New Bounds for the α -Indices of Graphs

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Abstract: Let G be a graph, for any real $0 \leq \alpha \leq 1$, Nikiforov defines the matrix $A_\alpha(G)$ as $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and diagonal matrix of degrees of the vertices of G . This paper presents some extremal results about the spectral radius $\rho_\alpha(G)$ of the matrix $A_\alpha(G)$. In particular, we give a lower bound on the spectral radius $\rho_\alpha(G)$ in terms of order and independence number. In addition, we obtain an upper bound for the spectral radius $\rho_\alpha(G)$ in terms of order and minimal degree. Furthermore, for $n > l > 0$ and $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$, let $G_p \cong K_l \vee (K_p \cup K_{n-p-l})$ be the graph obtained from the graphs K_l and $K_p \cup K_{n-p-l}$ and edges connecting each vertex of K_l with every vertex of $K_p \cup K_{n-p-l}$. We prove that $\rho_\alpha(G_{p+1}) < \rho_\alpha(G_p)$ for $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor - 1$.

Keywords: spectral radius; minimal degree; independence number; α -adjacency matrix

1. Introduction

Let $G = (V(G), E(G))$ be a simple and undirected connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. To simplify the notation, we represent a vertex just by u and then an edge is just represented by uv , we say that u is adjacent to v , or that u and v are neighbors and we write $u \sim v$. The adjacency matrix of G , denoted by $A(G) = (a_{ij})$ is a symmetric matrix of order n such that $a_{ij} = 1$ if v_i and v_j are adjacent, and 0 otherwise. Let $d_i = d_{v_i} = d_G(v_i)$ be the degree of vertex v_i in G . The largest and smallest vertex degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The degree matrix of G , denoted by $D(G)$, is the diagonal matrix with diagonal entries the vertex degrees of G . A k -regular graph is a graph where every vertex has degree k . The complement of a graph G is represented by $\overline{G}$. The spectrum of a matrix M , will be denoted by $Sp(M)$. In this paper, the complete graph of order n is denoted by K_n . An independent set is a set of vertices in a graph, where no two vertices are adjacent. The independence number of a graph G is the number of vertices of the largest independent set in G , denoted by $\gamma(G)$ or just γ if there is no ambiguity. The line graph of a graph G , denoted by L_G , is the graph whose vertex set is the edge set of G , where two vertices of L_G are adjacent, if and only if, the corresponding edges are incident in G . The signless Laplacian matrix and Laplacian matrix of G are defined as $Q(G) = D(G) + A(G)$ and $L(G) = D(G) - A(G)$, respectively. We denote by μ_1 the Laplacian spectral radius of G , by q_1 the signless Laplacian spectral radius of G , and by λ_1 the adjacency spectral radius or spectral radius of G . V. Nikiforov in Reference [1], define the matrix

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G), \text{ with } 0 \leq \alpha \leq 1.$$

Its straightforward verified that

$$A_0(G) = A(G), 2A_{\frac{1}{2}}(G) = Q(G), A_1 = D(G),$$

and

$$A_\alpha(G) - A_\beta(G) = (\alpha - \beta)L(G).$$

We denote by $\rho_\alpha(G)$ to the spectral radius of $A_\alpha(G)$ or well called the α -index of G . The join of two vertex disjoint graphs G_1 and G_2 is the graph obtained from the disjoint union $G_1 \cup G_2$ by adding new edges from each vertex in G_1 to every vertex in G_2 . It is usually denoted by $G_1 \vee G_2$. This graph operation can be generalized in the following way: Let H be a graph of order k and $V(H) = \{1, 2, \dots, k\}$. Let $\mathcal{F} = \{G_1, G_2, \dots, G_k\}$ be a set of pairwise vertex disjoint graphs. Here, each vertex $j \in V(H)$ is assigned to the graph $G_j \in \mathcal{F}$. Let G be the graph obtained from the graphs $G_1, G_2, \dots, G_k$ and the edges connecting each vertex of G_i with all the vertices of G_j , for all edge $ij \in E(H)$. That is, G is the graph with vertex set

$$V(G) = \bigcup_{i=1}^k V(G_i)$$

and edge set

$$E(G) = \left(\bigcup_{i=1}^k E(G_i) \right) \cup \left(\bigcup_{ij \in E(H)} \{uv : u \in V(G_i), v \in V(G_j)\} \right).$$

This graph is designated the H -join (or *generalized composition*) of the graphs $G_1, G_2, \dots, G_k$ [2–4], and it is denoted by

$$G = \bigvee_H \{G_j : 1 \leq j \leq k\}.$$

In Reference [5], the authors determine the unique graph with maximal α -index among all connected graphs with diameter d , and determine the unique graph with minimal α -index among all connected graphs with given clique number. In Reference [6], the extremal graph with maximal α -index with fixed order and cut vertices and the extremal tree which attains the maximal α -index with fixed order and matching number are characterized. In Reference [7], the authors obtain the extremal graphs with maximal α -index with fixed order and diameter at least k . In References [8,9], are characterized the graphs which have the minimal spectral radius among all the connected graphs of order n and some values of the independence number γ . In Reference [10], Nikiforov et al. shown that if T_Δ is a tree of maximal degree Δ , then the spectral radius of $A_\alpha(T_\Delta)$ satisfies the tight inequality

$$\rho_\alpha(T_\Delta) < \alpha\Delta + 2(1 - \alpha)\sqrt{\Delta - 1}.$$

In Reference [11], the authors obtain the following sharp upper bound for the spectral radius of G .

Theorem 1 ([11]). Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\lambda_1(G)$ be the spectral radius of the adjacency matrix of G . Then,

$$\lambda_1(G) \leq \frac{\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - \delta n)}}{2}.$$

Equality holds, if and only if, G is either a regular graph or a bidegreed graph where each vertex has degree either δ or $n - 1$.

In Reference [12], is presented a sharp lower bound on the signless Laplacian spectral radius of a graph in terms of independence number.

Theorem 2 ([12]). *Let G be a graph of order n and $\gamma(G) = \gamma$. Then*

$$q_1(G) \geq \frac{2(n - \gamma)}{\gamma}$$

with equality, if and only if, $n = k\gamma$ and $G \cong \gamma K_k$.

This paper is organized in the following way. In Section 1, besides the main concepts and notation used throughout the paper we present some recent work that motivated the authors. In Section 2, for $n = t\gamma + s$ where $0 \leq s < \gamma$ and $0 \leq \alpha < 1$, a lower bound for the α -index of graphs of order n and independence number γ is given, further it is shown equality cases.

In Section 3, for $0 \leq \alpha < 1$, an upper bound for the α -index of graphs of order n , edge number m and minimal degree δ is given, we demonstrate that the equality holds, if and only if, is either G a regular graph or a bidegreed graph where each vertex is of degree either δ or $n - 1$. In Section 4, for $0 \leq \alpha < 1$, we present an ordering of α -index in the class of the graphs $K_l \vee (K_p \cup K_{n-p-l})$ where $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$.

2. Spectral Radius and Independence Number

In this section, we present a lower bound for the spectral radius of graphs in terms of the order and independence number which generalizes and improves the lower bound presented in Theorem 2. We first present some lemmas used in the proof of our result. For general properties of the matrices $A_\alpha(G)$ we refer the reader to Reference [1]. In particular, we frequently use the facts that $\rho_\alpha(G)$ is non-decreasing in α ([1, Proposition 4]).

Lemma 1 ([1]). *Let G be a graph of order n and H be any subgraph of G . If $0 \leq \alpha \leq 1$ then*

$$\rho_\alpha(H) \leq \rho_\alpha(G).$$

Remark 1 ([12]). *If $S \subseteq V(G)$ is a maximal independent set of G , then for each $u \in V(G) - S$, there exists some vertex $v \in S$ such that $uv \in E(G)$. Moreover, for any $w \in V(G) - (S \cup \{u\})$ with $uw \notin E(G)$ and $wv \in E(G)$, there exists some vertex $v_1 \in S - v$ such that $uv_1 \in E(G)$ or $wv_1 \in E(G)$.*

Remark 2 ([12]). *For a graph G on n vertices non-isomorphic to the graph $\overline{K}_n$, we have*

$$\gamma(G) \leq \gamma(G - e) \leq \gamma(G) + 1.$$

Lemma 2 ([13]). *Let G be a graph of order n and m edges. Let $0 \leq \alpha < 1$. Then,*

$$\rho_\alpha(G) \geq \frac{2m}{n}$$

with equality, if and only if, G is a regular graph.

Lemma 3. *Let G be a graph of order n and independence number γ with $n = t\gamma + s$ where $0 \leq s < \gamma$ and $1 < \gamma < n$. Then $G \cong sK_{t+1} \cup (\gamma - s)K_t$, if and only if, in G the following conditions are verified*

- (a) *there exists t maximal independent sets $S_1, \dots, S_t$ and an independent set S_{t+1} of cardinality s where $S_i \cap S_j = \emptyset$ for $i, j = 1, \dots, t + 1, i \neq j$;*

(b) each vertex of S_i is adjacent to a single vertex of S_j for $i, j = 1, \dots, t+1, i \neq j$.

Proof. Let G be a graph of order n and independence number γ that verifies the conditions (a) and (b). If $t = 1$ and $s > 0$ or $t = 2$ and $s = 0$ then $G \cong sK_{t+1} \cup (\gamma - s)K_t$, the result holds. Otherwise, let $x \in S_k$, $y \in S_l$ and $z \in S_m$ where $k = 1, \dots, t+1; l, m = 1, \dots, t$ and $k \neq m \neq l \neq k$. Suppose $xy \in E(G)$ and $xz \in E(G)$. We claim $yz \in E(G)$. Suppose $yz \notin E(G)$ then by (b) $zw \in E(G)$ for some $w \in S_l, w \neq y$. By (a) $x \neq w$. Then, by (b) $(S_m - \{z\}) \cup \{x, w\}$ would be an independent set in G of cardinality $\gamma + 1$ which is a contradiction. By repeated applications of the above argument we can conclude that $G \cong sK_{t+1} \cup (\gamma - s)K_t$. Conversely, if $G \cong sK_{t+1} \cup (\gamma - s)K_t$ then G verify the conditions (a) and (b). The proof is complete. $\square$

Lemma 4. Let $G \cong sK_{t+1} \cup (\gamma - s)K_t$ be a graph of order n and independence number γ with $n = t\gamma + s$ where $0 \leq s < \gamma$ and $1 < \gamma < n$, then G has a proper subgraph H of order n and independence number γ .

Proof. Suppose $G \cong sK_{t+1} \cup (\gamma - s)K_t$ then G does not verify some of the conditions (a) or (b) of Lemma 3. Suppose G does not verify the condition (a), then the following two cases may occur:

- (i) G has exactly ℓ maximal independent sets $S_1, \dots, S_\ell$ where $\ell < t$ and $S_i \cap S_j = \emptyset$ for $i, j = 1, \dots, \ell, i \neq j$. Then, $\gamma(G - \bigcup_{i=1}^{\ell} S_i) < \gamma$ (otherwise, if $\gamma(G - \bigcup_{i=1}^{\ell} S_i) = \gamma$ then G would have $\ell + 1$ two by two disjoint independent sets which would be a contradiction). Then, by Remark 2, we can constructed a new graph $H_1 \cong G - e$ on n vertices and independence number γ , where $e \in E(G - \bigcup_{i=1}^{\ell} S_i)$, the result holds.
- (ii) G has exactly t maximal independent sets $S_1, \dots, S_t$ and $S_i \cap S_j = \emptyset$ for $i, j = 1, \dots, t, i \neq j, s > 0$ and $G_1 \cong G - \bigcup_{i=1}^t S_i \not\cong \overline{K}_s$ then we can constructed a new graph $H_2 \cong G - e$ on n vertices and independence number γ , where $e \in E(G_1)$, the result holds.

Now, suppose that G verify condition (a) but does not verify condition (b), this is, we assume that G has t maximal independent sets $S_1, \dots, S_t$ and an independent set S_{t+1} of cardinality s such that $S_i \cap S_j = \emptyset$ for $i, j = 1, \dots, t+1$ with $i \neq j$. As G does not check condition (b) there exists $v \in S_i$ and $u, w \in S_j$ for some $i, j = 1, \dots, t+1, i \neq j$ and $u \neq w$ such that $uv \in E(G)$ and $wv \in E(G)$.

- (i) For $i = 1, \dots, t$, by Remark 1, there exists some vertex $v_1 \in S_i - \{v\}$ such that $uv_1 \in E(G)$ or $wv_1 \in E(G)$. Suppose $uv_1 \in E(G)$. Then, we constructed a new graph $H_3 \cong G - uv$ on n vertices and independence number γ , the result holds.
- (ii) Suppose that each vertex of S_i is adjacent to a single vertex of S_j for $i = 1, \dots, t$ and $j = 1, \dots, t+1, i \neq j$, then we use the same techniques applied in the proof from Lemma 3, we can see that $G - S_{t+1} \cong \gamma K_t$. Now, let $i = t+1$ and $G - S_{t+1} \cong \gamma K_t$. We claim, v is adjacent to all the vertices of a connected component of $G - S_{t+1}$ isomorphic to K_t . Otherwise, there exists v_m in each connected component of $G - S_{t+1}$ isomorphic to K_t such that $vv_m \notin E(G)$ for $m = 1, \dots, \gamma$. So, $S = \{v, v_1, \dots, v_\gamma\}$ would be an independent set in G of cardinality $\gamma + 1$ which is a contradiction. Then, there exists $Z = \{v_1, \dots, v_t\}$ such that $v_p \in V(G) - S_{t+1}, vv_p \in E(G)$ for $p = 1, \dots, t$ and $v_p v_q \in E(G)$ for $p, q = 1, \dots, t$. Thus, $u \notin Z$ or $w \notin Z$. If $u \notin Z$ then we constructed a new graph $H_4 \cong G - uv$ of order n and independence number γ .

The proof is complete. $\square$

Let G and H be two graphs of order n , we will say that G and H are *comparable*, if and only if, H is subgraph of G or G is subgraph of H .

Lemma 5. Let G be a graph of order n and independence number γ with $n = t\gamma + s$ where $0 \leq s < \gamma$ and $1 < \gamma < n$, then $sK_{t+1} \cup (\gamma - s)K_t$ is a subgraph of G .

Proof. Let $G_1 \cong sK_{t+1} \cup (\gamma - s)K_t$ then by Lemma 3 in G_1 the following conditions are verified (a) there exists t maximal independent sets $S_1, \dots, S_t$ and an independent set S_{t+1} of cardinality s where $S_i \cap S_j = \emptyset$ for $i, j = 1, \dots, t+1, i \neq j$ and (b) each vertex of S_i is adjacent to a single vertex of S_j for $i, j = 1, \dots, t+1, i \neq j$. Let $G_2 \cong G_1 - uv$ where $u \in S_i, v \in S_j$ and $uv \in E(G_1)$ for some $i, j = 1, \dots, t+1, i \neq j$. Thus, in G_2 some of the sets $S_i \cup \{v\}$ or $S_j \cup \{u\}$ is an independent set of cardinality $\gamma + 1$. Since G is a graph of order n and independence number γ then G is not a proper subgraph of G_1 . Now, suppose G and G_1 are not comparable graphs. By Lemma 4, G has a proper subgraph H_1 of order n and independence number γ . Clearly, $H_1 \not\cong G_1$. Then by Lemma 4, H_1 has a proper subgraph H_2 of order n and independence number γ . Clearly, $H_2 \not\cong G_1$. By repeated applications of previous argument, we can conclude $\bar{K}_n$ is a graph of order n and independence number γ where $1 < \gamma < n$ which is a contradiction. Hence, G and G_1 are comparable graphs. Since G is not a proper subgraph of G_1 then G_1 is a subgraph of G . The proof is complete. $\square$

Below we present the main result of the section which is a lower bound for the spectral radius of graphs in terms of order and independence number.

Theorem 3. Let G be a graph of order n and independence number γ with $n = t\gamma + s$ where $0 \leq s < \gamma$. Let $0 \leq \alpha < 1$ then

$$\rho_\alpha(G) \geq \begin{cases} \frac{n-s}{\gamma} & \text{if } s > 0, \\ \frac{n-\gamma}{\gamma} & \text{if } s = 0. \end{cases}$$

- If $s = 0$, then the equality holds, if and only if, $G \cong \gamma K_t$.
- If $s > 0$, then the equality holds if $G \cong sK_{t+1} \cup (\gamma - s)K_t$.

Proof. Let G be a graph of order n and independence number γ with $n = t\gamma + s$ where $0 \leq s < \gamma$. First consider $\gamma = 1$ then $s = 0$ and $n = t$. By Remark 1, $G \cong K_n \cong \gamma K_t$ the result holds. Now, let $\gamma = n$ then $s = 0$ and $t = 1$. Then, $G \cong \bar{K}_n \cong \gamma K_t$ the result is true. Now, we assume $1 < \gamma < n$. By Lemma 5, $sK_{t+1} \cup (\gamma - s)K_t$ is a subgraph of G . Clearly, for $G^* \cong sK_{t+1} \cup (\gamma - s)K_t$, we have $\rho_\alpha(G^*) = \frac{n-s}{\gamma}$ if $s > 0$, and $\rho_\alpha(G^*) = \frac{n-\gamma}{\gamma}$ if $s = 0$. By Lemma 1,

$$\rho_\alpha(G) \geq \begin{cases} \frac{n-s}{\gamma} & \text{if } s > 0, \\ \frac{n-\gamma}{\gamma} & \text{if } s = 0. \end{cases}$$

Suppose the equality holds, $s = 0$ and $G \not\cong G^*$ then the edge number m of G is greater than $\frac{\gamma t(t-1)}{2}$. By Lemma 2, $\rho_\alpha(G) \geq \frac{2m}{n} > \frac{\gamma t(t-1)}{n} = \frac{n-\gamma}{\gamma}$ which is a contradiction. The proof is complete. $\square$

The following result which is a direct consequence of Theorem 3, presents a lower bound for the signless Laplacian spectral radius and this in turn improves the bound of Theorem 2.

Corollary 1. Let G be a graph of order n and independence number γ with $n = t\gamma + s$ where $0 \leq s < \gamma$, then

$$q_1(G) \geq \begin{cases} \frac{2(n-s)}{\gamma} & \text{if } s > 0, \\ \frac{2(n-\gamma)}{\gamma} & \text{if } s = 0. \end{cases}$$

- If $s = 0$, then the equality holds, if and only if, $G \cong \gamma K_t$.
- If $s > 0$, then the equality holds if $G \cong sK_{t+1} \cup (\gamma - s)K_t$.

Proof. Taking $\alpha = \frac{1}{2}$ then $2A_\alpha(G) = Q(G)$, by Theorem 3 the result holds. $\square$

3. Spectral Radius and Minimal Degree

In this section, we provide the necessary definitions and lemmas on which our main results rely. We begin with some simple matrix results. Let $B = (b_{ij})$ be an $m \times n$ matrix. Then $s_i(B)$ will denote the i -th row sum of B , that is, $s_i(B) = \sum_{j=1}^n b_{ij}$, where $1 \leq i \leq m$.

Lemma 6 ([14]). Let B be a real symmetric $n \times n$ matrix and λ be an eigenvalue of B , with an eigenvector x all of whose entries are nonnegative. Then

$$\min_{1 \leq i \leq n} s_i(B) \leq \lambda \leq \max_{1 \leq i \leq n} s_i(B).$$

Moreover, if the row sums of B are not all equal and if all entries of x are positive, then both inequalities above are strict.

An immediate consequence is the following result.

Lemma 7. Let G be a simple connected graph of order n with $A_\alpha(G) = A_\alpha$ where $0 \leq \alpha < 1$. Let P be any polynomial and $R_v(P(A_\alpha))$ be the row sum of $P(A_\alpha)$ corresponding to $v \in V(G)$, then

$$\min_{v \in V(G)} R_v(P(A_\alpha)) \leq P(\rho_\alpha(G)) \leq \max_{v \in V(G)} R_v(P(A_\alpha)).$$

Both equalities above holds, if and only if, the row sums of $P(A_\alpha)$ are all equal.

As a consequence of Lemma 7, we obtain the following result which generalize Theorem 1.

Theorem 4. Let G be a simple connected graph of order n and minimal degree δ with m edges and $0 \leq \alpha < 1$. Then

$$\rho_\alpha(G) \leq \frac{\delta - 1 + \alpha + \sqrt{(\delta + 1 - \alpha)^2 + 4(2m - n\delta)(1 - \alpha)}}{2}.$$

The equality holds, if and only if, G is either a regular graph or a bidegreed graph where each vertex has degree either δ or $n - 1$.

Proof. Let G be a graph of order n , minimal degree δ and m edges and $R_v(A_\alpha)$ be the row sum of A_α corresponding to $v \in V(G)$ where A_α denote the matrix $A_\alpha(G)$. We can easily see that $R_v(A_\alpha) = d_v$ for $v \in V(G)$. Calculating the row sum of A_α^2 for $v \in V(G)$, we have

$$R_v(A_\alpha^2) = \alpha d_v^2 + (1 - \alpha) \sum_{u \sim v} d_u = \alpha d_v^2 + (1 - \alpha) \left(2m - d_v - \sum_{u \not\sim v, u \neq v} d_u \right). \quad (1)$$

Thus,

$$R_v(A_\alpha^2) \leq \alpha d_v^2 + (1 - \alpha) (2m - d_v - (n - d_v - 1) \delta).$$

Thereby,

$$R_v(A_\alpha^2) - ((1 - \alpha)(\delta - 1) + \alpha d_v) R_v(A_\alpha) \leq (1 - \alpha) (2m - (n - 1) \delta).$$

For the linearity,

$$R_v(A_\alpha^2 - ((1 - \alpha)(\delta - 1) + \alpha d_v) A_\alpha) \leq (1 - \alpha) (2m - (n - 1) \delta).$$

Taking $P(x) = x^2 - ((1 - \alpha)(\delta - 1) + \alpha d_v)x$ in Lemma 7, we have

$$\rho_\alpha^2(G) - ((1 - \alpha)(\delta - 1) + \alpha d_v) \rho_\alpha(G) - (1 - \alpha) (2m - (n - 1) \delta) \leq 0.$$

In particular,

$$\rho_\alpha^2(G) - (\delta - 1 + \alpha) \rho_\alpha(G) - (1 - \alpha) (2m - (n - 1) \delta) \leq 0. \quad (2)$$

Solving the inequality (2) for $\rho_\alpha(G)$, we obtain

$$\rho_\alpha(G) \leq \frac{\delta - 1 + \alpha + \sqrt{(\delta + 1 - \alpha)^2 + 4(2m - n\delta)(1 - \alpha)}}{2}. \quad (3)$$

Suppose the equality in (3) holds, then all the inequalities in the above argument must be equalities. This implies that

$$2m - d_v - (n - d_v - 1) \delta = \sum_{u \sim v} d_u,$$

for all $v \in V(G)$, that is,

$$\sum_{u \not\sim v, u \neq v} d_u = (n - d_v - 1) \delta. \quad (4)$$

for all $v \in V(G)$. Now, suppose u_1 and v_1 are two non-adjacent vertices in G then $d_{v_1} < n - 1$. By (4), we have $d_{u_1} = \delta$. Analogously, we can prove that $d_{v_1} = \delta$. Hence, the affirmation "Any pair of non-adjacent vertices in G have degree δ " is true. Now, if there exists $v \in V(G)$ such that $d_v > \delta$ then by above affirmation, we conclude v is adjacent to u for all $u \in V(G)$ with $u \neq v$, this is, $d_v = n - 1$ which implies either G is a regular graph of degree δ or G is a bidegreed graph where each vertex has degree either δ or $n - 1$. Conversely, if G is a regular graph then the equality holds. Now, suppose G is a bidegreed graph where each vertex has degree δ or $n - 1$.

- Let $v \in V(G)$ such that $d_v = n - 1$ then $\sum_{u \not\sim v, u \neq v} d_u = 0$, since $R_v(A_\alpha) = d_v$, by (1) we have

$$\begin{aligned} R_v(A_\alpha^2) - ((1 - \alpha)(\delta - 1) + \alpha d_v) R_v(A_\alpha) &= (1 - \alpha)(2m - d_v - (\delta - 1)d_v) \\ &= (1 - \alpha)(2m - (n - 1)\delta). \end{aligned}$$

- Let $v \in V(G)$ such that $d_v = \delta$, since $R_v(A_\alpha) = d_v$, by (1) and (4) we have

$$\begin{aligned} R_v(A_\alpha^2) - ((1 - \alpha)(\delta - 1) + \alpha d_v) R_v(A_\alpha) &= (1 - \alpha)(2m - d_v - \sum_{u \not\sim v, u \neq v} d_u - (\delta - 1)d_v) \\ &= (1 - \alpha)(2m - \delta - (n - 1 - \delta)\delta) - (\delta - 1)\delta \\ &= (1 - \alpha)(2m - (n - 1)\delta). \end{aligned}$$

Now, taking $P(x) = x^2 - ((1 - \alpha)(\delta - 1) + \alpha d_v)x$ in Lemma 7, we have

$$\rho_\alpha^2(G) - ((1 - \alpha)(\delta - 1) + \alpha d_v) \rho_\alpha(G) - (1 - \alpha)(2m - (n - 1)\delta) = 0.$$

Hence,

$$\rho_\alpha(G) = \frac{\delta - 1 + \alpha + \sqrt{(\delta + 1 - \alpha)^2 + 4(2m - n\delta)(1 - \alpha)}}{2}.$$

The proof is complete. $\square$

Corollary 2. Let G be a graph of order n and minimal degree δ with m edges then

$$q_1 \leq \frac{2\delta - 1 + \sqrt{8(2m - n\delta) + (2\delta + 1)^2}}{2}.$$

Equality holds, if and only if, G is either a regular graph or a bidegreed graph where each vertex has degree either δ or $n - 1$.

Proof. Taking $\alpha = \frac{1}{2}$ then $2A_\alpha(G) = Q(G)$, by Theorem 4 the result holds. $\square$

Lemma 8 ([15,16]). Let G be a graph of order n with $m \geq 1$ edges. Let q_i be the i -th greatest signless Laplacian eigenvalue of G and $\lambda_i(L_G)$ be the i -th greatest eigenvalue of the line graph of G . Then

$$q_i = \lambda_i(L_G) + 2,$$

for $i = 1, 2, \dots, k$, where $k = \min\{n, m\}$. In addition, if $m > n$, then $\lambda_i(L_G) = -2$ for $i \geq n + 1$ and if $n > m$, then $q_i = 0$ for $i \geq m + 1$.

As a direct consequence of Corollary 2 and Lemma 8, we obtain the following result.

Corollary 3. Let G be a graph of order n and minimal degree δ with m edges then

$$\lambda_1(L_G) \leq \frac{2\delta - 5 + \sqrt{8(2m - n\delta) + (2\delta + 1)^2}}{2}.$$

Equality holds, if and only if, G is either a regular graph or a bidegreed graph where each vertex has degree δ or $n - 1$.

4. Monotonicity of the α -Indices of Graphs with $0 \leq \alpha < 1$

In this section, for $0 \leq \alpha < 1$, we present an ordering of α -indices in the class of the graphs $G_p \cong K_l \vee (K_p \cup K_{n-p-l})$ where $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$. The graphs G_p plays an important role in the representation of graphs that relate the join operation between an arbitrary family of graphs, in particular maximize some topological indices of graphs in terms of edge connectivity and vertices connectivity such as Energy, Estrada index, Spread (see References [17–19]). Furthermore, in graphs of communication or transportation networks, the edge connectivity is an important measure of reliability, the study of the line graph of these graphs is relevant. In Theorem 8 [20], for $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$ is presented an ordering between the spectral radii of the distance matrices $\mathcal{D}_\alpha(G_p)$ where $\alpha \in (\bar{\alpha}, 1)$, being $\bar{\alpha} = \max\{\alpha_p : 2 \leq p \leq \lfloor \frac{n-l}{2} \rfloor\}$ and α_p the unique zero of the function $e(x, \alpha) = (4\alpha - 3)x + (\alpha^2(2l - 4n) + \alpha(3n - 3l + 4) + l - 3)$ in the interval $(\frac{3}{4}, \frac{3n-l}{4n-2l})$. In this section, we prove that this same order is conserved for the spectral radii of the matrices $A_\alpha(G_p)$ with $\alpha \in [0, 1)$. We begin presenting a result for the spectrum of the matrices $A_\alpha(G_p)$ with $\alpha \in [0, 1)$. In Theorem 5 [2], the spectrum of the adjacency matrix of the H -join of regular graphs is obtained. The version of this result for the matrices A_α with $0 \leq \alpha < 1$ is given below, its proof is similar.

Theorem 5. Let $G \cong \vee_H \{G_j : j \in V(H)\}$ where H is a graph of order k and G_j is a r_j -regular graph of order n_j for $j = 1, \dots, k$. Then, the spectrum of the matrix $A_\alpha(G)$ with $0 \leq \alpha < 1$ is

$$\bigcup_{G_j \neq K_1} \{\lambda : \lambda \in Sp(A_\alpha(G_j) + \alpha s_j I_{n_j}) - \{r_j + \alpha s_j\}\} \cup Sp(M(G))$$

where $M(G)$ is the matrix of order k given by

$$M(G) = \begin{bmatrix} \alpha s_1 + r_1 & \delta_{12} \sqrt{n_1 n_2} & \cdots & \delta_{1k} \sqrt{n_1 n_k} \\ \delta_{12} \sqrt{n_1 n_2} & \alpha s_2 + r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{(k-1)k} \sqrt{n_{k-1} n_k} \\ \delta_{1k} \sqrt{n_1 n_k} & \cdots & \delta_{(k-1)k} \sqrt{n_{k-1} n_k} & \alpha s_k + r_k \end{bmatrix} \quad (5)$$

with

$$\delta_{ij} = \begin{cases} 1 - \alpha & \text{if } ij \in E(H) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$s_i = \sum_{ji \in E(H)} n_j.$$

Let l and n be fixed positive integer and p be a positive integer such that $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$, by (5) we have

$$M(G_p) = \begin{bmatrix} \alpha(n-l) + l - 1 & (1-\alpha)\sqrt{lp} & (1-\alpha)\sqrt{l(n-l-p)} \\ (1-\alpha)\sqrt{lp} & \alpha l + p - 1 & 0 \\ (1-\alpha)\sqrt{l(n-l-p)} & 0 & n - p - (1-\alpha)l - 1 \end{bmatrix}.$$

We remember that the eigenvalues of $A_\alpha(K_s)$ are $s - 1$ and $\alpha s - 1$ with multiplicity $s - 1$. The following result is a direct consequence of Theorem 5.

Corollary 4. The spectrum of the matrix $A_\alpha(G_p)$ with $0 \leq \alpha < 1$ and $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$ has the eigenvalues

$$\alpha n - 1 \quad l - 1 \text{ times,}$$

$$\alpha(l + p) - 1 \quad p - 1 \text{ times,}$$

$$\alpha(n - p) - 1 \quad n - l - p - 1 \text{ times,}$$

and the eigenvalues of the matrix $M(G_p)$.

Below we present Rayleigh's principle for real symmetric matrices.

Lemma 9 ([21]). Let β_1 and β_n be the largest and smallest eigenvalues of a real symmetric matrix H , respectively, then

$$\beta_1(G) = \max\{\mathbf{x}^t H \mathbf{x} : \mathbf{x}^t \mathbf{x} = 1\} \text{ and } \beta_n(G) = \min\{\mathbf{x}^t H \mathbf{x} : \mathbf{x}^t \mathbf{x} = 1\}.$$

Moreover, if $\mathbf{x}$ is a unit n -vector then $\beta_1(G) = \mathbf{x}^t H \mathbf{x}$, if and only if, $\mathbf{x}$ is an associated eigenvector to $\beta_1(G)$; and $\beta_n(G) = \mathbf{x}^t H \mathbf{x}$, if and only if, $\mathbf{x}$ is an associated eigenvector to $\beta_n(G)$.

The following remark is an immediate consequence of Lemma 9.

Remark 3. For $0 \leq \alpha < 1$ and $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$, the spectral radius of the matrix $M(G_p)$ is not less than its diagonal entries.

As a direct consequence of Corollary 4 and Remark 3, we have

Lemma 10. For $0 \leq \alpha < 1$ and $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$, the spectral radius of the matrix $M(G_p)$ is the α -index of G_p .

One can easily verify the following result.

Lemma 11. The characteristic polynomial of $M(G_p)$ with $0 \leq \alpha < 1$ and $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor$, is given by

$$\begin{aligned} f_{p,\alpha}(x) &= x^3 + (3 - n - \alpha(l + n))x^2 + (\alpha^2 nl + \alpha((n - 2)(l + n)) + n(p - 2) \\ &\quad - p(l + p) + 3)x + l\alpha^2(2p(n - l - p) - n(n - 1)) \\ &\quad + \alpha((n + l)(n - 1) - p(n + 3l)(n - p - l)) + p(l + 1)(n - p - l) - n + 1. \end{aligned}$$

Theorem 6. Let $0 \leq \alpha < 1$ and $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor - 1$. Then,

$$\rho_\alpha(G_{p+1}) < \rho_\alpha(G_p).$$

Proof. Since $M(G_p)$ and $M(G_{p+1})$ are nonnegative irreducible matrices then its spectral radii are simple eigenvalues, we can assume $\rho_\alpha(G_p) = \eta_1 > \eta_2 \geq \eta_3$ and $\rho_\alpha(G_{p+1}) = \gamma_1 > \gamma_2 \geq \gamma_3$ are the eigenvalues of the matrices $M(G_p)$ and $M(G_{p+1})$, respectively.

Hence,

$$\begin{aligned} f_{p+1,\alpha}(x) - f_{p,\alpha}(x) &= \prod_{j=1}^3 (x - \gamma_j) - \prod_{j=1}^3 (x - \eta_j) \\ &= (n - l - 2p - 1)(x + 2l\alpha^2 - \alpha(n + 3l) + l + 1). \end{aligned} \quad (6)$$

We claim $\rho_\alpha(G_p) > \rho_\alpha(G_{p+1})$. Otherwise, if $\rho_\alpha(G_p) \leq \rho_\alpha(G_{p+1})$ then $\rho_\alpha(G_{p+1}) \geq \eta_j$, for all j . Taking $x = \rho_\alpha(G_{p+1})$ in (6), we obtain

$$\begin{aligned} 0 &\geq - \prod_{j=1}^3 (\rho_\alpha(G_{p+1}) - \eta_j) \\ &= (n - l - 2p - 1)(\rho_\alpha(G_{p+1}) + 2l\alpha^2 - \alpha(n + 3l) + l + 1) \\ &\geq (n - l - 2p - 1)(\alpha(n - l) + l - 1 + 2l\alpha^2 - \alpha(n + 3l) + l + 1) \\ &\geq 2l(n - l - 2p - 1)(\alpha - 1)^2 \\ &> 0 \end{aligned} \quad (7)$$

which is a contradiction. Then, we conclude $\rho_\alpha(G_p) > \rho_\alpha(G_{p+1})$ for all $1 \leq p \leq \lfloor \frac{n-l}{2} \rfloor - 1$. $\square$

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