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Multiple Periodic Solutions and Fractal Attractors of Differential Equations with n-Valued Impulses

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Abstract: Ordinary differential equations with n-valued impulses are examined via the associated Poincaré translation operators from three perspectives: (i) the lower estimate of the number of periodic solutions on the compact subsets of Euclidean spaces and, in particular, on tori; (ii) weakly locally stable (i.e., non-ejective in the sense of Browder) invariant sets; (iii) fractal attractors determined implicitly by the generating vector fields, jointly with Devaney's chaos on these attractors of the related shift dynamical systems. For (i), the multiplicity criteria can be effectively expressed in terms of the Nielsen numbers of the impulsive maps. For (ii) and (iii), the invariant sets and attractors can be obtained as the fixed points of topologically conjugated operators to induced impulsive maps in the hyperspaces of the compact subsets of the original basic spaces, endowed with the Hausdorff metric. Five illustrative examples of the main theorems are supplied about multiple periodic solutions (Examples 1–3) and fractal attractors (Examples 4 and 5).

Keywords: impulsive differential equations; n-valued maps; Hutchinson-Barnsley operators; multiple periodic solutions; topological fractals; Devaney's chaos on attractors; Poincaré operators; Nielsen number

MSC: 28A20; 34B37; 34C28; Secondary 34C25; 37C25; 58C06

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1. Introduction

The theory of impulsive differential equations and inclusions has been systematically developed (see e.g., the monographs [1–4], and the references therein), among other things, especially because of many practical applications (see e.g., References [1,4–11]). These applications concern fluctuations of pendulum systems under impulsive effects, remittent oscillators, population dynamics, oxygen-driven self-cycling fermentation process, nutrient-driven self-cycling fermentation process, various impulsive drug effects, optimal impulsive vaccination for an SIR control model, an SEIRS epidemic model, malaria vector model, impulsive insecticide spraying, HIV induction-maintenance therapy, and so forth.

The impulsive maps can be deterministic or stochastic (random), crisp or fuzzy, state dependent or independent, time dependent or independent, single-valued or multivalued. Here, we will be exclusively interested in multivalued, deterministic non-fuzzy state and time independent impulses in subsets of Euclidean spaces. A particular attention will be paid to a subclass of n-valued maps (see Definition 1 below), whence the title of our article. For other sorts of multivalued impulses, see e.g., (Chapter 11 in Reference [12]), References [13–18].

As far as we know, the differential equations with n-valued impulses have been tendentiously considered only in Reference [19] for multiple periodic solutions. On the other hand, the recent research of n-valued maps is very active in the topological (i.e., mainly Nielsen) fixed point theory (see e.g., References [20,21] and the earlier survey article of Brown in the handbook [22]). This research is far from being trivial, because there is for instance nothing known about the lower estimates for the number of periodic points of such maps, especially because the number of points of their iterates can be quite arbitrary in general. Moreover, it seems to be difficult to find conditions under which the iterates have an exact given number of points.

Relaxing the strict requirement of exactly n-values in Definition 1 below, we can consider the union operators of n single-valued maps, called the Hutchinson-Barnsley operators. These operators play a crucial role in constructing the fractals as attractors of iterated function systems (see References [23,24]). This relaxation makes the study in a certain sense more liberal, but the n-valued Hutchinson-Barnsley operators then become nothing else but split n-valued maps, which might not be so interesting. Nevertheless, the application of the deep results for the iterated function systems, including the chaotic dynamics on the Hutchinson-Barnsley (fractal) attractors, to impulsive differential equations via the Poincaré translation operators along the trajectories is quite original.

Besides these two novelty applications, our research in this field can be justified by a simple argument that the n-valued impulses extend with no doubts the variability in practical applications. For instance, the repeated vaccination need not be always the same, but they can differ each from other just by finitely many possibilities (e.g., when the doctors have at the same time to their disposal vaccines made by n different producers).

Our paper is organized as follows. After the useful definitions in Preliminaries, we will recall the basic properties and results about n-valued maps (in Section 3) and the Hutchinson-Barnsley operators (in Section 4). These results are neither new, but (in case of n-valued maps) nor so well known. New and original are the applications of these results to impulsive differential equations in $\mathbb{R}^n$ (in Section 5) and $\mathbb{R}^n/\mathbb{Z}^n$ (in Section 6), jointly with the obtained theorems about topological fractals and deterministic chaos in the sense of Devaney (in Section 7). Several illustrative examples are supplied in Sections 5 and 6, jointly with concluding remarks in Section 8.

2. Preliminaries

In the entire text all topological spaces will be metric. A space X is an *absolute neighbourhood retract* (written $X \in$ ANR) if, for every space Y and every closed subset $A \subset Y$, each continuous map $f\colon A \to X$ is extendable over some open neighbourhood U of A in Y. A space X is an *absolute retract* (written $X \in$ AR) if each $f\colon A \to X$ is extendable over Y. Evidently, if $X \in$ AR, then $X \in$ ANR.

By a *polyhedron*, we understand as usually a triangulable space. It is well known that every polyhedron is an ANR-space. An important example of a compact polyhedron will be for us a torus. By the *n-torus* $\mathbb{T}^n$, $n \geq 1$, we will mean here either the factor space $\mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n$ or the Cartesian product $\underbrace{S^1 \times \ldots \times S^1}_{n\text{-times}}$, where $\mathbb{R}$ denotes the set of reals, $\mathbb{Z}$ denotes the set of integers, and

$$S^1 := \{x \in \mathbb{R}^2 \mid |x| = 1\} = \{z = \mathrm{e}^{2\pi s\,\mathrm{i}} \mid s \in [0,1]\}.$$

In particular, for $n = 1$, $\mathbb{T}^1 = S^1$ becomes a *circle*.

If not explicitly specified, we will not distinguish between the additive and multiplicative notations, because the logarithm map $\mathrm{e}^{2\pi s\,\mathrm{i}} \to s$, $s \in [0,1]$, establishes an isomorphism between these two representations.

Let us also note that the relation between the Euclidean space $\mathbb{R}^n$ and its factorization $\mathbb{R}^n/\mathbb{Z}^n$ can be realized by means of the natural projection, sometimes also called a canonical mapping, $\tau\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $x \to [x]$, where the symbol $[x] := \{y \in \mathbb{R}^n \mid (y - x) \in \mathbb{Z}^n\}$ stands for the equivalent class of elements with x in $\mathbb{R}^n/\mathbb{Z}^n$, that is, $\mathbb{R}^n/\mathbb{Z}^n := \{[x] \mid x \in \mathbb{R}^n\}$, where $[x] = x + \mathbb{Z}^n$, $x \in [0,1)^n$.

By a *multivalued map* $\varphi\colon X \multimap Y$, we understand $\varphi\colon X \to 2^Y \setminus \{\emptyset\}$. In the entire text, we will still assume that φ has closed values.

A multivalued map $\varphi\colon X \multimap Y$ is said to be *continuous* if, for every open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X and at the same time if, for every closed $V \subset Y$, the set $\{x \in X \mid \varphi(x) \subset V\}$ is closed in X.

Obviously, in the single-valued case, if $f\colon X \to Y$ is continuous in a multivalued sense, then it is continuous in the usual (single-valued) sense. Furthermore, every continuous map $\varphi\colon X \multimap Y$ has a closed *graph* $\Gamma_\varphi := \{(x,y) \in X \times Y \mid y \in \varphi(x)\}$, but not vice versa. If $\varphi\colon X \multimap Y$ is continuous with compact values and $A \subset X$ is compact, then $\varphi(A)$ is compact, too. The composition $\psi \circ \varphi\colon X \multimap Z$ of two continuous maps with compact values, $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$, is again continuous with compact values. For more details, see for example, References [25,26].

For the single-valued compact continuous maps $f\colon X \to X$, where $X \in$ ANR, we can define the global topological invariants, namely the *Lefschetz number* $L(f) \in \mathbb{Z}$ and the *Nielsen number* $N(f) \in \mathbb{N} \cup \{0\}$. If, in particular, $X \in$ AR, then $L(f) = 1$ and $N(f) = 1$.

For the single-valued continuous maps on tori, $f\colon \mathbb{T}^n \to \mathbb{T}^n$, the Anosov-type equality $N(f) = |L(f)|$ holds. For the single-valued continuous maps on the circle ($n = 1$), $f\colon S^1 \to S^1$, we can also define their degree $\deg(f) := 1 - L(f)$. If $S^1 = \mathbb{R}/\mathbb{Z}$, then $\deg(f) = \widetilde{f}(1) - \widetilde{f}(0)$, where $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ denotes the lift of f.

Besides their *existence property*, when the existence of a fixed point $x = f(x)$ of a compact continuous $f\colon X \to X$, $X \in$ ANR, is implied by $L(f) \neq 0$, resp. by $N(f) > 0$, or in particular for $X = S^1$ by $\deg(f) \neq 1$ (i.e., $\widetilde{f}(1) - \widetilde{f}(0) \neq 1$), all these numbers are *invariant* under a compact continuous *homotopy*, namely $L(f_0) = L(f_1) = L(f_\mu)$, $N(f_0) = N(f_1) = N(f_\mu)$, and (for $X = S^1$) $\deg(f_0) = \deg(f_1) = \deg(f_\mu) = f_0(1) - f_0(0) = f_1(1) - f_1(0) = f_\mu(1) - f_\mu(0)$, for all $\mu \in [0,1]$.

The Nielsen number $N(f)$ of a continuous map $f\colon X \to X$, where $X \in$ ANR, gives still the lower estimate of the number of fixed points, that is, $N(f) \leq \#\{x \in X \mid x = f(x)\}$, where the symbol # stands for the cardinality of the fixed point set $\{x \in X \mid x = f(x)\}$.

For the definitions and more details, see e.g., References [27,28].

Since in Sections 4 and 7 we will also consider hyperspaces endowed with the Hausdorff metric d_H, it will be convenient to recall finally their definitions. Hence, if (X, d) is a metric space endowed with the metric d, then the induced *hyperspace* $(K(X), d_H)$ is defined as

$$K(X) := \{Y \subset X \mid Y \text{ is a compact subset of } X\},$$

and the Hausdorff metric $d_H(\cdot,\cdot) := K(X) \times K(X) \to [0,\infty)$ is induced by d as follows:

$$d_H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\} = \max\left\{\sup_{a\in A}(\inf_{b\in B} d(a,b)), \sup_{b\in B}(\inf_{a\in A} d(a,b))\right\},$$

where $d(a,B)$ and $d(A,b)$ stand for the distances between points a, b and sets A, B, respectively. For more details, see for example, References [24,25,29,30].

3. N-Valued Maps

The topological fixed point theory for multivalued maps has been developed in two main directions: (i) for admissible maps (in the sense of Górniewicz) and their particular cases like acyclic maps, R_δ-maps, and so forth (see e.g., References [22,25,26], and the references therein), and (ii) for n-valued maps (see e.g., References [20–22,31–48]) and their generalizations like n-acyclic maps (see e.g., Reference [49]) and weighted maps (see e.g., Reference [50]). In the present paper, we will be exclusively interested in the second class of n-valued maps whose research made a big progress in the recent years.

Let us recall their definition and some basic properties.

Definition 1. *An* n-valued map $\varphi\colon X \multimap Y$ *is a continuous multivalued mapping that associates to each* $x \in X$ *an unordered subset of exactly* n *points of* Y*. We say that an* n*-valued map* φ *is* split *if there are single-valued continuous maps* $f_1, \ldots, f_n\colon X \to Y$ *such that* $\varphi(x) = \{f_1(x), \ldots, f_n(x)\}$*, for all* $x \in X$*.*

One can readily check that unlike to admissible maps, where the sets of values are compact and connected (i.e., continua), those of n-valued maps are disconnected. Moreover, unlike to the above definition of general multivalued maps, for the continuity of n-valued maps is sufficient if, for every closed $V \subset Y$, the set $\{x \in X \mid \varphi(x) \subset V\}$ is closed in X.

Lemma 1 (splitting lemma; cf. References [44,47,51])**.** *If X is simply connected and locally path-connected, then every n-valued map $\varphi\colon X \multimap Y$ is split.*

Let us recall that X is simply connected if and only if it is path-connected and its fundamental (first homotopy) group is trivial.

Lemma 2 (cf. Reference [33], Theorem 2.1)**.** *Any n-valued map $\varphi_1\colon X \multimap Y$ which is homotopic in an n-valued way (i.e., via a continuous mapping $\varphi_t\colon X \times [0,1] \multimap Y$ such that $\varphi_t(x)$ has exactly n-points, for all $(x,t) \in X \times [0,1]$) to a split n-valued map $\varphi_0\colon X \multimap Y$ (written $\varphi_t \sim \varphi_0$) is also split.*

If X is a compact ANR- space and $f\colon X \to X$ is a single-valued continuous self-map, then the Nielsen number $N(f)$ of f is well defined (see e.g., References [27,28]). Since a (compact) ANR-space is (uniformly) locally contractible, and so locally path-connected, if X is still simply connected, then all fixed points of f belong to a single (same) fixed point class, whose index is $L(f)$, where $L(f)$ stands for the *Lefschetz number* of f. For its definition and more details, see for example, Reference [27]. Therefore, if X is still simply connected, then $N(f) = 1$ if $L(f) \neq 0$, and $N(f) = 0$ if $L(f) = 0$.

Summing up, if X is a simply connected compact ANR-space, then the Nielsen relation of any self-map $g_k\colon X \to X$ homotopic to $f_k\colon X \to X$ (written $g_k \sim f_k$) is trivial, that is, $N(g_k) = N(f_k) \leq 1$, for $k = 1, \ldots, n$. Subsequently, the *Nielsen numbers* $N(\psi)$ and $N(\varphi)$ of homotopic n-valued maps $\psi = \{g_1, \ldots, g_n\} : X \multimap X$ and $\varphi = \{f_1, \ldots, f_n\} : X \multimap X$ can be simply defined and calculated by the formula:

$$N(\psi) = N(\varphi) := \sum_{k=1}^{n} N(f_k) = \#\{k = 1, \ldots, n \mid L(f_k) \neq 0\}, \tag{1}$$

where the symbol # denotes the cardinality of a given set.

If, in particular, X is a compact AR-space, then formula (1) takes the form

$$N(\psi) = N(\varphi) := \sum_{k=1}^{n} N(f_k) = n, \tag{2}$$

because $L(f_k) = 1$, for all $k = 1, \ldots, n$.

Remark 1. *Let us note that (compact) simply connected ANR-spaces are not necessarily (compact) AR-spaces. For example, every n-dimensional unit sphere S^n, where $n \geq 2$, is a simply connected ANR-space, but not contractible, and so an AR-space. Moreover, $L(\mathrm{id}_{S^2}) = 2$ but, according to (1), $N(\mathrm{id}_{S^2}) = 1$.*

Remark 2. *According to the example due to Jezierski [43], there exists a continuous map $\varphi\colon \mathcal{K}^2 \multimap \mathcal{K}^2$, where $\mathcal{K}^2 \subset \mathbb{C}$ is a two-dimensional closed ball in the complex plane $\mathbb{C}$, whose values consist of* 1 *or* 2 *or* 3 *points, which is fixed point free.It justifies the assumption of exactly n-valued maps in Definition 1.*

Of course, the assumption of a simple connectedness of X is not necessary for the splitting of n-valued self maps $\varphi\colon X \multimap X$. For instance, if $X = S^1 = \mathbb{R}/\mathbb{Z}$, then an n-valued map $\varphi\colon S^1 \multimap S^1$ of degree $\mathrm{Deg}(\varphi)$ (for its definition, see for example, Theorem 2.1 in Reference [33]) is split if and only if $\mathrm{Deg}(\varphi)$ is a multiple of n (see Corollary 5.1 in Reference [33]).

In the case of splitting, we have to our disposal the following lemma.

Lemma 3 (cf. Theorems 2.2 and 2.3 in Reference [33]). *If $\varphi\colon S^1 \multimap S^1$ is a split n-valued map, then its degree $\mathrm{Deg}(\varphi)$ equals n-times the classical degree of the maps in the splitting. Furthermore, if $\psi\colon S^1 \to S^1$ is homotopic in an n-valued way to φ ($\psi \sim \varphi$), then*

$$\mathrm{Deg}(\psi) = \mathrm{Deg}(\varphi) = n\deg(f_1) = \cdots = n\deg(f_n) = n\left[\widetilde{f}_1(1) - \widetilde{f}_1(0)\right], \tag{3}$$

where $\widetilde{f}_1\colon \mathbb{R} \to \mathbb{R}$ is the lift of $f_1\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

In the non-split case, the situation becomes obviously more delicate. Nevertheless, for $X = S^1 = \mathbb{R}/\mathbb{Z}$, we have even the *Wecken property*.

Lemma 4 (cf. Theorem 5.1 in Reference [33]). *If $\varphi\colon S^1 \multimap S^1$ is an n-valued map of degree $\mathrm{Deg}(\varphi)$, then $N(\varphi) := |n - \mathrm{Deg}(\varphi)|$ holds for the Nielsen number $N(\varphi)$ of φ, and there is an n-valued map, say ψ, homotopic to φ (i.e., $\psi \sim \varphi$), that has exactly $|n - \mathrm{Deg}(\varphi)| = |n - \mathrm{Deg}(\psi)|$ fixed points (i.e., the Wecken property).*

Remark 3. *Observe that the equalities* (3) *in Lemma* 3 *can be expressed in terms of the Nielsen numbers as follows:*

$$N(\psi) = N(\varphi) = |n - \mathrm{Deg}(\varphi)| = n\,|1 - \deg(f_1)| = n\left|1 - \left[\widetilde{f}_1(1) - \widetilde{f}_1(0)\right]\right|, \tag{4}$$

because $N(f_1) = |L(f_1)| = |1 - \deg(f_1)| = \left|1 - \left[\widetilde{f}_1(1) - \widetilde{f}_1(0)\right]\right|$.

Now, we will briefly sketch the definition and basic properties of the Nielsen number for n-valued maps on compact polyhedra, which is essentially due to Schirmer [46] (see also the recent Reference [20]). The restriction to compact polyhedra is caused by the application of such a fixed point index in Reference [46]. For more general indices, see for example, References [40,48,49]. On the other hand, it will be quite sufficient for our needs in applications.

Using the fact that n-valued maps are locally (but not globally) equivalent to n single-valued continuous functions, which is important for the choice of a suitable index of isolated fixed points, Schirmer proceeded in Reference [46] analogously to a single-valued case ($n = 1$). Of course, all the technicalities (especially those related to the fixed point index) must have been appropriately elaborated there. For an alternative approach via the lifting classes, see Reference [20].

Hence, let X be a compact polyhedron and $\varphi\colon X \multimap X$ be an n-valued self-map. To obtain the Nielsen number $N(\varphi)$ of φ, the fixed points of φ were at first divided into finitely many equivalent classes (see Theorem 5.2 in Reference [46]), called *fixed point classes* or *Nielsen classes*. Then a suitable fixed point index was associated with each fixed point class. The (Nielsen) classes with non-zero index are called *essential*. The *Nielsen number* $N(\varphi)$ of φ is the number of essential (Nielsen) fixed point classes.

The Nielsen number $N(\varphi)$ gives the *lower estimate* of the number of fixed points of φ, that is, that any n-valued self-map $\varphi\colon X \multimap X$ has at least $N(\varphi)$ fixed points (cf. Theorem 5.4 in Reference [46]). Furthermore, φ satisfies the *homotopy invariance*, that is, if $\varphi_t\colon X \times [0,1] \multimap X$ is an n-valued homotopy, then $N(\varphi_0) = N(\varphi_1)$ (cf. Theorem 6.5 in Reference [46]).

Remark 4. *Although the* Anosov property, *namely that $N(\varphi) = |L(\varphi)|$, holds for n-valued maps on S^1, because $\mathrm{Deg}(\varphi) = n - L(\varphi)$, it is no longer true for higher dimensional tori $\mathbb{T}^n$, where $n > 1$, which complicates the calculations. Nevertheless, if $\varphi = \{f_1, \ldots, f_n\} : X \multimap X$ is a split n-valued self-map on a compact polyhedron X, then $N(\varphi) = \sum_{k=1}^{n} N(f_k)$ (cf. Corollary 7.2 in Reference [46]). If, in particular, φ is an n-valued constant, then $N(\varphi) = n$ (cf. Corollary 7.3 in Reference [46]).*

Remark 5. *Observe that, unlike to equalities* (1) *and* (2)*, compact polyhedra in Remark* 4 *need not be simply connected. Let us note that every compact ANR-space is homotopically equivalent to some polyhedron (see e.g., Reference [26]). On the other hand, since any two continuous maps $f, g\colon X \to X$, where X is a compact absolute retract, are homotopic and $L(f) = 1$ (see again e.g., Reference [26]), we get that $N(f) = N(g) = 1$, and subsequently we can put $N(\varphi) = \sum_{k=1}^{n} N(f_k) = n$, as in* (2)*.*

4. Hutchinson-Barnsley's Operators

As far as we know, there are no nontrivial results about the periodic point theory, or more precisely periodic orbit theory, for n-valued maps with $n > 1$. The reason consists in an almost uncontrollable enormous variability of their iterates, by which the existence (and even worse multiplicity) problems of nontrivial periodic orbits seem to be a difficult task.

On the other hand, the theory of iterated function systems (IFS), originated by Hutchinson [23] and extended and popularized by Barnsley [24], allows us to construct compact invariant subsets $A \subset X$ of a complete metric space (X, d) of the *Hutchinson-Barnsley operators*

$$\varphi := \bigcup_{k=1}^{n} f_k\colon X \multimap X, \quad \text{i.e.,} \quad A = \varphi(A) = \bigcup_{k=1}^{n} \bigcup_{x \in A} f_k(x),$$

provided $f_k\colon X \to X$, $k = 1, \dots, n$, are contractions, that is,

$$\exists L_k \in [0,1)\colon d(f_k(x), f_k(y)) \le L_k d(x,y) \quad \forall x, y \in X, k = 1, \dots, n.$$

Since a unique A can be obtained as the limit $\lim_{m\to\infty} \varphi^m(A_0) = A$, that is, $\lim d_H(A_m, A) = 0$, where $A_0 \subset X$ is an arbitrary compact subset, $A_m = \varphi(A_{m-1})$, $m = 1, 2, \dots$, and d_H stands for the Hausdorff metric (see Section 2), A is called the (global) *attractor* of the *iterated function system* (IFS) $\{X; f_1, \dots, f_n\}$. Moreover, the inequality

$$d_H(A_m, A) \le \frac{L^m}{1-L} d_H(A_0, \varphi(A_0)), \tag{5}$$

where $L = \max_{k=1,\dots,n} L_k$, holds for the m-th iterate $\varphi^m(A_0) = A_m$ of φ, for all $m = 1, 2, \dots$

The attractor A has usually a fractal structure, whose fractal dimension $\dim A$ can be estimated from above (i.e., to get its upper bound $\dim A \le D$) by means of the *Moran-Hutchinson formula* (cf. References [23,24])

$$\sum_{k=1}^{n} L_k^D = 1 \quad \left(\text{for } L = L_1 = \dots = L_n\colon D = \frac{\log n}{\log \frac{1}{L}}\right), \tag{6}$$

provided the sets $f_k(A)$, $k = 1, \dots, n$, are either *totally disconnected*, that is,

$$f_j(A) \cap f_k(A) = \emptyset, \quad \text{for all } j, k = 1, \dots, n; j \neq k, \tag{7}$$

or *just touching* (i.e., neither (7), nor with overlaps).

If, in particular, f_k are *similitudes*, that is,

$$d(f_k(x), f_k(y)) = L_k d(x,y), \quad \text{for all } x, y \in X, k = 1, \dots, n,$$

then we get $\dim A = D$.

Observe that condition (7) is much stronger than the one required in Definition 1 for n-valued maps, because $\varphi|_A\colon A \multimap A$ is certainly continuous and every image set $\varphi|_A(x)$ must contain exactly n-points, for each $x \in A$. In other words, $\varphi|_A$ is in particular a split n-valued map on a compact invariant set $A \subset X$.

Moreover, one can prove that every $f_k|_A$, $k = 1, \ldots, n$, must be injective, because otherwise if there are two points, say $a_1, a_2 \in A$, such that $f_k(a_1) = f_k(a_2) = a \in A$, then $a \in A$ would have two addresses, which is impossible (see e.g., Reference [24]). Therefore, we can assume without any loss of generality that $f_k|_A$ are under (7) invertible, for all $k = 1, \ldots, n$. Furthermore, every inversion $f_k^{-1}\colon f_k(A) \to A$ is a continuous map, for every $k = 1, \ldots, n$.

Hence, we can define the discrete dynamical system (A, S) by the mapping $S\colon A \to A$, where

$$S(a) := \bigcup_{k=1}^{n} f_k^{-1}(a), \tag{8}$$

which can be uniquely defined by $S(a) := f_k^{-1}(a)$, provided $a \in f_k(A)$. The system (A, S) is called the *shift dynamical system*, associated with the iterated function system $\{A; f_1, \ldots, f_n\}$. It can be proved that it is *chaotic in the sense of Devaney* (see e.g., References [24,30]), that is,

(i) S is sensitive to initial conditions,
(ii) S is transitive (if $U, V \subset A$ are open subsets, then there exists an integer n^* such that $U \cap S^{n^*}(V) \neq \emptyset$),
(iii) the set of periodic points of S is dense in A.

We can extend the definition of S to the *hyperspace* $K(A) := \{B \subset A \mid B \text{ is a compact subset of } A\}$, endowed with the Hausdorff metric d_H. Hence, let us define the *hypermap* $S^*\colon K(A) \to K(A)$, where

$$S^*(B) := \bigcup_{k=1}^{n} f_k^{-1}(B), \qquad B = \bigcup_{x \in B} \in K(A). \tag{9}$$

It is well known (see e.g., References [23,24]) that $(K(A), d_H)$ is also compact and S^* is continuous in the metric d_H.

Haase Theorem 1 in Reference [52] has proved that the system $(K(A), S^*)$ is also chaotic in the sense of Devaney with respect to the Hausdorff metric d_H, provided (5) holds.

Summing up, we can formulate the following two well known propositions.

Proposition 1 (cf. References [23,24])**.** *The iterated function system $\{X; f_1, \ldots, f_n\}$, where (X, d) is a complete metric space and $f_k\colon X \to X$ are contractions, for all $k = 1, \ldots, n$, admits a unique global attractor $A \in K(X)$, which can be obtained as the limit $\lim_{m\to\infty} \varphi^m(A_0) = A$, that is, $\lim_{m\to\infty} d_H(A_m, A) = 0$, where $A_0 \subset X$ is an arbitrary compact subset and $A = \varphi(A)$, $A_m = \varphi^m(A_0)$, $\varphi := \cup_{k=1}^n f_k\colon X \multimap X$.*

Moreover, the inequality (5) holds for the successive approximations A_m, $m = 1, 2, \ldots$, of the attractor A, whose fractal dimension $\dim A$ can be estimated from above (i.e., $\dim A \le D$) by means of the Moran-Hutchinson formula (6).

Proposition 2 (cf. References [24,52])**.** *Let $\{X; f_1, \ldots, f_n\}$ be an iterated function system as in Proposition 1. Assume, furthermore, that the sets $f_k(A)$ are totally disconnected, for all $k = 1, \ldots, n$ (see (7)). Then the shift dynamical system (A, S), where $S\colon A \to A$ is defined in (8), which is associated with the iterated function system $\{A; f_1, \ldots, f_n\}$, is chaotic in the sense of Devaney. The same is true for the system $(K(A), S^*)$, where the hypermap $S^*\colon K(A) \to K(A)$ is defined in (9), with respect to the Hausdorff metric d_H.*

Remark 6. *Since every contraction f_k on a complete metric space X has, for every $k = 1, \ldots, n$, exactly one fixed point, the Hutchinson-Barnsley operator $\varphi := \bigcup_{k=1}^n f_k\colon X \multimap X$ must have at most n fixed points. Since the attractor $A \subset X$ is compact, and so complete, all the fixed points of the restricted contractions $f_k|_A\colon A \to A$, $k = 1, \ldots, n$, as well as of $\varphi|_A\colon A \multimap A$ must belong to A.*

The existence of a compact invariant subset of the Hutchinson-Barnsley operator can be also obtained in the frame of topological fixed point theory (unlike to "metric" Proposition 1) as follows.

Let us recall here that a metric space X is *locally continuum-connected* if, for each neighbourhood U of each point $x \in X$, there is a neighbourhood $V \subset U$ of x such that each point of V can be connected with x by a subcontinuum (i.e., a compact, connected subset) of U.

Proposition 3 (cf. References [29,53,54])**.** *Let X be a connected, locally continuum-connected metric space and $f_k\colon X \to X$ be compact continuous maps, for all $k = 1, 2, \dots, n$. Then the Hutchinson-Barnsley operator*

$$\varphi := \bigcup_{k=1}^{n} f_k\colon X \multimap X$$

possesses at least one compact invariant set, say $X_0 \in K(X)$, such that $X_0 = \varphi(X_0)$.

If, in particular, X is a Peano continuum (i.e., compact, connected and locally connected) and $f_k\colon X \to X$ are continuous maps, for all $k = 1, 2, \dots, n$, then φ possesses at least one compact invariant set $\varphi(X_0) = X_0 \subset X$ which is non-ejective in the sense of Browder, that is,

$$\begin{gathered} \forall \varepsilon > 0 \exists X_1 \in K(X), X_1 \neq X_0, \text{ and } d_H(X_0, X_1) < \delta: \\ \varphi^m(X_1) = \left[\bigcup_{x \in X_0} \bigcup_{k=1}^{n} f_k(x)\right]^m \subset \{Y \in K(X) \mid d_H(X_0, Y) < \varepsilon\}, \forall m \geq 1. \end{gathered} \tag{10}$$

Remark 7. *The Hutchinson-Barnsley operator in Propositions 1 and 3 need not (but can) be exactly n-valued. Furthermore, the contractions f_k in Proposition 1 can be more generally replaced, for the existence of a unique attractor $A \in K(X)$, by multivalued contractions with compact values. In Proposition 3, compact continuous maps f_k can be also replaced, for the existence of a compact invariant set $A \in K(X)$ of φ (without uniqueness, but including* (10)*), by compact continuous multivalued maps with compact values. For more details and further possibilities (see References [29,53–55]).*

Remark 8. *Let us emphasize that the maps f_k as well as the Hutchinson-Barnsley operator φ can be, under the assumptions of Proposition 3, fixed point free. For instance, rotations on the circle $X = S^1$ can be so. On the other hand, the non-ejectivity* (10) *can be regarded as a weak local stability of an invariant set $A \in K(X)$ of φ.*

5. Application to Impulsive Differential Equations in $\mathbb{R}^n$

In this section, the presented results for n-valued maps will be applied to impulsive differential equations in $\mathbb{R}^n$.

Consider the vector differential equation

$$x' = F(t, x), \tag{11}$$

where $F\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the *Carathéodory mapping* such that $F(t, x) \equiv F(t + \omega, x)$, for some given $\omega > 0$, that is,

(i) $F(\cdot, x)\colon [0, \omega] \to \mathbb{R}^n$ is measurable, for every $x \in \mathbb{R}^n$,
(ii) $F(t, \cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous, for almost all (a.a.) $t \in [0, \omega]$.

Let, furthermore (11) satisfy a *uniqueness condition* (e.g., a locally Lipschitz condition) and all solutions of (11) entirely exist on the whole line $(-\infty, \infty)$.

By a *(Carathéodory) solution* $x(\cdot)$ of (11), we understand a locally absolutely continuous function, that is, $x \in AC_{loc}(\mathbb{R}, \mathbb{R}^n)$, which satisfies (11) for a.a. $t \in \mathbb{R}$.

We can associate to (11) the *Poincaré translation operator* $T_\omega\colon \mathbb{R}^n \to \mathbb{R}^n$ along its trajectories as follows:

$$T_\omega(x_0) := \{x(\omega)\colon x(\cdot) \text{ is a solution of (11) such that } x(0) = x_0\}. \tag{12}$$

It is well known (see e.g., Chapter 1.1 in Reference [56]) that T_ω is a homeomorphism such that $T_\omega^k = T_{k\omega}$, (i.e., the semi-group property), for every $k \in \mathbb{N}$.

One can easily detect the one-to-one correspondence between the $k\omega$-periodic solutions of (11), that is, $x(t) \equiv x(t+k\omega)$ but $x(t) \not\equiv x(t+j\omega)$ for $j<k$, and k-periodic points of T_ω, i.e., $x_0 = T_\omega^k(x_0)$ but $x_0 \neq T_\omega^j(x_0)$ for $j<k$, where $x_0 = x(0)$ and j,k are positive integers.

Consider also the vector impulsive differential equation

$$\begin{cases} x' = F(t,x),\ t \neq t_j := j\omega,\ \text{for some given } \omega > 0, \\ x(t_j^+) \in I(x(t_j^-)),\ j \in \mathbb{Z}, \end{cases} \tag{13}$$

where $F\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is the Carathéodory mapping such that $F(t,x) \equiv F(t+\omega,x)$, Equation (11) satisfies a uniqueness condition and a global existence of all its solutions on $(-\infty,\infty)$. Let, furthermore, $I\colon \mathbb{R}^n \multimap \mathbb{R}^n$ be a continuous impulsive mapping.

The solutions of impulsive differential Equation (13) will be also understood in the Carathéodory sense, that is, $x \in AC\,[t_j, t_{j+1}]$, $j \in \mathbb{Z}$. For multivalued impulses I, there need not be any longer one-to-one correspondence between the $k\omega$-periodic solutions of (13) and k-periodic orbits of the composition $I \circ T_\omega$. Nevertheless, every k-periodic orbit of $I \circ T_\omega$ implies the existence of a related $k\omega$-periodic solution of (13), and vice versa.

The first application deals with compact m-valued impulses I in (13).

Theorem 1. *Assume that $I\colon \mathbb{R}^n \multimap \mathbb{R}^n$ is a compact m-valued map such that $I(K_0) = K_0$, where $K_0 := \overline{I(\mathbb{R}^n)}$ is a simply connected ANR-space such that $K_0 \subset K_1 := T_\omega(K_0)$. Then $I|_{K_0}$ is an m-valued split map of the form $I|_{K_0} = \{i_1,\ldots,i_m\}$, and subsequently the number of ω-periodic solutions of* (13) *is at least equal to $\#\{k = 1,\ldots,m \mid L(i_k) \neq 0\}$, where L is the ordinary Lefschetz number.*

Proof. Since I is compact and T_ω is a homeomorphism, $K_0 := \overline{I(\mathbb{R}^n)}$ and $K_1 := T_\omega(K_0)$ must be compact sets. Since every ANR-space is locally path-connected, K_0 is a compact simply connected and locally path-connected ANR-space, and $I|_{K_0}\colon K_0 \multimap K_0$ is (according to Lemma 1) a split m-valued map.

Furthermore, since $T_{\mu\omega}\colon \mathbb{R}^n \to \mathbb{R}^n$ is also a homeomorphism, for every $\mu \in [0,1]$, the composition $I \circ T_{\mu\omega}|_{K_0}\colon K_0 \multimap K_0$, $\mu \in [0,1]$, is an m-valued homotopy between ($\mu = 0$) $I \circ T_0|_{K_0} = I \circ \mathrm{id}\,|_{K_0} = I|_{K_0}\colon K_0 \to K_0$, and ($\mu = 1$) $I \circ T_\omega|_{K_0}\colon K_0 \multimap K_0$. According to Lemma 2, $I \circ T_\omega|_{K_0}\colon K_0 \multimap K_0$ is a split m-valued map, too.

Letting $I|_{K_0} = \{i_1,\ldots,I_m\}$ and applying formula (1), we get that $N(I \circ T_\omega|_{K_0}) = N(I|_{K_0}) = \sum_{k=1}^m N(i_k) = \#\{k = 1,\ldots,m \mid L(i_k) \neq 0\}$. It means that the mapping $I \circ T_\omega|_{K_0}$ has at least such a number of fixed points, which determine the same number of ω-periodic solutions of (13), as claimed. This completes the proof. □

Corollary 1. *Let the assumptions of Theorem 1 be satisfied. Assume additionally that K_0 is a (compact) AR-space. Then Equation* (13) *admits at least m ω-periodic solutions.*

Proof. The claim follows immediately from Theorem 1, when replacing in the proof formula (1) by formula (2) (cf. Remark 1). □

We will supply Corollary 1 by two simple illustrative examples. The first example concerns the scalar case ($n = 1$).

Example 1. *Consider the semi-linear impulsive equation*

$$\begin{cases} x' = p(t,x)x + q(t,x), \quad t \neq t_j := j\omega, \quad \textit{for some given } \omega > 0, \\ x(t_j^+) \in I(x(t_j^-)), \quad j \in \mathbb{Z}, \end{cases} \tag{14}$$

where $p, q\colon \mathbb{R}^2 \to \mathbb{R}$ *are Carathéodory functions such that* $p(t,x) \equiv p(t+\omega, x)$, $q(t,x) \equiv q(t+\omega, x)$, *and the compact m-valued function* $I\colon \mathbb{R} \multimap \mathbb{R}$ *satisfies* $\overline{I(\mathbb{R})} = [0,1]$, *and* $I([0,1]) = [0,1]$.

The solutions $x_0(\cdot), x_1(\cdot)$ *of* $x' = p(t,x)x + q(t,x)$ *such that* $x_0(0) = 0$, $x_1(0) = 1$ *can be implicitly expressed as*

$$x_0(t) = \int_0^t \mathrm{e}^{\int_s^t p(r,x_0(r))\,\mathrm{d}r}\, q(s, x_0(s))\,\mathrm{d}s,$$
$$x_1(t) = \mathrm{e}^{\int_0^t p(s,x_1(s))\,\mathrm{d}s} + \int_0^t \mathrm{e}^{\int_s^t p(r,x_1(r))\,\mathrm{d}r}\, q(s, x_1(s))\,\mathrm{d}s.$$

Hence, the required inclusion $K_0 := [0,1] \subset K_1 := [T_\omega(0), T_\omega(1)]$ *in Corollary 1* ($K_0 := [0,1]$ *is obviously a compact AR-space), takes the form*

$$0 \geq \int_0^\omega \mathrm{e}^{\int_s^\omega p(r,x_0(r))\,\mathrm{d}r}\, q(s, x_0(s))\,\mathrm{d}s,$$
$$1 \leq \mathrm{e}^{\int_0^\omega p(t,x_1(t))\,\mathrm{d}t} + \int_0^\omega \mathrm{e}^{\int_s^\omega p(r,x_1(r))\,\mathrm{d}r}\, q(s, x_1(s))\,\mathrm{d}s.$$

In order to satisfy the first inequality, we can assume that $q(t,x) \leq 0$, *for a.a.* $t \in [0,\omega]$ *and all* $x \in \mathbb{R}$. *The second inequality can be then more restrictively rewritten into*

$$\mathrm{e}^{\int_0^\omega p(t,x_1(t))\,\mathrm{d}t} \geq 1 + \left| \int_0^\omega \mathrm{e}^{\int_s^\omega p(r,x_1(r))\,\mathrm{d}r}\, q(s, x_1(s))\,\mathrm{d}s \right|.$$

Assuming still the existence of real constants p_0, p_1, q_1 *such that*

$$0 < p_0 \leq p(t,x) \leq p_1 \text{ and } |q(t,x)| \leq q_1, \text{ for a.a. } t \in [0,\omega] \text{ and all } x \in \mathbb{R},$$

we still require that

$$q_1 \leq \frac{\mathrm{e}^{p_0\omega} - 1}{\omega\,\mathrm{e}^{p_1\omega}},$$

that is, jointly with $q(t,x) \leq 0$,

$$-\frac{\mathrm{e}^{p_0\omega} - 1}{\omega\,\mathrm{e}^{p_1\omega}} \leq q(t,x) \leq 0, \text{ for a.a. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}, \tag{15}$$

where $0 \leq p_0 \leq p(t,x)$, *for a.a.* $t \in [0,\omega]$ *and all* $x \in \mathbb{R}$.

Thus, the semi-linear impulsive Equation (14) *admits, according to Corollary 1, at least* m ω*-periodic solutions, provided* (15) *holds jointly with* $I\colon \mathbb{R} \multimap \mathbb{R}$ *being a compact m-valued function such that* $\overline{I(\mathbb{R})} = [0,1]$ *and* $I([0,1]) = [0,1]$.

Now, we would like to apply Corollary 1 to the nonlinear vector impulsive differential Equation (13).

Example 2. *Consider* (13), *where F is as above, and assume that the inequalities*

$$\begin{cases} f_j(t,\ldots,x_j,\ldots) > 0 & \text{holds for all } x_j \geq b_j,\ j = 1,\ldots,n, \\ f_j(t,\ldots,x_j,\ldots) < 0 & \text{holds for all } x_j \leq a_j,\ j = 1,\ldots,n, \end{cases} \tag{16}$$

hold uniformly for a.a. $t \in [0,\omega]$ *and all the remaining components of* $x = (x_1,\ldots,x_n)$, *where* $F(t,x) = (f_1(t,x),\ldots,f_n(t,x))^T$. *Let* $I\colon \mathbb{R}^n \multimap \mathbb{R}^n$ *be a compact m-valued map such that* $\left(\overline{I(\mathbb{R}^n)} =\right) K_0 := [a_1,b_1] \times \cdots \times [a_n,b_n]$, $I(K_0) = K_0$. *One can readily check that* K_0 *is a compact AR-space.*

Since, in view of (16), *the inequalities* $x_j(\omega, a_j) \leq a_j$ *and* $x_j(\omega, b_j) \geq b_j$, $j = 1,\ldots,n$, *hold for all the components of the solutions* $x(\cdot,a)$ *and* $x(\cdot,b)$ *such that* $x(0,a) = a$ *and* $x(0,b) = b$, *where* $a = (a_1,\ldots,a_n)$,

$b = (b_1, \dots, b_n)$, the particular inclusion $K_0 \subset K_1$ required in Corollary 1 is satisfied, where $K_0 := [a_1, b_1] \times \dots \times [a_n, b_n]$ and $K_1 := T_\omega(K_0)$.

Therefore, the nonlinear impulsive Equation (13) admits, according to Corollary 1, at least m ω-periodic solutions, provided (16) holds, jointly with $I \colon \mathbb{R}^n \multimap \mathbb{R}^n$ being a compact m-valued map such that $\overline{I(\mathbb{R}^n)} = K_0 := [a_1, b_1] \times \dots \times [a_n, b_n]$ and $I(K_0) = K_0$.

In the non-splitting case (see Remark 5), we can state the following rather theoretical result.

Theorem 2. *Assume that $I \colon \mathbb{R}^n \multimap \mathbb{R}^n$ is a compact m-valued map such that $I(K_0) = K_0$, where $K_0 := \overline{I(\mathbb{R}^n)}$ is a (compact) polyhedron such that $K_0 \subset K_1 := T_\omega(K_0)$. Then Equation (13) has at least $N(I|_{K_0})$ ω-periodic solutions, where the Nielsen number $N(I|_{K_0})$ of the restriction $I|_{K_0} \colon K_0 \multimap K_0$ was defined in Reference [46] by Schirmer (for the sketch, see Section 3).*

Proof. We can proceed quite analogously as in the proof of Theorem 1, when avoiding the parts guaranteeing the split arguments, and use the homotopy invariance to get $N(I \circ T_\omega|_{K_0}) = N(I|_{K_0})$. □

Remark 9. *As already pointed out in Remark 4, to calculate the Nielsen number $N(I|_{K_0})$ is not an easy task in general. That is why Theorem 2 is rather theoretical than practical. In the splitting case, its calculations can be much easier (see again Remark 4). On the other hand, although compact polyhedra are only special ANR-spaces, they need not be simply connected, as required in Lemma 1, Theorem 1 and Corollary 1 (cf. Remark 5).*

6. Application to Impulsive Differential Equations on $\mathbb{R}^n/\mathbb{Z}^n$

We will also apply the presented results for m-valued maps to impulsive differential equations on tori.

Hence, consider the Equation (11) with the same assumptions as in Section 5.

Assuming still that

$$F(t, \dots, x_j, \dots) \equiv F(t, \dots, x_j + 1, \dots), \quad j = 1, \dots, n, \tag{17}$$

where $x = (x_1, \dots, x_n)$, we can also consider (11) on the torus (factor space) $\mathbb{R}^n/\mathbb{Z}^n$, which can be endowed with the metric

$$\widehat{d}(x, y) := \min \{d_{Eucl}(a, b) \colon a \in [x], b \in [y]\},$$

for all $x, y \in \mathbb{R}^n/\mathbb{Z}^n$, where $d_{Eucl}(a, b) := \sqrt{\sum_{j=1}^n (a_j - b_j)^2}$, for all $a, b \in \mathbb{R}^n$.

The solutions of (11), considered on $\mathbb{R}^n/\mathbb{Z}^n$, will be also understood in the same Carathéodory sense.

The associated *Poincaré translation operator* $\widehat{T}_\omega \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ along the trajectories of (11), considered on $\mathbb{R}^n/\mathbb{Z}^n$, takes the form $\widehat{T}_\omega := \tau \circ T_\omega$, where T_ω was defined in (12), and $\tau \colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $x \to [x] := \{y \in \mathbb{R}^n \colon (y - x) \in \mathbb{Z}^n\}$ is the natural (canonical) projection. It is well known (see e.g., Chapter XVII in Reference [57]) that $\widehat{T}_\omega$ is also a homeomorphism such that $\widehat{T}_\omega^k = \widehat{T}_{k\omega}$ (i.e., the semi-group property), for every $k \in \mathbb{N}$. In particular, for $n = 1$, $\widehat{T}_\omega$ is an orientation-preserving homeomorphism.

The same one-to-one correspondence holds between $k\omega$-periodic solutions $\widehat{x}(\cdot) := \tau \circ x(\cdot)$ of (11), considered on $\mathbb{R}^n/\mathbb{Z}^n$, and k-periodic points $\widehat{x}_0 = \tau \circ x_0$ of $\widehat{T}_\omega := \tau \circ T_\omega$, where $\widehat{x}_0 = \widehat{x}(0)$.

We will still consider, under (17), the impulsive differential Equation (13), where this time

$$I(\dots, x_j, \dots) \equiv I(\dots, x_{j+1}, \dots) (\text{mod } 1), j = 1, \dots, n, \tag{18}$$

$x = (x_1, \dots, x_n)$, by which $\widehat{I} := \tau \circ I \colon \mathbb{R}^n/\mathbb{Z}^n \multimap \mathbb{R}^n/\mathbb{Z}^n$.

Every k-periodic orbit of the composition $\widehat{I \circ T_\omega} := \widehat{I} \circ \widehat{T_\omega} \colon \mathbb{R}^n/\mathbb{Z}^n \multimap \mathbb{R}^n/\mathbb{Z}^n$ implies then again the existence of a related $k\omega$-periodic solution of (13) on $\mathbb{R}^n/\mathbb{Z}^n$ on $\mathbb{R}^n/\mathbb{Z}^n$, and vice versa.

The following application is for $n > 1$, like in Theorem 2, rather theoretical than practical in the non-splitting case.

Theorem 3. *Assume that* $I\colon \mathbb{R}^n \multimap \mathbb{R}^n$ *is an* m*-valued* (mod 1) *map, satisfying* (18)*. Then Equation* (13) *admits, under* (17)*, at least* $N(\widehat{I})$ ω*-periodic* (mod 1) *solutions, where the Nielsen number* $N(\widehat{I})$ *of* $\widehat{I} = \tau \circ I\colon \mathbb{R}^n/\mathbb{Z}^n \multimap \mathbb{R}^n/\mathbb{Z}^n$ *was defined in Reference [46] by Schirmer (for the sketch, see Section 3).*

Proof. Since the torus $\mathbb{R}^n/\mathbb{Z}^n$ is a special compact polyhedron and $\widehat{I} = \tau \circ I\colon \mathbb{R}^n/\mathbb{Z}^n \multimap \mathbb{R}^n/\mathbb{Z}^n$ is a (compact) m-valued map, it follows that the composition $\widehat{I} \circ \widehat{T_{\mu\omega}}\colon \mathbb{R}^n/\mathbb{Z}^n \multimap \mathbb{R}^n/\mathbb{Z}^n$, $\mu \in [0,1]$, of a homeomorphism $\widehat{T_{\mu\omega}}$ with $\widehat{I}$ must be also a (compact) m-valued map on a compact polyhedron, for every $\mu \in [0,1]$, that is, an m-valued homotopy on $\mathbb{R}^n/\mathbb{Z}^n$.

Thus, $N(\widehat{I} \circ \widehat{T_\omega}) = N(\widehat{I})$ holds for the Nielsen numbers, because of the invariance under homotopy $\widehat{I} \circ \widehat{T_{\mu\omega}}$, for $\mu = 0, 1$, that is, $N(\widehat{I} \circ \widehat{T_\omega}) = N(\widehat{I} \circ \widehat{T_0}) = N(\widehat{I} \circ \mathrm{id}\,|_{\mathbb{R}^n/\mathbb{Z}^n}) = N(\widehat{I})$.

The composition $\widehat{I} \circ \widehat{T_\omega}$ has therefore at least $N(\widehat{I})$ fixed points, and each of them determines the existence of an ω-periodic solution on $\mathbb{R}^n/\mathbb{Z}^n$, that is, an ω-periodic (mod 1) solution of (13), as claimed. □

Remark 10. *In the splitting case, the Nielsen number* $N(\widehat{I})$ *in Theorem 3 is much easier for calculation by the formula*

$$N(\widehat{I}) = \sum_{k=1}^{m} N(\widehat{i_k}) = \sum_{k=1}^{m} \left| L(\widehat{i_k}) \right| = \sum_{k=1}^{m} |\det(J - A_k)| = \sum_{k=1}^{m} \left| \prod_{j=1}^{n} (1 - \lambda_j^{(k)}) \right|, \tag{19}$$

where $\widehat{I} := \left\{ \widehat{i}_1, \dots, \widehat{i}_m \right\}$, $\widehat{i}_k\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ *are endomorphisms defined by integer matrices* A_k, $k = 1, \dots, m$, *with eigenvalues* $\lambda_j^{(k)}$, $j = 1, \dots, n$; $k = 1, \dots, m$. *If, in particular,* I *is an* m*-valued constant* (mod 1)*, that is,* $\widehat{I}$ *is an* m*-valued constant on* $\mathbb{R}^n/\mathbb{Z}^n$*, then* $N(\widehat{I}) = m$ *(cf. Remark 4).*

For $n = 1$, the difficult calculation of the Nielsen number $N(\widehat{I})$ in Theorem 3 can be simplified (even in the non-splitting case) by means of Lemmas 3 and 4 as follows.

Corollary 2. *Assume that* $I\colon \mathbb{R} \multimap \mathbb{R}$ *is an* m*-valued* (mod 1) *map such that* $I(x) \equiv I(x+1)$ (mod 1)*. Then the scalar* ($n = 1$) *Equation* (13) *admits, under* (17) *with* $n = 1$*, at least* $\left| m - \mathrm{Deg}(\widehat{I}) \right|$ ω*-periodic* (mod 1) *solutions, where the degree* $\mathrm{Deg}(\widehat{I})$ *of* $\widehat{I} = \tau \circ I\colon \mathbb{R}^n/\mathbb{Z}^n \multimap \mathbb{R}^n/\mathbb{Z}^n$ *was defined in Reference [33] by Brown (cf. Section 3). If* $\widehat{I}$ *is still split, that is,* $\widehat{I} := \left\{ \widehat{i}_1, \dots, \widehat{i}_m \right\}$*, then* (13) *admits, under the same assumptions, at least* $m\,|1 - [i_1(1) - i_1(0)]|$ ω*-periodic* (mod 1) *solutions, where* $I_1\colon \mathbb{R} \to \mathbb{R}$ *is the lift of* $\widehat{i}_1$*, that is, the first component of* I.

Proof. According to Theorem 3, (13) admits at least $N(\widehat{I})$ ω-periodic (mod 1) solutions. In view of Lemma 4, we get that $N(\widehat{I}) = \left| m - \mathrm{Deg}(\widehat{I}) \right|$.

If $\widehat{I}$ is still split, then by means of (3) in Lemma 3 we have that $\mathrm{Deg}(\widehat{I}) = m\,[i_1(1) - i_1(0)]$, and so we arrive at $N(\widehat{I}) = m\,|1 - [i_1(1) - i_1(0)]|$, that is, (4) in Remark 3, which completes the proof. □

Corollary 2 can be illustrated by the following simple example.

Example 3. *Consider the scalar* ($n = 1$) *Equation* (13)*, satisfying* (17) *with* $n = 1$*. Let* $I := \{T_1, T_2\} : \mathbb{R} \multimap \mathbb{R}$*, where* T_1 *is the* 1*-periodically extended standard tent map, that is,* $T_1(x) \equiv T_1(x+1)$*, where*

$$T_1(x) := \begin{cases} 1x, & \text{for } x \in [0, \frac{1}{2}], \\ 2(1-x), & \text{for } x \in [\frac{1}{2}, 1], \end{cases}$$

and $T_2 = T_1 + \frac{1}{2}$.

Although $\widehat{I}\colon \mathbb{R}/\mathbb{Z} \multimap \mathbb{R}/\mathbb{Z}$ *has evidently four fixed points* $0, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$, *we get that* $N(\widehat{I}) = 2\,|1 - [T_1(1) - T_1(0)]| = 2$. *It is not surprising, because the both tent maps* T_1, T_2 *can be easily homotopically deformed to the maps having just one fixed point for each (i.e., that* 0 *and* $\frac{1}{2}$ *are not essential).*

Observe that, according to (19), *we also get that (for* $m = 2, n = 1$*)* $N(\widehat{I}) = \left|1 - \lambda_1^{(1)}\right| + \left|1 - \lambda_1^{(2)}\right| = 2$, *because obviously* $\lambda_1^{(1)} = \lambda_1^{(2)} = 0$.

In any way, since $N(\widehat{I} \circ \widehat{T_\omega}) = N(\widehat{I}) = 2$, *the scalar Equation* (13) *possesses, under* (17) *with* $n = 1$, *at least two* ω*-periodic* (mod 1) *solutions.*

7. Fractals and Chaos Determined by Impulsive Differential Equations

In this section, we would like to apply Propositions 1–3 to fractals and chaos, determined implicitly by impulsive differential Equations (13).

Hence, consider (13) and assume the same as in Section 5. We will suppose that I takes the form of the Hutchinson-Barnsley operator, namely

$$I := \bigcup_{k=1}^{m} i_k \colon \mathbb{R}^n \multimap \mathbb{R}^n, \tag{20}$$

where $i_k\colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, m$, are at least continuous functions. The continuous impulsive mapping $I\colon \mathbb{R}^n \multimap \mathbb{R}^n$ need not (but can) be any longer m-valued (cf. Remark 7).

We start with the application of Proposition 3.

Theorem 4. *Let* $I\colon \mathbb{R}^n \multimap \mathbb{R}^n$ *be still a compact map and* $X \subset \mathbb{R}^n$ *be a connected and locally connected subset containing* $I(\mathbb{R}^n)$*, that is,* $I(\mathbb{R}^n) \subset X$*. Then the composition* $I \circ T_\omega|_X \colon X \multimap X$*, where* T_ω *is the Poincaré translation operator along the trajectories of* (11)*, defined in* (12)*, possesses at least one compact invariant set* $X_0 \subset X$*, that is,* $I \circ T_\omega|_X(X_0) = X_0$*. If* X *is still compact, then* $I \circ T_\omega|_X$ *possesses at least one compact invariant set* $X_0 \subset X$ *which is non-ejective in the sense of Browder, that is,*

$$\begin{array}{l} \forall \varepsilon > 0 \exists X_1 \in K(X), X_1 \neq X_0, \text{ and } d_H(X_0, X_1) < \varepsilon : \\ (I \circ T_\omega|_X)^m(X_1) \subset \{Y \in K(X) \mid d_H(X_0, Y) < \varepsilon\}, \forall m \geq 1. \end{array} \tag{21}$$

The same is true for the hypersystem $(K(X), (I \circ T_\omega|_X)^*)$*, where* $(I \circ T_\omega|_X)^*(A) := \bigcup_{a \in A} I \circ T_\omega|_X(a)$.

Proof. Since I is by the hypothesis a compact map, $\overline{I(\mathbb{R}^n)}$ must be compact, and there certainly exists a connected and locally connected subset $X \subset \mathbb{R}^n$ such $I(\mathbb{R}^n) \subset X$. Since $\mathbb{R}^n$ is locally compact, X is also locally continuum-connected, that is, for each neighbourhood U of each point $x \in X$, there is a neighbourhood $V \subset U$ of x such that each point of V can be connected with x by a subcontinuum of U. If X is still compact, then it is a Peano's continuum, that is, compact connected and locally connected.

Then the composition $I \circ T_\omega|_X \colon X \multimap X$ is a compact continuous map, which possesses according to Proposition 3 at least one compact invariant set $X_0 \in K(X)$.

If X is still compact, and so a Peano's continuum, then $I \circ T_\omega|_X$ possesses, again according to Proposition 3, at least one compact invariant set $X_0 \in K(X)$ which is non-ejective in the sense of Browder, that is, (21), which completes the proof.

The proof for the hypersystem $(K(X), (I \circ T_\omega|_X)^*)$ follows directly by the arguments in References [29,54]. □

Remark 11. *The set* X_0 *can be called a* topological fractal *in the sense of References [29,53], because it was obtained by means of the Lefschetz-type fixed point theorem as a fixed point in the hyperspace* $(K(X), d_H)$*, that is, in the frame of the topological fixed point theory (cf. e.g., Reference [22]). Let us note that our terminology* *differs from the one in e.g., References [58–60] where the notion of a topological fractal is understood in a different way.*

Remark 12. *The meaning of Theorem 4 can be also interpreted geometrically in terms of impulsive differential Equation (13). Every solution $x(\cdot)$ of (13) such that $x(0) \in X_0$ satisfies $x(k\omega) \in X_0$, for all $k = 1, 2, \ldots$ If X is still compact, then there moreover exist trajectories starting at any ε-neighbourhood $\mathcal{N}_\varepsilon(X_0)$ of X_0 such that $x(k\omega) \in \mathcal{N}_\varepsilon(X_0)$, for all $k = 1, 2, \ldots$*

Now, we will consider (13), where $i_k \colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, m$, in (20) are contractions. Hence, consider (13) with the same assumptions as above and suppose additionally that $i_k \colon \mathbb{R}^n \to \mathbb{R}^n$ in (20) are contractions with factors $L_k \in [0, 1)$, for every $k = 1, 2, \ldots, m$. Let A be a unique global compact IFS-attractor of $\{\mathbb{R}^n; i_1, \ldots, i_m\}$, guaranteed by Proposition 1. Let furthermore $T_\omega \colon \mathbb{R}^n \to \mathbb{R}^n$ be the Poincaré translation operator along the trajectories of (11), defined in (12).

Letting $I' := \bigcup_{k=1}^m i'_k \colon \mathbb{R}^n \multimap \mathbb{R}^n$, where $i'_k := T_\omega \circ i_k \circ T_\omega{}^{-1}$, $k = 1, \ldots, m$, and $I'' := \bigcup_{k=1}^m i''_k \colon \mathbb{R}^n \multimap \mathbb{R}^n$, where $i''_k := T_\omega{}^{-1} \circ i_k \circ T_\omega$, $k = 1, \ldots, m$, we can formulate the following theorem.

Theorem 5. *There exists a unique global compact attractor $B := T_\omega(A) \in K(\mathbb{R}^n)$ of the system $\{\mathbb{R}^n; i'_1, \ldots, i'_m\}$, that is, $B = I'(B)$ such that $B = \lim_{j\to\infty} I'^j(B_0)$ holds for any $B_0 \in K(\mathbb{R}^n)$.*

There also exists a unique global compact attractor $C = T_\omega{}^{-1}(A) \in K(\mathbb{R}^n)$ of the system $\{\mathbb{R}^n; i''_1, \ldots, i''_m\}$, that is, $C = I''(C)$ such that $C = \lim_{j\to\infty} I''^j(C_0)$ holds for any $C_0 \in K(\mathbb{R}^n)$.

The fractal (Hausdorff) dimensions $\dim B$ and $\dim C$ of both B and C can be estimated in the same way from above, that is, $\dim B \le D$ and $\dim C \le D$, by means of a unique solution D of the equation $\sum_{k=1}^m L_k^D = 1$. If i_k are similitudes, for all $k = 1, \ldots, m$, then $\dim B = \dim C = D$.

Proof. The iterated function system $\{\mathbb{R}^n; i_1, \ldots, i_m\}$ has, according to Proposition 1 a unique global compact attractor $A \in K(\mathbb{R}^n)$ such that $A = I(A)$ (cf. (20)). It can be obtained as the limit $\lim_{j\to\infty} I^j(A_0)$, for $A_0 \in K(\mathbb{R}^n)$. The inequality (cf. (5))

$$d_H(I^j(A_0), A) \le \frac{L^j}{1-L} d_H(A_0, I(A_0)),$$

$L = \max_{k=1,\ldots,m} L_k$, holds for the upper estimate of the Hausdorff distance between A and its successive approximations $I^j(A_0)$. The fractal (Hausdorff) dimension $\dim A$ of A can be estimated from above, that is, $\dim A < D$, by means of a unique solution D of the Moran-Hutchinson equation $\sum_{k=1}^m L_k^D = 1$. In case of similitudes i_k, for all $k = 1, \ldots, m$, we have the precise value $\dim A = D$.

Now, consider the associated (via the Poincaré operator T_ω) systems $\{\mathbb{R}^n; i'_1, \ldots, i'_m\}$ and $\{\mathbb{R}^n; i''_1, \ldots, i''_m\}$. It is known (see e.g., Lemma 2.8 in Reference [61]) that these associated systems have unique global compact attractor $B := T_\omega(A)$ and $C := T_\omega{}^{-1}(A)$, respectively, that is, $B = I'(B)$ and $C = I''(C)$, where $B = \lim_{j\to\infty} I'^j(B_0)$ holds for any $B_0 \in K(\mathbb{R}^n)$ and $C = \lim_{j\to\infty} I''^j(C_0)$ holds for any $C_0 \in K(\mathbb{R}^n)$.

Furthermore, the global attractivity of B, C is preserved from the global attractivity of A, jointly with their fractal (Hausdorff) dimension.

Since $A = I(A)$, the existence of unique compact invariant sets B under I' and C under I'' follows easily from the commutative diagrams:

$$\begin{tikzcd} A \arrow[r, "I", -o] \arrow[d, "T_\omega"'] & A \arrow[d, "T_\omega"] \\ B \arrow[r, "I'", -o] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "I", -o] & A \\ C \arrow[u, "T_\omega"] \arrow[r, "I''", -o] & C \arrow[u, "T_\omega"'] \end{tikzcd}$$

because

$$I'(B) = T_\omega \circ I \circ T_\omega{}^{-1}(T_\omega(A)) = T_\omega \circ I(A) = T_\omega(A) = B,$$
$$I''(C) = T_\omega{}^{-1} \circ I \circ T_\omega(T_\omega{}^{-1}(A)) = T_\omega{}^{-1} \circ I(A) = T_\omega{}^{-1}(A) = C.$$

Since being an IFS-attractor is known (see e.g., Reference [59]) to be a bi-Lipschitz invariant and even an equi-Hölder invariant, we can simplify the proof of the attractivity of B and C by showing that the associated Poincaré operator T_ω is bi-Lipschitz. Since T_ω and $T_\omega{}^{-1}$ are homeomorphisms, it is enough to show that the restrictions $T_\omega|_{K_1}$ as well as $T_\omega{}^{-1}|_{K_2}$ are Lipschitz on any compact subsets $K_1, K_2 \subset \mathbb{R}^n$. It is true, provided the right-hand side F in (11) is locally Lipschitz, which is the most common uniqueness condition. The same is then true for T_ω (see e.g., Reference [56]).

Since the Hausdorff dimension is also a bi-Lipschitz invariant (see e.g., Corollary 2.4 on p. 32 in Reference [62], Chapter 8.3 in Reference [63]), the proof is complete. □

Remark 13. *The IFS-attractors B, C in Theorem 5 can be called* topological fractals *in the sense of References [58–60] and, jointly with a* metric fractal A *in the sense of References [29,53], they can be called more precisely* Banach fractals *in the sense of References [58,59].*

Remark 14. *Although the fixed points $\bar{x} \in A$ of I, described in Remark 6, correspond to the fixed points $T_\omega(\bar{x}) \in B$ of I' and $T_\omega{}^{-1}(\bar{x}) \in C$ of I'', they need not be anyhow related to ω-periodic solutions of (13), as in Sections 5 and 6. Nevertheless, if i_k in (20) are constants, then they determine ω-periodic solutions of (13) (cf. Remark 4) and their images under T_ω and $T_\omega{}^{-1}$ are the fixed points of I' and I''.*

The application of Theorem 5 can be illustrated by the following example.

Example 4. *Consider the impulsive pendulum system under a 1-periodic ($\omega = 1$) forcing:*

$$\begin{cases} x' = y, \quad y' = -\sin x + \cos 2\pi t, \quad t \neq t_j := j, \\ (x(t_j^+), y(t_j^+)) \in I(x(t_j^-), y(t_j^-)), \quad j \in \mathbb{Z}, \end{cases} \tag{22}$$

where $I\colon \mathbb{R}^2 \multimap \mathbb{R}^2$ is the affine system of disconnected similitudes such that

$$I(x, y) := i_1(x, y) \cup i_2(x, y) \cup i_3(x, y),$$

$$i_1(x, y) = (0.45x, 0.45y),$$
$$i_2(x, y) = (0.45x, 0.45y + 0.55),$$
$$i_3(x, y) = (0.45x + 0.55, 0.45y).$$

The associated Poincaré translation operator $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ takes the form

$$T_1(x_0, y_0) := \Big\{(x(1), y(1)) \mid (x(\cdot), y(\cdot)) \text{ is a solution of the system } x' = y,\ y' = -\sin x + \cos 2\pi t \text{ such that } (x(0), y(0)) = (x_0, y_0)\Big\}.$$

Its inversion $T_1^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ can be defined as

$$T_1^{-1}(x_1, y_1) := \Big\{(x(0), y(0)) \mid (x(\cdot), y(\cdot)) \text{ is a solution of the system } x' = y,\ y' = -\sin x + \cos 2\pi t \text{ such that } (x(1), y(1)) = (x_1, y_1)\Big\}.$$

The Sierpiński-like attractor $A = I(A)$ of the system $\{\mathbb{R}^2; i_1, i_2, i_3\}$, the attractor $B = T_1(A)$ of the system $\{\mathbb{R}^2; i'_1, i'_2, i'_3\}$, and the attractor $C = T_1^{-1}(A)$ of the system $\{\mathbb{R}^2; i''_1, i''_2, i''_3\}$ are depicted in Figure 1. The fractal (Hausdorff) dimension D of the attractors A, B, C can be easily calculated as

$$D = \dim A = \dim B = \dim C = \frac{\log 3}{\log \frac{1}{0.45}} \doteq 1.376.$$

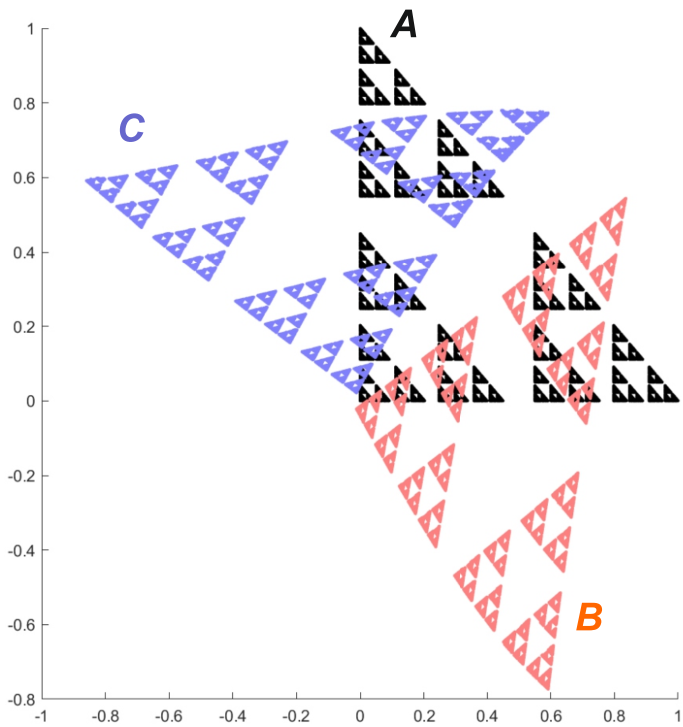
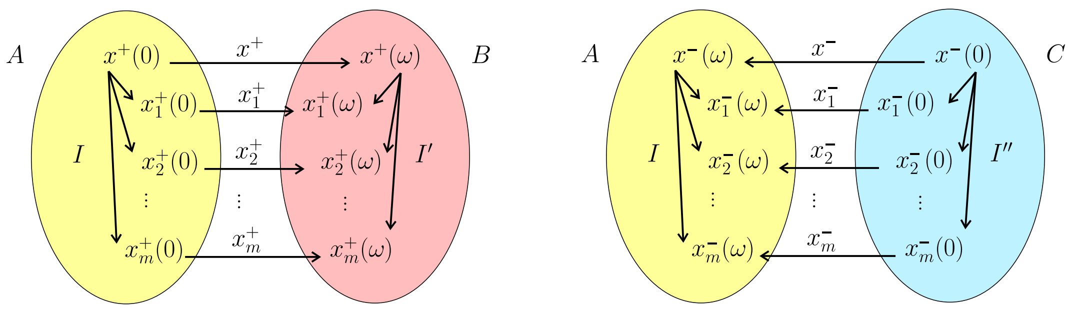


Figure 1. Attractor $A = I(A)$, attractor $B = T_1(A)$ and attractor $C = T_1^{-1}(A)$.

Remark 15. *The meaning of the illustrative Example 4 can be better understood by means of the following commutative diagrams ($m = 3$, $\omega = 1$):*



where $x^+, x_1^+, \ldots, x_m^+$ are the solutions of (22) such that $x^+(0)$, $x_1^+(0), \ldots, x_m^+(0) \in A$ and $x^+(\omega), x_1^+(\omega), \ldots, x_m^+(\omega) \in B$, while $x^-, x_1^-, \ldots, x_m^-$ are the solutions of (22) such that $x^-(0), x_1^-(0), \ldots, x_m^-(0) \in C$ and $x^-(\omega), x_1^-(\omega), \ldots, x_m^-(\omega) \in A$. The attractors A, $B = T_\omega(A)$, $C = T_\omega{}^{-1}(A)$ such that $A = I(A)$, $B = I'(B)$, $C = I''(C)$ are depicted in Figure 1.

Now, consider (13) with the same assumptions as above and suppose additionally that $i_k : \mathbb{R}^n \to \mathbb{R}^n$ in (20) are contractions such that the sets $i_k(A)$ are totally disconnected, for all $k = 1, \ldots, m$ (see (7)), where $A \in K(\mathbb{R}^n)$ is a unique global attractor of the iterated function system $\{\mathbb{R}^n; i_1, \ldots, i_m\}$ guaranteed by Proposition 1. Let $T_\omega\colon \mathbb{R}^n \to \mathbb{R}^n$ be the Poincaré translation operator along the trajectories of (11), defined in (12). Then we can define the shift dynamical systems (B, S_1) and (C, S_2), associated with the systems $\left\{B; i'_1|_B, \dots, i'_m|_B\right\}$ and $\left\{C; i''_1|_C, \dots, i''_m|_C\right\}$, defined in Theorem 5, where $B := T_\omega(A)$ and $C := T_\omega{}^{-1}(A)$.

Concretely, $S_1\colon B \to B$ is a single-valued continuous mapping such that

$$S_1(b) := \bigcup_{k=1}^{m} i'^{-1}_k(b)$$

and $S_1(b) := i'^{-1}_k(b) = T_\omega \circ i_k^{-1} \circ T_\omega{}^{-1}(b)$, provided $b \in i'_k(B)$, $k = 1, \dots, m$. Respectively, $S_2\colon C \to C$ is a single-valued continuous mapping such that

$$S_2(c) := \bigcup_{k=1}^{m} i''^{-1}_k(c)$$

and $S_2(c) := i''^{-1}_k(c) = T_\omega{}^{-1} \circ i_k^{-1} \circ T_\omega(c)$, provided $c \in i''_k(C)$, $k = 1, \dots, m$.

Theorem 6. *The shift dynamical systems (B, S_1) and (C, S_2) are chaotic in the sense of Devaney (cf. Section 4). The same is true for the induced hypersystems $(K(B), S_1^*)$ and $(K(C), S_2^*)$ in the hyperspaces $(K(B), d_H)$ and $(K(C), d_H)$, where $S_1^*(\mathcal{B}) := \bigcup_{b\in\mathcal{B}} S_1(b)$ and $S_2^*(\mathcal{C}) := \bigcup_{c\in\mathcal{C}} S_2(c)$.*

Proof. According to Theorem 5, B and C are unique global compact attractors of the respective systems $\{\mathbb{R}^n; i'_1, \dots, i'_m\}$ and $\{\mathbb{R}^n; i''_1, \dots, i''_m\}$. Thus, $I'|_B\colon B \multimap B$ and $I''|_C\colon C \multimap C$, and we can restrict ourselves to the systems $\left\{B; i'_1|_B, \dots, i'_m|_B\right\}$ and $\left\{C; i''_1|_C, \dots, i''_m|_C\right\}$.

Since $i_k^{-1}\colon A \to A$ were shown in Section 4 to be injective and invertible, and so single-valued and continuous, the latter must be true for $i'^{-1}_k = T_\omega \circ i_k^{-1} \circ T_\omega{}^{-1}$ as well as $i''^{-1}_k = T_\omega{}^{-1} \circ i_k^{-1} \circ T_\omega$, for all $k = 1, \dots, m$.

Furthermore, since the sets $i'_k(B)$ and $i''_k(C)$ must be also totally disconnected, the single-valued continuous Hutchinson-Barnsley operators $S_1\colon B \to B$ and $S_2\colon C \to C$ can be uniquely defined by

$$S_1(b) := i'^{-1}_k(b) = T_\omega \circ i_k^{-1} \circ T_\omega{}^{-1}(b), \text{ provided } b \in i'_k(B),$$

and

$$S_2(c) := i''^{-1}_k(c) = T_\omega{}^{-1} \circ i_k^{-1} \circ T_\omega(c), \text{ provided } c \in i''_k(C),$$

for all $k = 1, \dots, m$.

Since Devaney's chaos is well known to be a topological invariant (i.e., an invariant under a topological conjugacy) on a compact metric space (see e.g., Reference [64]), the both systems (B, S_1) and (C, S_2) must be chaotic in the sense of Devaney, because the same was recalled to be true for the system $(A, i_1^{-1}, \dots, i_m^{-1})$ in Section 4.

Since the induced maps $S_1^*\colon K(B) \to K(B)$ and $S_2^*\colon K(C) \to K(C)$ in the hyperspaces $(K(B), d_H)$ and $(K(C), d_H)$, where

$$S_1^*(\mathcal{B}) := \bigcup_{b\in\mathcal{B}} S_1(b), \mathcal{B} \in K(B), \text{ and } S_2^*(\mathcal{C}) := \bigcup_{c\in\mathcal{C}} S_2(c), \mathcal{C} \in K(C),$$

are single-valued continuous, because so are the maps $S_1\colon B \to B$ and $S_2\colon C \to C$ (see Reference [53]), we can also consider the shift dynamical hypersystems $(K(B), S_1^*)$ and $(K(C), S_2^*)$.

Since the induced Poincaré operators $T_\omega{}^*\colon K(\mathbb{R}^n) \to K(\mathbb{R}^n)$ remain homeomorphisms and $S_1^* = T_\omega{}^* \circ (i_k^{-1})^* \circ (T_\omega{}^{-1})^*$, $S_2^* = (T_\omega{}^{-1})^* \circ (i_k^{-1})^* \circ T_\omega{}^*$, the hypersystems $(K(B), S_1^*)$ and $(K(C), S_2^*)$

must be by the same arguments in Reference [64], in view of the last part of Proposition 2, chaotic in the sense of Devaney, too. $\square$

Remark 16. *It is well known that the density of periodic points in the basic space implies the one for induced maps in hyperspaces (see e.g., Reference [30]). Therefore, since the transitivity together with density of periodic points imply a sensitive dependence on initial conditions (i.e., (ii), (iii) $\Rightarrow$ (i), for the conditions in the definition of Devaney's chaos in Section 4), provided the basic space is infinite (see again e.g., Reference [30]), we would have to restrict ourselves to transitivity in the alternative proof of Devaney's chaos for the hypersystems $(K(B), S_1^*)$ and $(K(C), S_2^*)$. The transitivity in hyperspaces is however not implied in general by the one in the original basic space (see Reference [30]).*

Example 5 (continued Example 4)**.** *Consider again systems (22). In order to apply Theorem 6, let us define the shift dynamical system (B, S_1) and (C, S_2), where $B = T_1(A)$, $C = T_1^{-1}(A)$, $A = I(A)$,*

$$S_1(b) := \begin{cases} T_1 \circ \left(\frac{x}{0.45}, \frac{y}{0.45}\right) \circ T_1^{-1}(b), & \text{for } b \in i_1'(B), \\ T_1 \circ \left(\frac{x}{0.45}, \frac{y-0.55}{0.45}\right) \circ T_1^{-1}(b), & \text{for } b \in i_2'(B), \\ T_1 \circ \left(\frac{x-0.55}{0.45}, \frac{y}{0.45}\right) \circ T_1^{-1}(b), & \text{for } b \in i_3'(B), \end{cases}$$

and

$$S_2(c) := \begin{cases} T_1^{-1} \circ \left(\frac{x}{0.45}, \frac{y}{0.45}\right) \circ T_1(c), & \text{for } c \in i_1''(B), \\ T_1^{-1} \circ \left(\frac{x}{0.45}, \frac{y-0.55}{0.45}\right) \circ T_1(c), & \text{for } c \in i_2''(B), \\ T_1^{-1} \circ \left(\frac{x-0.55}{0.45}, \frac{y}{0.45}\right) \circ T_1(c), & \text{for } c \in i_3''(B). \end{cases}$$

The shift dynamical systems (B, S_1) and (C, S_2) are, according to Theorem 6, chaotic in the sense of Devaney.

The same is true, according to Theorem 6, for the induced hypersystems $(K(B), S_1^*)$ and $(K(C), S_2^*)$, where $S_1^*(\mathcal{B}) := \bigcup_{b \in \mathcal{B}} S_1(b)$ and $S_1^*(\mathcal{C}) := \bigcup_{c \in \mathcal{C}} S_2(c)$.

8. Concluding Remarks

As already pointed out in the Introduction, no Nielsen periodic point theory so far exists for n-valued maps. In case of any progress for at least $n = 2$, we could think about multiple subharmonic (i.e., $k\omega$-periodic with $k > 1$) solutions of impulsive differential equations.

If a uniqueness condition is omitted at given (impulsive) differential equations or, more generally, at (impulsive) differential inclusions, then the associated Poincaré translation operators become multivalued and, in particular, admissible in the sense of Górniewicz (see Reference [26]). Then, however, the needed information about their compositions with n-valued impulsive maps would be lost or, even worse, useless. On the other hand, in the splitting case, their compositions with single-valued selections might be useful for at least partial answers. As a starting point, one can therefore study the same problems as above for differential inclusions with single-valued impulses, as indicated e.g., in Reference [19].

Another problem is to randomize at least some obtained results or to extend some random results like those in References [15,65] along the lines discussed above. Unfortunately, the randomization technique which we have to our disposal (see e.g., Reference [15], and the references therein) is based on the existence of measurable selections whose multiplicity can be lost. In other words, no Nielsen theory so far exists for random fixed points.

A more promissible situation concerning a possible extension and application of the obtained results seems to be for the Hutchinson-Barnsley operators (see Remark 7). The appropriate application can be possible, for instance, for a fractal image compression of pictures. The results might be also randomized.

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References

1. Bainov, D.; Simeonov, P. *Impulsive Differential Equations: Periodic Solutions and Applications*; Pitman Monographs and Surveys in Pure and Applied Mathematics; Longman Scientific&Technical: Marlow, UK, 1993; Volume 66.
2. Perestyuk, N.A.; Plotnikov, V.A.; Somoilenko, A.M.; Skripnik, N.V. *Differential Equations with Impulsive Effects. Multivalued Right-hand Sides with Discontinuities*; De Gruyter Studies in Mathematics; De Gruyter: Berlin, Germany, 2011; Volume 40.
3. Samoilenko, A.M.; Stanzhytskyi, O. *Qualitative and Asymptotic Analysis of Differential Equations with Random Perturbations*; World Scientific Series on Nonlinear Science Series A; World Scientific Publishing: Singapore, 2011; Volume 78.
4. Song, X.; Gno, H.; Shi, X. *Theory and Applications of Impulsive Differential Equations*; Science Press: Beijing, China, 2011.
5. Dishlieva, K. Impulsive differential equations and applications. *J. Appl. Comput. Math.* **2012**, *1*, 1–3. [CrossRef]
6. Li, X.; Bohner, M.; Wang, C.K. Impulsive differential equations: Periodic solutions and applications. *Automatica* **2015**, *52*, 173–178. [CrossRef]
7. Church, K.E.; Smith, R.J. Comparing malaria surveillance with periodic spraying in the presence of insecticide-resistant mosquitoes: Should we spray regularly or based on human infections? *Math. Biosci.* **2016**, *276*, 145–163. [CrossRef] [PubMed]
8. Miron, R.E.; Smith, R.J. Modelling imperfect adherence to HIV induction therapy. *BMC Infect. Dis.* **2010**, *10*, 6. [CrossRef]
9. Smith, R.J.; Wolkowicz, G.S.K. Analysis of a model of the nutrient driven self-cycling fermentation process. *Dyn. Contin. Discret. Impuls. Syst. Ser. B Appl. Algorithms* **2004**, *11*, 239–265.
10. Wang, S.; Huang, Q. Bifurcation of nontrivial periodic solutions for a food chain Beddington–DeAngelis interference model with impulsive effect. *Int. J. Bifurc. Chaos* **2018**, *28*, 1850131. [CrossRef]
11. Zhang, Q.; Tang, B.; Cheng, T.; Tang, S. Bifurcation analysis of a generalized impulsive Kolmogorov model with applications to pest and disease control. *SIAM J. Appl. Math.* **2020**, *80*, 1796–1819. [CrossRef]
12. Abbas, S.; Benchohra, M. *Advanced Functional Evolution Equations and Inclusions*; Springer: Berlin, Germany, 2015.
13. Abada, N.; Agarwal, R.P.; Benchohra, M.; Hammouchi, H. Impulsive demilinear neutral functional differential inclusions with multivalued jumps. *Appl. Math.* **2011**, *56*, 227–250. [CrossRef]
14. Andres, J. The standard Sharkovsky cycle coexistence theorem applies to impulsive differential equations: Some notes and beyond. *Proc. Am. Math. Soc.* **2019**, *147*, 1497–1509. [CrossRef]
15. Andres, J. Application of the randomized Sharkovsky-type theorems to random impulsive differential equations and inclusions. *J. Dyn. Diff. Equ.* **2019**, *31*, 2127–2144. [CrossRef]
16. Benedetti, J. An existence result for impulsive functional differential inclusions in Banach spaces. *Discuss. Math. Diff. Incl. Control Optim.* **2004**, *24*, 13–30. [CrossRef]
17. Malyutina, E.V. On control system with multivalued impulses and delay. *Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta* **2012**, *1*, 92–93. (In Russian)
18. Ning, H.; Liu, B. Existence and controllability results for infinite delay partial functional differential systems with multi-valued impulses in Banach spaces. *Asian-Eur. J. Math.* **2010**, *3*, 631–646. [CrossRef]
19. Andres, J. Nielsen number, impulsive differential equations and problem of Jean Leray. *Topol. Meth. Nonlinear Anal.* **2020**, in press. [CrossRef]
20. Brown, R.F.; Deconinck, C.; Dekimpe, K.; Staecker, P.C. Lifting classes for the fixed point theory of n-valued maps. *Topol. Appl.* **2020**, *274*, in press. [CrossRef]
21. Brown, R.F.; Gonçalves, D.L. On the topology of n-valued maps. *Adv. Fixed Point Theory* **2018**, *8*, 205–220.

22. Brown, R.F.; Furi, M.; Górniewicz, L.; Jiang, B. (Eds.) *Fixed Point Theory of Multivalued Weighted Maps*; Springer: Dordrecht, The Netherlands, 2005.
23. Hutchinson, J.E. Fractals and self similarity. *Indiana Univ. Math. J.* **1981**, *30*, 713–747. [CrossRef]
24. Barnsley, M.F. *Fractals Everywhere*; Academic Press: New York, NY, USA, 1988.
25. Andres, J.; Górniewicz, L. *Topological Fixed Point Principles for Boundary Value Problems*; Kluwer: Dordrecht, The Netherlands, 2003.
26. Górniewicz, L. *Topological Fixed Point Theory of Multivalued Mappings*, 2nd ed.; Springer: Berlin, Germany, 2006.
27. Brown, R.F. *The Lefschetz Fixed Point Theorem*; Scott-Foresman and Co.: Glenview, IL, USA, 1971.
28. Jiang, B. *Lectures on Nielsen Fixed Point Theory*; Contemporary Mathematics 14; The American Mathematical Society: Providence, RI, USA, 1983.
29. Andres, J.; Rypka, M. Multivalued fractals and hyperfractals. *Int. J. Bifurc. Chaos* **2012**, *22*, 125009. [CrossRef]
30. Fedeli, A. On chaotic set-valued discrete dynamical systems. *Chaos Solitons Fractals* **2005**, *23*, 1381–1384. [CrossRef]
31. Better, J. Equivariant Nielsen fixed point theory for n-valued maps. *Topol. Appl.* **2010**, *157*, 1804–1814. [CrossRef]
32. Better, J. A Wecken theorem for n-valued maps. *Topol. Appl.* **2012**, *159*, 3707–3715. [CrossRef]
33. Brown, R.F. Fixed points of n-valued maps of the circle. *Bull. Pol. Acad. Sci. Math.* **2006**, *54*, 153–162. [CrossRef]
34. Brown, R.F. The Lefschetz number of an n-valued multimap. *J. Fixed Point Theory Appl.* **2007**, *1*, 53–60.
35. Brown, R.F. Nielsen numbers of n-valued fiber maps. *J. Fixed Point Theory Appl.* **2008**, *2*, 183–201. [CrossRef]
36. Brown, R.F. Construction of multiply fixed n-valued maps. *Topol. Appl.* **2015**, *196*, 249–259. [CrossRef]
37. Brown, R.F.; Kolahi, K. Nielsen coincidence, fixed point and root theories of n-valued maps. *J. Fixed Point Theory Appl.* **2013**, *14*, 309–324. [CrossRef]
38. Brown, R.F.; Lin, J.T.L.K. Coincidences of projections and linear n-valued maps of tori. *Topol. Appl.* **2010**, *157*, 1990–1998. [CrossRef]
39. Brown, R.F.; Nan, J. Stabilizers of fixed point classes and Nielsen numbers of n-valued maps. *Bull. Belg. Math. Soc. Adv. Fixed Point Theory* **2017**, *24*, 523–535. [CrossRef]
40. Crabb, M.C. Lefschetz indices for n-valued maps. *J. Fixed Point Theory Appl.* **2015**, *17*, 153–186. [CrossRef]
41. Gonçalves, D.L.; Guaschi, J. Fixed points of n-valued maps on surfaces and the Wecken property—A configuration space approach. *Sci. China Math.* **2017**, *60*, 1561–1574. [CrossRef]
42. Gonçalves, D.L.; Guaschi, J. Fixed points of n-valued maps, the fixed point property and the case of surfaces—A braid approach. *Indag. Math.* **2018**, *29*, 91–124. [CrossRef]
43. Jezierski, J. An example of finitely-valued fixed point free map. *Zesz. Nauk (Univ. Grańsk)* **1987**, *6*, 87–93.
44. O'Neil, B. Induced homology homomorphism for set-valued maps. *Pac. J. Math.* **1957**, *7*, 1179–1184. [CrossRef]
45. Schirmer, H. Fix-finite approximations of n-valued multifunctions. *Fund. Math.* **1984**, *121*, 73–80. [CrossRef]
46. Schirmer, H. An index and a Nielsen number for n-valued multifunctions. *Fund. Math.* **1984**, *124*, 207–219. [CrossRef]
47. Schirmer, H. A minimum theorem for n-valued multifunctions. *Fund. Math.* **1985**, *126*, 83–92. [CrossRef]
48. Staecker, P.C. Axioms for the fixed point index of n-valued maps, and some applications. *J. Fixed Point Theory Appl.* **2018**, *20*, 61. [CrossRef]
49. Dzedzej, Z. *Fixed Point Index Theory for a Class of Nonacyclic Multivalued Maps*; Dissertationes Math. 253; PWN: Warsaw, Poland, 1985.
50. Pejsachowicz, J.; Skiba, R. Fixed Point Theory of Multivalued Weighted Maps. In *Handbook of Topological Fixed Point Theory*; Brown, R.F., Furi, M., Górniewicz, L., Jiang, B., Eds.; Springer: Dordrecht, The Netherlands, 2005; pp. 217–263. [CrossRef]
51. Banach, S.; Mazur, S. Über mehrdeutige stetige Abbildungen. *Stud. Math.* **1934**, *5*, 174–178. [CrossRef]
52. Haase, H. Chaotic maps in hyperspaces. *Real Anal. Exch.* **1995**, *21*, 689–695. [CrossRef]
53. Andres, J.; Fišer, J. Metric and topological multivalued fractals. *Int. J. Bifurc. Chaos* **2004**, *14*, 1277–1289. [CrossRef]
54. Andres, J.; Górniewicz, L. Note on nonejective topological fractals on Peano's continua. *Int. J. Bifurc. Chaos* **2014**, *24*, 1450148. [CrossRef]

55. Andres, J.; Väth, M. Calculation of Lefschetz and Nielsen numbers in hyperspaces for fractals and dynamical systems. *Proceed. Am. Math. Soc.* **2007**, *135*, 479–487. [CrossRef]
56. Krasnoselskii, M.A. *The Operator of Translation along the Trajectories of Differential Equations*; Translations of Mathematical Monographs; American Mathematical Society: Providence, RI, USA, 1968; Volume 19.
57. Codington, E.A.; Levinson, N. *Theory of Differential Equations*; McGraw-Hill: New York, NY, USA, 1985.
58. Banakh, T.; Nowak, M.; Strobin, F. Detecting topological and Banach fractals among zero-dimensional spaces. *Topol. Appl.* **2015**, *196*, 22–30. [CrossRef]
59. Banakh, T.; Nowak, M.; Strobin, F. Embedding fractals in Banach, Hilbert or Euclidean spaces. *J. Fractal Geom.* **2018**, in press.
60. Montiel, M.E.; Aguado, A.S.; Zaluska, E. Topology in fractals. *Chaos Solitons Fractals* **1996**, *7*, 1187–1207. [CrossRef]
61. Barnsley, M.F.; Harding, B.; Rypka, M. Measure preserving fractal homeomorphisms. In *Fractals, Wavelets, and Their Applications*; Bandt, C., Barnsley, M., Devaney, R., Falconer, K.J., Kannan, V., Vinod Kumar P.B., Eds.; Springer: Cham, Switzerland, 2014; pp. 79–102.
62. Falconer, K. *Fractal Geometry: Mathematical Foundations and Applications*; J. Wiley & Sons: New York, NY, USA, 1990.
63. Falconer, K. *Techniques in Fractal Geometry*; J. Wiley & Sons: New York, NY, USA, 1997.
64. Kirchgraber, U.; Hoffer, D. On the definition of chaos. *Z. Angew. Math. Mech.* **1989**, *69*, 175–185. [CrossRef]
65. Shang, Y. Emergence in random noisy environment. *Int. J. Math. Anal.* **2010**, *4*, 1205–1215.