A)\right)$ for each $A \in \mathcal{B}(R)$, then $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S} - \text{measurable}\}$.

Remark 1. *Sometimes we need to work with n-dimensional IF-observable $x : \mathcal{B}(R^n) \to \mathcal{F}$ defined as a mapping with the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A \cap B = \emptyset$, $A, B \in \mathcal{B}(R^n)$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$*;*
(iii) *if* $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$ *for each* $A, A_n \in \mathcal{B}(R^n)$.

If $n = 1$ *we simply say that* x *is an IF-observable.*

Similarly as in the classical case the following theorem can be proved (see [6,17]).

Theorem 2. *Let* $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *be an IF-observable,* $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ *be an IF-state. Define the mapping* $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ *by the formula*

$$\mathbf{m}_x(C) = \mathbf{m}(x(C)).$$

Then $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ *is a probability measure.*

Proof. In [17] *Proposition 3.1.* □

In [3] we introduced the notion of product operation on the family of IF-events $\mathcal{F}$ as follows:

Definition 5. *We say that a binary operation* $\cdot$ *on* $\mathcal{F}$ *is a product if it satisfies the following conditions:*

(i) $(1_\Omega, 0_\Omega) \cdot (a_1, a_2) = (a_1, a_2)$ *for each* $(a_1, a_2) \in \mathcal{F}$*;*
(ii) *the operation* $\cdot$ *is commutative and associative;*
(iii) *if* $(a_1, a_2) \odot (b_1, b_2) = (0_\Omega, 1_\Omega)$ *and* $(a_1, a_2), (b_1, b_2) \in \mathcal{F}$, *then* $(c_1, c_2) \cdot \big((a_1, a_2) \oplus (b_1, b_2)\big) = \big((c_1, c_2) \cdot (a_1, a_2)\big) \oplus \big((c_1, c_2) \cdot (b_1, b_2)\big)$ *and* $\big((c_1, c_2) \cdot (a_1, a_2)\big) \odot \big((c_1, c_2) \cdot (b_1, b_2)\big) = (0_\Omega, 1_\Omega)$ *for each* $(c_1, c_2) \in \mathcal{F}$*;*
(iv) *if* $(a_{1n}, a_{2n}) \searrow (0_\Omega, 1_\Omega)$, $(b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$ *and* $(a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F}$, *then* $(a_{1n}, a_{2n}) \cdot (b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$.

In the following theorem is the example of product operation for IF-events.

Theorem 3. *The operation* $\cdot$ *defined by*

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$$

for each $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$ *is a product operation on* $\mathcal{F}$.

Proof. In [3] *Theorem 1.* □

In [15] B. Riečan defined the notion of a joint IF-observable as follows:

Definition 6. *Let* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *be two IF-observables. The joint IF-observable of the IF-observables* x, y *is a mapping* $h : \mathcal{B}(R^2) \to \mathcal{F}$ *satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$*;*
(ii) *if* $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$, *then* $h(A \cup B) = h(A) \oplus h(B)$ *and* $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$*;*
(iii) *if* $A, A_1, \ldots \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$, *then* $h(A_n) \nearrow h(A)$*;*
(iv) $h(C \times D) = x(C) \cdot y(D)$ *for each* $C, D \in \mathcal{B}(R)$.

Theorem 4. *For each two IF-observables* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *there exists their joint IF-observable.*

Proof. In [15] *Theorem 3.3.* □

Remark 2. *The joint IF-observable of IF-observables* x, y *from Definition 6 are two-dimensional IF-observables.*

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables, as follows:

Definition 7. *Let $x_1, \dots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be IF-observables, h_n be their joint IF-observable and let $g_n : R^n \to R$ be a Borel measurable function. Then the IF-observable $g_n(x_1, \dots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ is given by the formula*

$$g_n(x_1, \dots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$.

3. Conditional Intuitionistic Fuzzy Probability

In [6] B. Riečan defined the conditional probability for IF-case. He was inspired by classical case, in which a conditional probability (of A with respect to B) is the real number $P(A \mid B)$ such that

$$P(A \cap B) = P(B) \cdot P(A \mid B).$$

An alternative way of defining the conditional probability is

$$P(A \cap B) = \int_B P(A \mid B)\, dP.$$

The number $P(A \mid B)$ can be regarded as a constant function. The constant functions are measurable with respect to the σ-algebra $\mathcal{S}_0 = \{\emptyset, \Omega\}$.

Generally, $P(A \mid \mathcal{S}_0)$ can be defined for any σ-algebra $\mathcal{S}_0 \subset \mathcal{S}$ as an $\mathcal{S}_0$-measurable function such that

$$P(A \cap C) = \int_C P(A \mid \mathcal{S}_0)\, dP,\ C \in \mathcal{S}_0.$$

If $\mathcal{S}_0 = \mathcal{S}$, then we can put $P(A \mid \mathcal{S}_0) = \chi_A$, since χ_A is $\mathcal{S}_0$-measurable and

$$\int_C \chi_A\, dP = P(A \cap C).$$

An important example of $\mathcal{S}_0$ is the family of all pre-images of a random variable $\xi : \Omega \to R$:

$$\mathcal{S}_0 = \{\xi^{-1}(B);\ B \in \sigma(\mathcal{J})\}.$$

In this case we write $P(A \mid \mathcal{S}_0) = P(A \mid \xi)$, hence

$$\int_C P(A \mid \xi)\, dP = P(A \cap C),\ C = \xi^{-1}(B),\ B \in \sigma(\mathcal{J}).$$

By the transformation formula,

$$P\big(A \cap \xi^{-1}(B)\big) = \int_{\xi^{-1}(B)} g \circ \xi\, dP = \int_B g\, dP_\xi,\ B \in \sigma(\mathcal{J}).$$

B. Riečan in [6] used this formulation for the IF-case to define the conditional IF-probability:

Definition 8. *Let $y : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable, $\mathbf{A} \in \mathcal{F}$. Then the conditional IF-probability $\mathbf{p}(\mathbf{A} \mid y) = f$ is a Borel measurable function (i.e., $B \in \mathcal{B}(R) \Longrightarrow f^{-1}(B) \in \mathcal{B}(R)$) such that*

$$\int_B \mathbf{p}(\mathbf{A} \mid y)\, d\mathbf{m}_y = \mathbf{m}\big(\mathbf{A} \cdot y(B)\big)$$

for each $B \in \mathcal{B}(R)$.

Now we prove the properties of the conditional IF-probability.

Theorem 5. *Let $\mathcal{F}$ be family of IF-events, $\mathbf{A} \in \mathcal{F}$, and $y : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. Then $\mathbf{p}(\mathbf{A}|y)$ has the following properties:*

(i) $\mathbf{p}\big((0_\Omega, 1_\Omega)|y\big) = 0$, $\mathbf{p}\big((1_\Omega, 0_\Omega)|y\big) = 1$ *hold* $\mathbf{m}_y$*-almost everywhere;*

(ii) $0 \le \mathbf{p}(\mathbf{A}|y) \le 1$ *holds* $\mathbf{m}_y$*-almost everywhere;*

(iii) *if* $\bigodot_{i=1}^{\infty} \mathbf{A}_i = (0_\Omega, 1_\Omega)$, *then* $\mathbf{p}\Big(\bigoplus_{i=1}^{\infty} \mathbf{A}_i \Big| y\Big) = \sum_{i=1}^{\infty} \mathbf{p}(\mathbf{A}_i|y)$ *holds* $\mathbf{m}_y$*-almost everywhere;*

(iv) *if* $\mathbf{A}_n \nearrow \mathbf{A}$, *then the convergence* $\mathbf{p}(\mathbf{A}_n|y) \nearrow \mathbf{p}(\mathbf{A}|y)$ *holds* $\mathbf{m}_y$*-almost everywhere.*

Proof. By *Definition 8* we have $\mathbf{m}\big(\mathbf{A} \cdot y(B)\big) = \int_B \mathbf{p}(\mathbf{A}|y)\, d\mathbf{m}_y$.

(i) If $\mathbf{A} = (0_\Omega, 1_\Omega)$, then $\mathbf{m}\big((0_\Omega, 1_\Omega) \cdot y(B)\big) = \mathbf{m}((0_\Omega, 1_\Omega)) = 0 = \int_B 0\, d\mathbf{m}_y$. If $\mathbf{A} = (1_\Omega, 0_\Omega)$, then $\mathbf{m}\big((1_\Omega, 0_\Omega) \cdot y(B)\big) = \mathbf{m}(y(B)) = \int_B 1\, d\mathbf{m}_y$.

(ii) If $B \in \mathcal{B}(R)$, $\mathbf{A} \in \mathcal{F}$, then

$$0 = \mathbf{m}(\mathbf{A} \cdot y(\emptyset)) \le \mathbf{m}(\mathbf{A} \cdot y(B)) = \int_B \mathbf{p}(\mathbf{A}|y)\, d\mathbf{m}_y \le \mathbf{m}(\mathbf{A} \cdot y(R)) \le 1$$

and

$$\mathbf{m}_y\big(\{t \in R\,;\, \mathbf{p}(\mathbf{A}|y) < 0\}\big) = \mathbf{m}_y(B_0) = 0,$$
$$\mathbf{m}_y\big(\{t \in R\,;\, \mathbf{p}(\mathbf{A}|y) > 1\}\big) = \mathbf{m}_y(B_1) = 0.$$

We note that the cases $\mathbf{m}_y(B_0) > 0$, $\mathbf{m}_y(B_1) > 0$ lead to contradictions

$$\int_{B_0} \mathbf{p}(\mathbf{A}|y)\, d\mathbf{m}_y < 0\ ,\ \int_{B_1} \mathbf{p}(\mathbf{A}|y)\, d\mathbf{m}_y > 1,$$

respectively.

(iii) Let $\bigodot_{i=1}^{\infty} \mathbf{A}_i = (0_\Omega, 1_\Omega)$. Then using *Definition 5* and the properties of IF-state $\mathbf{m}$ we obtain

$$\begin{aligned}
\int_B \mathbf{p}\Big(\bigoplus_{i=1}^{\infty} \mathbf{A}_i \Big| y\Big)\, d\mathbf{m}_y &= \mathbf{m}\bigg(\Big(\bigoplus_{i=1}^{\infty} \mathbf{A}_i\Big) \cdot y(B)\bigg) = \mathbf{m}\bigg(\bigoplus_{i=1}^{\infty} (\mathbf{A}_i \cdot y(B))\bigg) = \sum_{i=1}^{\infty} \mathbf{m}(\mathbf{A}_i \cdot y(B)) \\
&= \sum_{i=1}^{\infty} \int_B \mathbf{p}(\mathbf{A}_i|y)\, d\mathbf{m}_y = \int_B \sum_{i=1}^{\infty} \mathbf{p}(\mathbf{A}_i|y)\, d\mathbf{m}_y.
\end{aligned}$$

(iv) Let $\mathbf{A}_n \nearrow \mathbf{A}$, $\mathbf{A}_n, \mathbf{A} \in \mathcal{F}$. Then $\mathbf{m}\big(\mathbf{A}_n \cdot y(B)\big) \nearrow \mathbf{m}\big(\mathbf{A} \cdot y(B)\big)$ holds for each $B \in \mathcal{B}(R)$. Therefore

$$\begin{aligned}
\int_B \lim_{n\to\infty} \mathbf{p}(\mathbf{A}_n \mid y)\, d\mathbf{m}_y &= \lim_{n\to\infty} \int_B \mathbf{p}(\mathbf{A}_n \mid y)\, d\mathbf{m}_y = \lim_{n\to\infty} \mathbf{m}\big(\mathbf{A}_n \cdot y(B)\big) = \mathbf{m}\big(\mathbf{A} \cdot y(B)\big) \\
&= \int_B \mathbf{p}(\mathbf{A} \mid y)\, d\mathbf{m}_y.
\end{aligned}$$

□

4. Martingale Convergence Theorem

Let us consider the probability space $(\Omega, \mathcal{S}, P)$, $A \in \mathcal{S}$, a random variable $\xi : \Omega \to R$ and the Borel measurable functions $g_n : R \to R$ $(n = 1, 2, \dots)$ such that $\lim\limits_{n\to\infty} g_n(t) = g(t)$ for each $t \in R$ and $g_n^{-1}\big(\mathcal{B}(R)\big) \nearrow g^{-1}\big(\mathcal{B}(R)\big)$. Then by the martingale convergence theorem we have

$$p(A \mid g_n \circ \xi) \to p(A \mid g \circ \xi),$$

where $p(A \mid g_n \circ \xi)$, $p(A \mid g \circ \xi)$ are the conditional probabilities (see [18]).

We show a version of the martingale convergence theorem for the conditional intuitionistic fuzzy probabilities $\mathbf{p}(\mathbf{A} \mid y \circ g_n^{-1})$, $\mathbf{p}(\mathbf{A} \mid y \circ g^{-1})$, i.e.,

$$\mathbf{p}(\mathbf{A} \mid y \circ g_n^{-1}) \to \mathbf{p}(\mathbf{A} \mid y \circ g^{-1})$$

for $\mathbf{A} \in \mathcal{F}$ and an *IF*-observable $y : \mathcal{B}(R) \to \mathcal{F}$.

Proposition 1. *Let $\mathbf{A} \in \mathcal{F}$, $y : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable and let an IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ be defined by*

$$x(B) = \begin{cases} (0_\Omega, 1_\Omega), & \text{if } B = \emptyset \\ \mathbf{A}, & \text{if } B = \{1\} \\ x(B \cap \{1\}), & \text{if } B \neq \emptyset, B \neq R, B \in \mathcal{B}(R) \\ (1_\Omega, 0_\Omega), & \text{if } B = R. \end{cases}$$

Let $h : \mathcal{B}(R^2) \to \mathcal{F}$ be the joint IF-observable of x and y, let $\mathbf{m} : \mathcal{F} \to [0,1]$ be an IF-state, $\Omega = R^2$, $\mathcal{S} = \mathcal{B}(R^2)$, $P = \mathbf{m} \circ h$, $\xi : R^2 \to R$ be such that $\xi(u,v) = v$ and $A = \{1\} \times R$. Then $(\Omega, \mathcal{S}, P)$ is a probability space, $A \in \mathcal{S}$, ξ is a random variable,

$$P_\xi = \mathbf{m}_y$$

and

$$\mathbf{p}(\mathbf{A} \mid y) = p(A \mid \xi)$$

holds $\mathbf{m}_y$-almost everywhere.

Proof. By definitions we obtain

$$\begin{aligned} P_\xi(B) &= P\big(\xi^{-1}(B)\big) = \mathbf{m} \circ h\big(\xi^{-1}(B)\big) = \mathbf{m}\big(h(R \times B)\big) = \mathbf{m}\big(x(R) \cdot y(B)\big) = \mathbf{m}\big((1_\Omega, 0_\Omega) \cdot y(B)\big) \\ &= \mathbf{m}\big(y(B)\big) = \mathbf{m}_y(B) \end{aligned}$$

for each $B \in \mathcal{B}(R)$ and

$$\begin{aligned} \int_B p(A \mid \xi)\, dP_\xi &= P\big(A \cap \xi^{-1}(B)\big) = \mathbf{m}\big(h(\{1\} \times B)\big) = \mathbf{m}\big(x(\{1\}) \cdot y(B)\big) = \mathbf{m}\big(\mathbf{A} \cdot y(B)\big) \\ &= \int_B \mathbf{p}(\mathbf{A} \mid y)\, d\mathbf{m}_y. \end{aligned}$$

Hence $\mathbf{p}(\mathbf{A} \mid y) = p(A \mid \xi)$ holds $\mathbf{m}_y$-almost everywhere. $\square$

Theorem 6. (Martingale Convergence Theorem). *Let $\mathcal{F}$ be a family of IF-events with product $\cdot$, $\mathbf{A} \in \mathcal{F}$, $y : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \to [0,1]$ be an IF-state and $g, g_n : R \to R$ $(n = 1, 2, \ldots)$ be the Borel measurable functions such that $g_n^{-1}\big(\mathcal{B}(R)\big) \nearrow g^{-1}\big(\mathcal{B}(R)\big)$. Then the convergence*

$$\mathbf{p}(\mathbf{A} \mid y \circ g_n^{-1}) \to \mathbf{p}(\mathbf{A} \mid y \circ g^{-1})$$

holds $\mathbf{m}_{y \circ g^{-1}}$-almost everywhere.

Proof. By *Proposition 1* we have the probability space $(\Omega, \mathcal{S}, P)$, $A \in \mathcal{S}$, a random variable ξ such that $P_\xi = \mathbf{m}_y$ and $\mathbf{p}(\mathbf{A} \mid y) = p(A \mid \xi)$ holds $\mathbf{m}_y$ - almost everywhere.

Put $\eta_n = g_n \circ \xi$ $(n = 1, 2, \ldots)$ and $\eta = g \circ \xi$. Then η_n, η are the random variables such that $\eta_n \nearrow \eta$ and

$$\mathcal{S}_n = \eta_n^{-1}\big(\mathcal{B}(R)\big) = \xi^{-1}\big(g_n^{-1}\big(\mathcal{B}(R)\big)\big) \nearrow \xi^{-1}\big(g^{-1}\big(\mathcal{B}(R)\big)\big) = \eta^{-1}\big(\mathcal{B}(R)\big) = \mathcal{S}_0.$$

Put

$$f_n = P(A \mid \mathcal{S}_n) = E(\chi_A \mid \mathcal{S}_n) \ (n = 1, 2, \ldots),$$

where $E(\chi_A \mid \mathcal{S}_n)$ are the conditional expectations. Then the sequence $(f_n, \mathcal{S}_n)_n$ is a martingale and the convergence $f_n \to f_\infty$ holds $\mathcal{S}_\infty$-almost everywhere, where

$$f_\infty = E(\chi_A \mid \mathcal{S}_\infty), \mathcal{S}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{S}_n\right) = \sigma(\mathcal{S}_0) = \mathcal{S}_0.$$

By a special type of martingale theorem we have that the convergence $P(A \mid \mathcal{S}_n) \to P(A \mid \mathcal{S}_0)$ holds $\mathcal{S}_0$ - almost everywhere, and hence the convergence

$$p(A \mid \eta_n) \to p(A \mid \eta)$$

holds P_η-almost everywhere.

Now we prove that

$$\begin{aligned}
&\mathbf{p}(\mathbf{A} \mid y \circ g_n^{-1}) = p(A \mid \eta_n) \text{ holds } \mathbf{m}_{y \circ g_n^{-1}} - \text{almost everywhere,}\\
&\mathbf{p}(\mathbf{A} \mid y \circ g^{-1}) = p(A \mid \eta) \text{ holds } \mathbf{m}_{y \circ g^{-1}} - \text{almost everywhere,}
\end{aligned}$$

and

$$\mathbf{m}_{y \circ g_n^{-1}} = P_{\eta_n}, \ \mathbf{m}_{y \circ g^{-1}} = P_\eta.$$

For each $B \in \mathcal{B}(R)$ we get

$$\begin{aligned}
P_{\eta_n}(B) &= P_{g_n \circ \xi} = P\big(\xi^{-1}\big(g_n^{-1}(B)\big)\big) = \mathbf{m} \circ h\big(\xi^{-1}\big(g_n^{-1}(B)\big)\big) = \mathbf{m}\big(h\big(R \times g_n^{-1}(B)\big)\big)\\
&= \mathbf{m}\big(x(R) \cdot y\big(g_n^{-1}(B)\big)\big) = \mathbf{m}\big((1_\Omega, 0_\Omega) \cdot y\big(g_n^{-1}(B)\big)\big) = \mathbf{m}\big(y\big(g_n^{-1}(B)\big)\big) = \mathbf{m}_{y \circ g_n^{-1}}(B)
\end{aligned}$$

and

$$\begin{aligned}
\int_B p(A \mid \eta_n)\, dP_{\eta_n} &= P\big(A \cap \eta_n^{-1}(B)\big) = P\big((\{1\} \times R) \cap \big(\xi^{-1}\big(g_n^{-1}(B)\big)\big)\big)\\
&= P\big((\{1\} \times R) \cap (R \times g_n^{-1}(B))\big) = P\big(\{1\} \times g_n^{-1}(B)\big) = \mathbf{m}\big(h(\{1\} \times g_n^{-1}(B))\big)\\
&= \mathbf{m}\big(x(\{1\}) \cdot y(g_n^{-1}(B))\big) = \mathbf{m}\big(\mathbf{A} \cdot y(g_n^{-1}(B))\big) = \int_B \mathbf{p}(\mathbf{A} \mid y \circ g_n^{-1})\, d\mathbf{m}_{y \circ g_n^{-1}}.
\end{aligned}$$

Hence $p(A \mid \eta_n) = \mathbf{p}(\mathbf{A} \mid y \circ g_n^{-1})$ holds $\mathbf{m}_{y \circ g_n^{-1}}$ − almost everywhere because $P_{\eta_n} = \mathbf{m}_{y \circ g_n^{-1}}$.

The assertion that $p(A \mid \eta) = \mathbf{p}(\mathbf{A} \mid y \circ g^{-1})$ holds $\mathbf{m}_{y \circ g^{-1}}$ − almost everywhere can be proved analogously.

Finally, we obtain that the convergence

$$\mathbf{p}(\mathbf{A} \mid y \circ g_n^{-1}) = p(A \mid \eta_n) \to p(A \mid \eta) = \mathbf{p}(\mathbf{A} \mid y \circ g^{-1})$$

holds $\mathbf{m}_{y \circ g^{-1}}$ − almost everywhere. □

5. Conclusions

The paper deals with the probability theory on intuitionistic fuzzy sets. We proved the properties of the conditional intuitionistic fuzzy probability induced by an intuitionistic fuzzy state. We formulated and proved the martingale convergence theorem for the conditional intuitionistic fuzzy probability, too. The next very interesting notion is the notion of a conditional expectation. In [19] V. Valenčaková defined a conditional expectation of intuitionistic fuzzy observables $E(x \mid y)$ using Gödel connectives $\vee, \wedge$ given by $\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$, $\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$. She proved the martingale convergence theorem for this conditional expectation. In future research directions one can try to formulate the definition of conditional intuitionistic fuzzy expectation using Lukasiewicz connectives $\oplus, \odot$ and to prove the version of the martingale convergence theorem in this context.

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Abbreviations

The following abbreviation is used in this manuscript:

IF Intuitionistic Fuzzy

References

1. Atanassov, K.T. Intuitionistic fuzzy sets. *Repr. Int. Bioautom.* **2016**, *20*, S1–S6.
2. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
3. Lendelová, K. Conditional IF-probability. *Adv. Soft Comput. Soft Methods Integr. Uncertain. Model.* **2006**, *37*, 275–283.
4. Valenčáková, V. A note on the conditional probability of IF-events. *Math. Slovaca* **2009**, *2*, 251–260. [CrossRef]
5. Riečan, B. Probability theory on IF-events. In *Algebraic and Proof-Theoretic Aspects of Non-classical Logics. Lecture Notes in Computer Science*, *4460*; Aguzzoli, S., Ciabattoni, A., Eds.; Springer: Berlin, Germany, 2007; pp. 290–308.
6. Riečan, B. Analysis of fuzzy logic models. In *Intelligent systems*; Koleshko, V., Ed.; INTECH: Rijeka, Croatia, 2012; pp. 219–244.
7. Čunderlíková, K.; Riečan, B. Convergence of Intuitionistic Fuzzy Observables. In *Uncertainty and Imprecision in Decision Making and Decision Support: New Challenges, Solutions and Perspectives. IWIFSGN 2018, Advances in Intelligent Systems and Computing*, *1081*; Atanassov, K.T., Ed.; Springer: Cham, Germany, 2021; pp. 29–39.
8. Bartková, R.; Čunderlíková, K. About Fisher-Tippett-Gnedenko Theorem for Intuitionistic Fuzzy Events. In *Advances in Fuzzy Logic and Technology. Advances in Intelligent Systems and Computing*, *641*; Kacprzyk, J., Ed.; Springer: Cham, Germany, 2018; pp. 125–135.
9. Nowak, P.; Hryniewicz, O. On generalized versions of central limit theorems for IF-events. *Inf. Sci.* **2016**, *355–356*, 299–313. [CrossRef]
10. Nowak, P.; Pawlowski, M. Pricing European options under uncertainty with application of Levy processes and the minimal L^q equivalent martingale measure. *J. Comput. Appl. Math.* **2019**, *345*, 416–433. [CrossRef]

11. Nowak, P.; Romaniuki, M. Application of Levy processes and Esscher transformed martingale measures for option pricing in fuzzy framework. *J. Comput. Appl. Math.* **2014**, *263*, 129–151. [CrossRef]
12. Nowak, P.; Romaniuki, M. Computing option price for Levy process with fuzzy parameters. *Eur. J. Oper. Res.* **2010**, *201*, 206–210. [CrossRef]
13. Atanassov, K. T. *Intuitionistic Fuzzy Sets : Theory and Applications*, 1st ed.; Physica Verlag: New York, NY, USA, 1999; pp. 1–137.
14. Atanassov, K. T. *On Intuitionistic Fuzzy Sets Theory*; Springer: Berlin, Germany, 2012; pp. 1–52.
15. Riečan, B. On the probability and random variables on IF events. In *Applied Artifical Intelligence: Proceedings of the 7th International FLINS Conference, Genova, Italy, 29–31 August 2006*; Ruan, D., D'hondt, P., Eds.; World Scientific: Singapore, 2006; pp. 138–145.
16. Riečan, B. On a problem of Radko Mesiar: General form of IF-probabilities. *Fuzzy Sets Syst.* **2006**, *152*, 1485–1490. [CrossRef]
17. Lendelová, K.; Riečan, B. Weak law of large numbers for IF-events. In *Current Issues in Data and Knowledge Engineering*; EXIT: Warszawa, Poland, 2004; pp. 309–314.
18. Riečan B.; Neubrunn T. *Integral, Measure and Ordering*; Kluwer: Dordrecht, The Netherlands, 1997; pp. 1–378.
19. Valenčáková, V. A Note on the Conditional Expectation of IF-Observables. In *Fuzzy Logic and Applications. Lecture Notes in Computer Science, 5571*; Di Gesú, V., Pal, S.K., Petrosino, A., Eds.; Springer: Berlin, Germany, 2009; pp. 85–92.