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On Special Spacelike Hybrid Numbers

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Abstract: Hybrid numbers are generalizations of complex, hyperbolic and dual numbers. A hyperbolic complex structure is frequently used in both pure mathematics and numerous areas of physics. In this paper we introduce a special kind of spacelike hybrid number, namely the $F(p,n)$-Fibonacci hybrid numbers and we give some of their properties.

Keywords: Fibonacci numbers; Fibonacci–Narayana numbers; recurrence relations; hybrid numbers

MSC: 11B37; 11B39

1. Definitions and Preliminary Results

In [1] Özdemir introduced hybrid numbers as a new type of numbers which generalize complex, hyperbolic and dual numbers. We recall this definition.

Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ε, $\mathbf{h}$ are operators for which

$$\mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1 \tag{1}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}. \tag{2}$$

Let $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ be arbitrary hybrid numbers. Then we define equality, addition, subtraction and multiplication by scalar $s \in \mathbb{R}$ in the following way:

$\mathbf{Z}_1 = \mathbf{Z}_2$ if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
$\mathbf{Z}_1 \pm \mathbf{Z}_2 = (a_1 \pm a_2) + (b_1 \pm b_2)\mathbf{i} + (c_1 \pm c_2)\varepsilon + (d_1 \pm d_2)\mathbf{h}$,
$s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\varepsilon + sd_1\mathbf{h}$.

Using equalities (1) and (2) we define the hybrid numbers multiplication. Moreover, by (1) and (2) we can find the product of any two hybrid units as presented in Table 1.

Table 1. The hybrid number multiplication.

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	$\mathbf{h}+1$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Rules given in Table 1. are helpful for the multiplication of hybrid numbers and it can be made analogously as multiplications of algebraic expressions.

The conjugate of a hybrid number $\mathbf{Z}$ is the hybrid number

$$\overline{\mathbf{Z}} = \overline{a + b\mathbf{i} + c\varepsilon + d\mathbf{h}} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}.$$

The real number

$$C(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2$$

is named as the character of the hybrid number $\mathbf{Z}$.

Hybrid numbers are classified as spacelike, timelike and lightlike according to its character. We say that a hybrid number $\mathbf{Z}$ is spacelike, timelike or lightlike if $C(\mathbf{Z}) < 0$, $C(\mathbf{Z}) > 0$ or $C(\mathbf{Z}) = 0$, respectively.

For the basics on hybrid number theory and also algebraic and geometric properties of hybrid numbers, see [1].

Hybrid numbers generalize complex, hyperbolic and dual numbers. Hyperbolic complex structure have many applications also in physics, see for example [2,3]. Hybrid numbers can be connected with the well-known numbers belonging to the family of Fibonacci type numbers.

We recall that the nth Fibonacci number F_n is defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = F_1 = 1$. Note that Fibonacci sequence $\{F_n\}$ starts also from $F_0 = 0$, $F_1 = 1$ but in this paper we put $F_0 = F_1 = 1$. The nth Lucas number L_n is defined recursively by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$.

Besides the usual Fibonacci and Lucas numbers many kinds of generalizations of these numbers have been presented in the literature. These generalizations are given by the pth order linear recurrence relations, see for their list [4]. Among many generalizations Kwaśnik and Włoch [5] generalized Fibonacci and Lucas numbers in the context of their interpretations in graph theory.

Let $p \geq 2$, $n \geq 0$ be integers. Generalized Fibonacci numbers $F(p,n)$ were defined as follows

$$\begin{aligned} F(p,n) &= n+1, \text{ for } n = 0,1,\ldots,p-1 \\ F(p,n) &= F(p,n-1) + F(p,n-p), \text{ for } n \geq p. \end{aligned}$$

For their graphs applications the sequence $\{F(p,n)\}$ starts from $F(p,0) = 1$.

Based on the definition of $F(p,n)$ generalized Lucas numbers $L(p,n)$ were defined as follows

$$\begin{aligned} L(p,n) &= n+1, \text{ for } n = 0,1,\ldots,2p-1 \\ L(p,n) &= L(p,n-1) + L(p,n-p), \text{ for } n \geq 2p. \end{aligned}$$

The same recurrence relations were introduced by Stakhov as "Fibonacci and Lucas p-numbers", see [6].

Note that for $n \geq 0$ we have that $F(2,n) = F_{n+1}$ and for $n \geq 2$ holds $L(2,n) = L_n$. Moreover, $F(3,n) = u_{n+2}$, where u_n is the well-known nth Fibonacci–Narayana number defined as follows $u_0 = u_1 = u_2 = 1$ and $u_n = u_{n-1} + u_{n-3}$ for $n \geq 3$, see for details in [7].

Numbers $F(p,n)$ and $L(p,n)$ were investigated in many papers with respect to their combinatorial and algebraic properties, see for example [8–15]. Fibonacci polynomials can be used as special generalization of Fibonacci numbers, and they are studied in the context of their roots, power series, matrix generators and also connections with Chebyshev polynomials, more details can be found in [16–19].

Fibonacci numbers have applications in studying topological indices (Hosoya index and Marrifield-Simmons index) related to variety of physicochemical properties of alkanes, for example their boiling points. These structure descriptors are used in the theory of conjugated π-electron systems of molecular-graphs, see [20].

For these reasons Fibonacci numbers and their generalizations are intensively studied both from the pure mathematical point of view and their applications. A new generalization of Fibonacci and Lucas hybrid numbers were presented quite recently in [21]. Another generalization of Fibonacci and Lucas hybrid numbers are the Fibonacci and Lucas hybrinomials, see [22].

Table 2 presents initial terms of generalized Fibonacci numbers and generalized Lucas numbers for special cases of n and p.

Table 2. The values of $F(p,n)$, $L(p,n)$, F_n, u_n and L_n.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
F_n	1	1	2	3	5	8	13	21	34	55	89
$F(2,n)$	1	2	3	5	8	13	21	34	55	89	144
u_n	1	1	1	2	3	4	6	9	13	19	28
$F(3,n)$	1	2	3	4	6	9	13	19	28	41	60
$F(4,n)$	1	2	3	4	5	7	10	14	19	26	36
$F(5,n)$	1	2	3	4	5	6	8	11	15	20	26
L_n	2	1	3	4	7	11	18	29	47	76	123
$L(2,n)$	1	2	3	4	7	11	18	29	47	76	123
$L(3,n)$	1	2	3	4	5	6	10	15	21	31	46
$L(4,n)$	1	2	3	4	5	6	7	8	13	19	26

For $F(p,n)$ and $L(p,n)$ some identities were given for example in [11,13]. We recall some of them.

Theorem 1 ([13])**.** *Let $p \geq 2$ be integer. Then for $n \geq p+1$*

$$\sum_{l=0}^{n-p} F(p,l) = F(p,n) - p. \tag{3}$$

Theorem 2 ([13])**.** *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} F(p,lp-1) + 1 = F(p,np). \tag{4}$$

Theorem 3 ([11])**.** *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} F(p,lp) = F(p,np+1) - F(p,1), \tag{5}$$

$$\sum_{l=1}^{n} F(p,lp+1) = F(p,np+2) - F(p,2), \tag{6}$$

$$\sum_{l=1}^{n} F(p,lp+2) = F(p,np+3) - F(p,3). \tag{7}$$

Theorem 4 ([13])**.** *Let $p \geq 2$, $n \geq 2p-2$ be integers. Then*

$$F(p,n) = \sum_{l=0}^{p-1} F(p,n-(p-1)-l). \tag{8}$$

Theorem 5 ([13])**.** *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} L(p, pl) = L(p, np+1) - (p+2). \tag{9}$$

Theorem 6 ([11])**.** *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} L(p, pl+1) = L(p, np+2) - L(p, p+2). \tag{10}$$

$$\sum_{l=2}^{n} L(p, pl+2) = L(p, np+3) - L(p, p+3). \tag{11}$$

$$\sum_{l=2}^{n} L(p, pl+3) = L(p, np+4) - L(p, p+4). \tag{12}$$

Theorem 7 ([13])**.** *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$L(p, n) = pF(p, n-(2p-1)) + F(p, n-p). \tag{13}$$

In spite of generalized Fibonacci numbers $F(p, n)$ and generalized Lucas numbers $L(p, n)$ have been studied, mainly with respect to their graph and combinatorial properties, they found applications also in the theory of quaternions [11] and bicomplex numbers [8]. In this paper we introduce and study $F(p, n)$-Fibonacci hybrid numbers and we describe their distinct properties.

2. $F(p, n)$-Fibonacci Hybrid Numbers

Let $n \geq 0$, $p \geq 2$ be integers. The nth $F(p, n)$-Fibonacci hybrid number FH_n^p and the nth $L(p, n)$-Lucas hybrid number LH_n^p are defined as

$$FH_n^p = F(p, n) + F(p, n+1)\mathbf{i} + F(p, n+2)\varepsilon + F(p, n+3)\mathbf{h}, \tag{14}$$
$$LH_n^p = L(p, n) + L(p, n+1)\mathbf{i} + L(p, n+2)\varepsilon + L(p, n+3)\mathbf{h}, \tag{15}$$

respectively.

For $p = 2$ we obtain $FH_n^2 = FH_{n+2}$ and $LH_n^2 = LH_n$, where FH_n and LH_n denote the nth Fibonacci hybrid number and the nth Lucas hybrid number, respectively (see [23]).

Defining the nth Fibonacci–Narayana hybrid number uH_n as

$$uH_n = u_n + u_{n+1}\mathbf{i} + u_{n+2}\varepsilon + u_{n+3}\mathbf{h}$$

we have that $FH_n^3 = uH_{n+2}$.

Theorem 8. *Let $p \geq 2$ be integer. Then for $n \geq p+1$*

$$\sum_{l=0}^{n-p} FH_l^p = FH_n^p - p - (p + F(p,0))\,\mathbf{i} - (p + F(p,0) + F(p,1))\,\varepsilon + \\ - (p + F(p,0) + F(p,1) + F(p,2))\,\mathbf{h}.$$

Proof. Using (3) and (14) we have

$$\begin{aligned}
\sum_{l=0}^{n-p} FH_l^p &= FH_0^p + FH_1^p + \cdots + FH_{n-p}^p \\
&= F(p,0) + F(p,1)\mathbf{i} + F(p,2)\varepsilon + F(p,3)\mathbf{h} \\
&+ F(p,1) + F(p,2)\mathbf{i} + F(p,3)\varepsilon + F(p,4)\mathbf{h} + \cdots \\
&+ F(p,n-p) + F(p,n-p+1)\mathbf{i} + F(p,n-p+2)\varepsilon + F(p,n-p+3)\mathbf{h} \\
&= F(p,0) + F(p,1) + \cdots + F(p,n-p) \\
&+ (F(p,1) + \cdots + F(p,n-p+1) + F(p,0) - F(p,0))\,\mathbf{i} \\
&+ (F(p,2) + \cdots + F(p,n-p+2) + F(p,0) + F(p,1) - F(p,0) - F(p,1))\,\varepsilon \\
&+ (F(p,3) + \cdots + F(p,n-p+3) + F(p,0) + F(p,1) + F(p,2) \\
&- F(p,0) - F(p,1) - F(p,2))\,\mathbf{h} \\
&= F(p,n) - p + (F(p,n+1) - p - F(p,0))\,\mathbf{i} \\
&+ (F(p,n+2) - p - F(p,0) - F(p,1))\,\varepsilon \\
&+ ((F(p,n+3) - p - F(p,0) - F(p,1) - F(p,2))\,\mathbf{h} \\
&= FH_n^p - p - (p + F(p,0))\,\mathbf{i} - (p + F(p,0) + F(p,1))\,\varepsilon \\
&- (p + F(p,0) + F(p,1) + F(p,2))\,\mathbf{h},
\end{aligned}$$

which ends the proof. □

Remark 1. *If $p = 2$ then we have*

$$\begin{aligned}
\sum_{l=0}^{n-2} FH_l^2 &= FH_n^2 - 2 - (2 + F(2,0))\,\mathbf{i} - (2 + F(2,0) + F(2,1))\,\varepsilon \\
&- (2 + F(2,0) + F(2,1) + F(2,2))\,\mathbf{h} \\
&= FH_n^2 - (2 + 3\mathbf{i} + 5\varepsilon + 8\mathbf{h}) \\
&= FH_n^2 - FH_1^2.
\end{aligned}$$

On the other hand $FH_n^2 = FH_{n+2}$, so

$$\begin{aligned}
\sum_{l=0}^{n} FH_l &= FH_0 + FH_1 + \cdots + FH_n \\
&= FH_0 + FH_1 + FH_0^2 + FH_1^2 + \cdots + FH_{n-2}^2 \\
&= FH_0 + FH_1 + FH_n^2 - FH_1^2 \\
&= FH_0 + FH_1 + FH_{n+2} - FH_3 \\
&= FH_0 + FH_1 + FH_{n+2} - (FH_2 + FH_1) \\
&= FH_0 + FH_{n+2} - FH_2 \\
&= FH_0 + FH_{n+2} - (FH_1 + FH_0) \\
&= FH_{n+2} - FH_1
\end{aligned}$$

and we obtain the known equality for the Fibonacci hybrid numbers FH_n (see [23])

$$\sum_{l=0}^{n} FH_l = FH_{n+2} - FH_1.$$

Remark 2. *If $p = 3$ then we have*

$$\begin{aligned}\sum_{l=0}^{n-3} FH_l^3 &= FH_n^3 - 3 - (3 + F(3,0))\,\mathbf{i} - (3 + F(3,0) + F(3,1))\,\varepsilon \\ &\quad - (3 + F(3,0) + F(3,1) + F(3,2))\,\mathbf{h} \\ &= FH_n^3 - (3 + 4\mathbf{i} + 6\varepsilon + 9\mathbf{h}) \\ &= FH_n^3 - FH_2^3.\end{aligned}$$

On the other hand $FH_n^3 = uH_{n+2}$, so

$$\begin{aligned}\sum_{l=0}^{n-1} uH_l &= uH_0 + uH_1 + \cdots + uH_{n-1} \\ &= uH_0 + uH_1 + FH_0^3 + FH_1^3 + \cdots + FH_{n-3}^3 \\ &= uH_0 + uH_1 + FH_n^3 - FH_2^3 \\ &= uH_0 + uH_1 + uH_{n+2} - uH_4 \\ &= uH_0 + uH_1 + uH_{n+2} - (uH_3 + uH_1) \\ &= uH_0 + uH_{n+2} - uH_3 \\ &= uH_0 + uH_{n+2} - (uH_2 + uH_0) \\ &= uH_{n+2} - uH_2\end{aligned}$$

and we obtain the equality for the Fibonacci–Narayana hybrid numbers uH_n

$$\sum_{l=0}^{n} uH_l = uH_{n+3} - uH_2.$$

Theorem 9. *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} FH_{lp-1}^p = FH_{np}^p - (F(p,0) + F(p,1)\mathbf{i} + F(p,2)\varepsilon + F(p,3)\mathbf{h})\,. \tag{16}$$

Proof. Using (14) we have

$$\begin{aligned}&\sum_{l=1}^{n} FH_{lp-1}^p = FH_{p-1}^p + FH_{2p-1}^p + \cdots + FH_{np-1}^p \\ &= F(p,p-1) + F(p,p)\mathbf{i} + F(p,p+1)\varepsilon + F(p,p+2)\mathbf{h} \\ &+ F(p,2p-1) + F(p,2p)\mathbf{i} + F(p,2p+1)\varepsilon + F(p,2p+2)\mathbf{h} + \cdots \\ &+ F(p,np-1) + F(p,np)\mathbf{i} + F(p,np+1)\varepsilon + F(p,np+2)\mathbf{h} \\ &= F(p,p-1) + F(p,2p-1) + \cdots + F(p,np-1) \\ &+ (F(p,p) + F(p,2p) + \cdots + F(p,np))\,\mathbf{i} \\ &+ (F(p,p+1) + F(p,2p+1) + \cdots + F(p,np+1))\,\varepsilon \\ &+ (F(p,p+2) + F(p,2p+2) + \cdots + F(p,np+2))\,\mathbf{h}.\end{aligned}$$

Writing (4) as $\sum_{l=1}^{n} F(p,lp-1) = F(p,np) - 1 = F(p,np) - F(p,0)$ and using (5)–(7) we obtain (16). □

Theorem 10. *Let $p \geq 2$, $n \geq 2p-2$ be integers. Then*

$$FH_n^p = \sum_{l=0}^{p-1} FH_{n-(p-1)-l}^p.$$

Proof. Using (8) and (14) we have

$$\begin{aligned}
&\sum_{l=0}^{p-1} FH_{n-(p-1)-l}^p = FH_{n-(p-1)}^p + FH_{n-(p-1)-1}^p + \cdots + FH_{n-(p-1)-(p-1)}^p \\
&= F(p, n-(p-1)) + F(p, n-(p-1)+1)\mathbf{i} \\
&\ + F(p, n-(p-1)+2)\varepsilon + F(p, n-(p-1)+3)\mathbf{h} \\
&\ + F(p, n-(p-1)-1) + F(p, n-(p-1))\mathbf{i} \\
&\ + F(p, n-(p-1)+1)\varepsilon + F(p, n-(p-1)+2)\mathbf{h} + \cdots \\
&\ + F(p, n-(p-1)-(p-1)) + F(p, n-(p-1)-(p-1)+1)\mathbf{i} \\
&\ + F(p, n-(p-1)-(p-1)+2)\varepsilon \\
&\ + F(p, n-(p-1)-(p-1)+3)\mathbf{h} \\
&= F(p,n) + F(p, n+1)\mathbf{i} + F(p, n+2)\varepsilon + F(p, n+3)\mathbf{h} = FH_n^p,
\end{aligned}$$

which ends the proof. □

Remark 3. *If $p=2$ and $n \geq 2$ then we obtain the basic equality for the Fibonacci hybrid numbers FH_n*

$$FH_n = FH_{n-1} + FH_{n-2}.$$

Remark 4. *If $p = 3$ and $n \geq 4$ then we obtain the basic equality for the Fibonacci–Narayana hybrid numbers uH_n*

$$\begin{aligned}
uH_n &= uH_{n-2} + uH_{n-3} + uH_{n-4} \\
&= uH_{n-3} + (uH_{n-2} + uH_{n-4}) = uH_{n-3} + uH_{n-1}.
\end{aligned}$$

Theorem 11. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} LH_{pl}^p = LH_{np+1}^p - LH_{p+1}^p. \tag{17}$$

Proof. Using (15) we have

$$\begin{aligned}
&\sum_{l=2}^{n} LH_{pl}^p = LH_{2p}^p + LH_{3p}^p + \cdots + LH_{np}^p \\
&= L(p,2p) + L(p,2p+1)\mathbf{i} + L(p,2p+2)\varepsilon + L(p,2p+3)\mathbf{h} \\
&\ + L(p,3p) + L(p,3p+1)\mathbf{i} + L(p,3p+2)\varepsilon + L(p,3p+3)\mathbf{h} + \cdots \\
&\ + L(p,np) + L(p,np+1)\mathbf{i} + L(p,np+2)\varepsilon + L(p,np+3)\mathbf{h} \\
&= L(p,2p) + L(p,3p) + \cdots + L(p,np) \\
&\ + (L(p,2p+1) + L(p,3p+1) + \cdots + L(p,np+1))\,\mathbf{i} \\
&\ + (L(p,2p+2) + L(p,3p+2) + \cdots + L(p,np+2))\,\varepsilon \\
&\ + (L(p,2p+3) + L(p,3p+3) + \cdots + L(p,np+3))\,\mathbf{h}.
\end{aligned}$$

Writing (9) as $\sum_{l=2}^{n} L(p, pl) = L(p, np+1) - L(p, p+1)$ and using (10)–(12) we obtain (17). □

Remark 5. *If $p = 2$ then we have*

$$\sum_{l=2}^{n} LH_{2l} = LH_{2n+1} - LH_3,$$

and

$$\begin{aligned}\sum_{l=0}^{n} LH_{2l} &= \sum_{l=2}^{n} LH_{2l} + LH_0 + LH_2 \\ &= LH_{2n+1} - LH_3 + LH_0 + LH_2 \\ &= LH_{2n+1} - (LH_2 + LH_1) + LH_0 + LH_2\end{aligned}$$

hence we obtain the known equality for the Lucas hybrid numbers LH_n (see [23])

$$\sum_{l=0}^{n} LH_{2l} = LH_{2n+1} + LH_0 - LH_1.$$

Theorem 12. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$LH_n^p = p \cdot FH_{n-(2p-1)}^p + FH_{n-p}^p.$$

Proof. Using (14) we have

$$\begin{aligned}FH_{n-(2p-1)}^p &= F(p, n-(2p-1)) + F(p, n-(2p-1)+1)\mathbf{i} \\ &\quad + F(p, n-(2p-1)+2)\varepsilon + F(p, n-(2p-1)+3)\mathbf{h}\end{aligned}$$

and

$$\begin{aligned}FH_{n-p}^p &= F(p, n-p) + F(p, n-p+1)\mathbf{i} \\ &\quad + F(p, n-p+2)\varepsilon + F(p, n-p+3)\mathbf{h},\end{aligned}$$

consequently

$$\begin{aligned}&p \cdot FH_{n-(2p-1)}^p + FH_{n-p}^p \\ &= p \cdot F(p, n-(2p-1)) + F(p, n-p) \\ &+ (p \cdot F(p, (n+1)-(2p-1)) + F(p, (n+1)-p))\, \mathbf{i} \\ &+ (p \cdot F(p, (n+2)-(2p-1)) + F(p, (n+2)-p))\, \varepsilon \\ &+ (p \cdot F(p, (n+3)-(2p-1)) + F(p, (n+3)-p))\, \mathbf{h}.\end{aligned}$$

Using (13) we have

$$\begin{aligned}&p \cdot FH_{n-(2p-1)}^p + FH_{n-p}^p \\ &= L(p, n) + L(p, n+1)\mathbf{i} + L(p, n+2)\varepsilon + L(p, n+3)\mathbf{h},\end{aligned}$$

which ends the proof. □

3. Concluding Remarks

Let $\{a_n\}$ be an increasing sequence of integer numbers. Then nth a_n-hybrid number aH_n is defined as $aH_n = a_n + a_{n+1}\mathbf{i} + a_{n+2}\varepsilon + a_{n+3}\mathbf{h}$. We shall show that a_n-hybrid numbers are spacelike. Since $\{a_n\}$ is increasing, so

$$\begin{aligned} C(aH_n) &= (a_n)^2 + (a_{n+1})^2 - 2a_{n+1}a_{n+2} - (a_{n+3})^2 \\ &< (a_n)^2 + (a_{n+1})^2 - 2(a_{n+1})^2 - (a_{n+3})^2 \\ &= (a_n)^2 - (a_{n+1})^2 - (a_{n+3})^2 < 0. \end{aligned}$$

From the above immediately follows that $F(p,n)$-Fibonacci hybrid numbers and $L(p,n)$-Lucas hybrid numbers are spacelike.

Among generalizations of Fibonacci type numbers the well-known is generalization given by Horadam, see [24].

Let $\mathbf{p}, \mathbf{q}, n$ be integers. For $n \geq 0$ the nth Horadam number $W_n = W_n(W_0, W_1; \mathbf{p}, \mathbf{q})$ is defined by

$$W_n = \mathbf{p} \cdot W_{n-1} - \mathbf{q} \cdot W_{n-2},$$

for $n \geq 2$ with fixed real numbers W_0, W_1.

The Horadam hybrid numbers were introduced in [25] as follows. The nth Horadam hybrid number H_n is defined as

$$H_n = W_n + W_{n+1}\mathbf{i} + W_{n+2}\varepsilon + W_{n+3}\mathbf{h}.$$

The character $C(H_n)$ of the Horadam hybrid number H_n is equal to

$$\begin{aligned} C(H_n) = W_n^2(1 - \mathbf{p}^2\mathbf{q}^2) &+ W_n W_{n+1}(2\mathbf{q} + 2\mathbf{p}^3\mathbf{q} - 2\mathbf{p}\mathbf{q}^2) \\ &+ W_{n+1}^2(1 - 2\mathbf{p} - \mathbf{p}^4 + 2\mathbf{p}^2\mathbf{q} - \mathbf{q}^2). \end{aligned}$$

The well known special case of Horadam numbers are Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, Mersenne numbers and many others. Since the corresponding sequences are increases, so the hybrid numbers based on these sequences are spacelike. However it seems to be interesting to describe which Horadam hybrid numbers are spacelike.

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