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On Minimal and Maximal Hyperideals in n -ary Semihypergroups

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Abstract: The concept of j -hyperideals, for all positive integers $1 \leq j \leq n$ and $n \geq 2$, in n -ary semihypergroups, is a generalization of the concept of left, lateral and right hyperideals in ternary semihypergroups. In this paper, we first introduce the concept of j -(0-)simple n -ary semihypergroups and discuss their related properties through terms of j -hyperideals. Furthermore, we characterize the minimality and maximality of j -hyperideals in n -ary semihypergroups and establish the relationships between the (0-)minimal, maximal j -hyperideals and the j -(0-)simple n -ary semihypergroups. Our main results are to extend and generalize the results on semihypergroups and ternary semihypergroups. Moreover, a related question raised by Petchkaew and Chinram is solved.

Keywords: semihypergroup; n -ary semihypergroup; hyperideal; j -hyperideal

1. Introduction

The notion of n -ary systems was introduced by Kasner [1] in 1904. In 1928, Dörnte [2] introduced and studied the notion of n -ary groups which is a generalization of the concept of groups. In 1963, Sioson [3] investigated the properties of j -ideals, for all positive integers $1 \leq j \leq n$, in n -ary semigroups, and used such ideals to characterize regular n -ary semigroups. Here, the notion of j -ideals can be considered as a generalization and an extension of the concept of (left, right) ideals in ordinary semigroups. Next, Dudek et al. [4,5] studied the further properties of ideals in n -ary semigroups. Pop [6] investigated the simple n -ary semigroups through the properties of ideals. Recently, algebraic n -ary systems have been developed by many mathematicians. In particular, ternary semigroups are the simplest algebraic n -ary systems for $n \geq 3$. Ternary semigroups are known as n -ary semigroups when $n = 3$. In 1995, Dixit and Dewan [7] introduced and studied the properties of left, lateral, right, bi- and quasi-ideals in ternary semigroups. Here, a right (lateral, left) ideal in ternary semigroups can be considered as a j -ideal in n -ary semigroups for a special case $j = 1$ (2 and 3, respectively). In 2007, Iampan [8] introduced the concept of lateral (0-)simple ternary semigroups and characterized the relationships among the (0-)minimal lateral ideals, the maximal lateral ideals and the lateral (0-)simple ternary semigroups. In 2010, he gave some characterizations of the minimality and maximality of right(left) ideals in ternary semigroups.

On the other hand, the theory of algebraic hyperstructure was first introduced in 1934 when Marty [9] defined hypergroups as a generalization of groups. Later on, a number of aspects hyperstructures were widely studied from the theoretical point of view and for their applications in many subjects of pure and applied mathematics. In particular, semihypergroup theory is useful for solving the problems in various areas of algebraic hyperstructure. Semihypergroups

are considered to be a natural extension of semigroups. Some basic notion and fundamental results of hypergroups and semihypergroups can be found in [10,11]. In 2009, Davvaz et al. [12] introduced a new class of algebraic hyperstructures called n -ary semihypergroups. Such an n -ary semihypergroup represents a generalization of semigroups, semihypergroups and n -ary semigroups. The structural properties of n -ary semihypergroups, especially the quotient structures, neutral elements and fundamental relation, have been already considered by Davvaz et al. in [12,13]. In 2013, Hila et al. [14] extended the concept of ideals to n -ary hyperstructure. Indeed, they defined the concept of j -hyperideals in n -ary hyperstructure and studied their properties. Such concept can be applied to study in n -ary semihypergroups too. A ternary semihypergroup is a particular case of an n -ary semihypergroup for $n = 3$. In [15], Davvaz and Leoreanu-Fotea studied some properties of compatible binary relations on ternary semihypergroups. Naka and Hila [16] studied some properties of right, lateral and left hyperideals in ternary semihypergroups and gave the characterization of the minimality and maximality of right (lateral, left) hyperideals in ternary semihypergroups. In 2017, the characterizations of regular, completely regular and intra-regular ternary semihypergroups in term of various hyperideals are presented in [17]. Recently, Petchkaew and Chinram [18] characterized (0-)minimal and maximal n -ideals (j -ideals, for case $j = n$) in n -ary semigroups. They investigated the relationships between the (0-)minimal n -ideals and the n -(0-)simple n -ary semigroups. In the last section of the article, they gave the open problem: "Can we generalize the results to general cases for $1 < j < n$?"

In the present paper, our motivation is to study the hyperideal theory in n -ary semihypergroups by using the idea of ideal theory (hyperideal theory) in a semigroup and a ternary semigroup (respectively, semihypergroup and ternary semihypergroup), and then solve the question raised by Petchkaew and Chinram in [18].

In this paper, we first discuss the properties of hyperideals, especially the properties of j -hyperideal for all positive integers $1 \leq j \leq n$, in n -ary semihypergroups. In Section 1, we recall some basic concepts of j -hyperideals in n -ary semihypergroups. In Section 2, we first introduce the concept of j -(0-)simple n -ary semihypergroups and study their properties. The characterization of the minimality of j -hyperideals and the relationships between the (0-)minimal j -hyperideals and the j -(0-)simple n -ary semihypergroups are established in Section 3. In Section 4, we investigate the connections between the maximality of j -hyperideals and the j -(0-)simple n -ary semihypergroups. Finally, the corresponding results in ordinary n -ary semigroups can be also obtained. This means that a question raised by Petchkaew and Chinram in [18] is solved.

2. Preliminaries

In this section, we recall some basic definitions and some results of n -ary semihypergroups [12–14].

Let S be a nonempty set and f a mapping $f : S \times \dots \times S \rightarrow \mathcal{P}^*(S)$, where S appears $n \geq 2$ times and $\mathcal{P}^*(S)$ is the set of all nonempty subsets of S . Here, f is called an n -ary hyperoperation and a pair (S, f) is called an n -ary hypergroupoid. Since we can identify the set $\{x\}$ with the element x , any n -ary groupoid is an n -ary hypergroupoid. Let us denote by x_k^l the sequence of elements $x_k, x_{k+1}, \dots, x_l$. For case $l < k$, x_k^l is the empty symbol. Then,

$$f(x_1, x_2, \dots, x_n) = f(x_1^n)$$

and

$$f(x_1, \dots, x_k, y_{k+1}, \dots, y_l, z_{l+1}, \dots, z_n) = f(x_1^k, y_{k+1}^l, z_{l+1}^n).$$

For case $x_1 = \dots = x_k = x$, we write the last expression in the form $f(x^k, y_{k+1}^l, z_{l+1}^n)$. Here, the symbol for the sequence of subsets of S is defined similarly. For any nonempty subsets $A_1, \dots, A_n$ of S , we define

$$f(A_1^n) = f(A_1, \dots, A_n) = \bigcup \{f(x_1^n) \mid x_k \in A_k, k = 1, \dots, n\}.$$

If $A_1 = \{x_1\}$, then we write $f(\{x_1\}, A_2^n)$ as $f(x_1, A_2^n)$, and analogously, in another such case we write $f(\{x_1\}, A_2^{n-1}, \{x_n\})$ as $f(x_1, A_2^{n-1}, x_n)$. In the case $A_1 = \dots = A_k = A$ and $A_{k+1} = \dots = A_n = B$, we write $f(A_1^n)$ as $f(A^k, B^{n-k})$.

An n -ary hyperoperation f is called (k, l) -associative if

$$f(x_1^{k-1}, f(x_k^{n+k-1}), x_{n+k}^{2n-1}) = f(x_1^{l-1}, f(x_l^{n+l-1}), x_{n+l}^{2n-1}) \quad (1)$$

holds for fixed $1 \leq k < l \leq n$ and for all $x_1^{2n-1} \in S$. If the Equation (1) satisfies for all $1 \leq k \leq l \leq n$, then f is called associative and an n -ary hypergroupoid (S, f) is called an n -ary semihypergroup.

Throughout this paper, we write S instead of an n -ary semihypergroup (S, f) . A nonempty subset H of S is called an n -ary subsemihypergroup of S if $f(H_1^n) \subseteq H$.

Let J be a nonempty subset of S . For any positive integer $1 \leq j \leq n$, a set J is called a j -hyperideal of S if $f(x_1^{j-1}, y, x_{j+1}^n) \subseteq J$ for all $x_1^{j-1}, x_{j+1}^n \in S$ and all $y \in J$. A set J is called a hyperideal of S if J is a j -hyperideal of S for all $j = 1, 2, \dots, n$. A j -hyperideal J of S is said to be a proper j -hyperideal of S if $J \neq S$. Clearly, every j -hyperideal (hyperideal) of S is always an n -ary subsemihypergroup of S . For a particular case $j = 1$ and $j = n$, every 1-hyperideal is called a right hyperideal and every n -hyperideal is called a left hyperideal.

Let H be an n -ary subsemihypergroup of S . For any positive integer $1 \leq j \leq n$, the intersection of all j -hyperideals of H containing a nonempty subset A of H is called a j -hyperideal of H generated by A and it is denoted by $M_H^j(A)$. Similarly, a left/right hyperideal of H generated by A is denoted by $L_H(A)/R_H(A)$ $I_H(A)$. For case $A = \{a\}$, we write $M_H^j(\{a\})$ as $M_H^j(a)$. For case $H = S$, we write $M_S^j(A)$ as $M^j(A)$. For other hyperideals, we define them analogously.

An element z of S , where S has at least two elements, is called a zero element of S if $f(x_1^{k-1}, z, x_{k+1}^n) = \{z\}$ for all $x_1^{k-1}, x_{k+1}^n \in S$ and for all $k = 1, 2, \dots, n$. Here, we denote it by 0. Clearly, if S has a zero element, then every j -hyperideal and hyperideal of S always contain a zero element each. We know that there exist some n -ary semihypergroups which have no zero elements, but their n -ary subsemihypergroups have a zero element. Here, we denote by 0_{sub} the zero element of its subsemihypergroup.

Example 1. Let $\mathbb{Z}$ be the set of all integers. Define $f : \mathbb{Z}^n \rightarrow \mathcal{P}^*(\mathbb{Z})$ by

$$f(x_1^n) = \{y \in \mathbb{Z} : y \leq \min\{x_1, \dots, x_n\}\}$$

for all $x_1^n \in \mathbb{Z}$. Then $(\mathbb{Z}, f)$ is an n -ary semihypergroup without zero. Let $\mathbb{N}$ be the set of all positive integers. Then $(\mathbb{N}, f)$ is an n -ary subsemihypergroup of $\mathbb{Z}$ with a zero element $0_{sub} = 1$.

Example 2. [Example 4.5 [13]] Let $S = \{a, b, c, d, \dots\}$ have at least four elements. Define $f : S^n \rightarrow \mathcal{P}^*(S)$ by

$$f(x_1^n) = \begin{cases} S \setminus \{a, b\} & \text{for } x_1 = \dots = x_n = a, \\ S \setminus \{a, c\} & \text{otherwise,} \end{cases}$$

for all $x_1^n \in S$. Then (S, f) is an n -ary semihypergroup without zero. Fix a positive integer $j = 1, 2, \dots, n$. Clearly, if $x_j \neq a$, then $f(x_1^n) = S \setminus \{a, c\}$ for all $x_1^{j-1}, x_{j+1}^n \in S$. Therefore, $A_1 = S \setminus \{a\}$ and $A_2 = S \setminus \{a, c\}$ are proper j -hyperideals of S .

Example 3. Let $S = \{1, 2, \dots, 100\}$. Define $f : S^n \rightarrow \mathcal{P}^*(S)$ by

$$f(x_1^n) = \{z \in S : z \geq \max\{x_1, \dots, x_n\}\}$$

for all $x_1^n \in S$. Then (S, f) is an n -ary semihypergroup with a zero element $0 = 100$. Let $B_1 = \{99, 100\}$ and $B_2 = S \setminus \{1\}$. Then B_1 and B_2 are proper j -hyperideals of S .

Example 4. [Applying Example 1.13 [16]] Let $S = \{a, b, c, d, e, g\}$. Define $f : S^n \rightarrow \mathcal{P}^*(S)$ by $f(x_1^n) = ((\dots((x_1 * x_2) * \dots) * x_n), \text{ for all } x_1^n \in S, \text{ where } * \text{ is defined by the following table.}$

$*$	a	b	c	d	e	g
a	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$
b	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$
c	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
d	$S \setminus \{d\}$	$S \setminus \{d\}$	$S \setminus \{d\}$	$S \setminus \{d\}$	$S \setminus \{d\}$	$S \setminus \{d\}$
e	$S \setminus \{e\}$	$S \setminus \{e\}$	$S \setminus \{e\}$	$S \setminus \{e\}$	$S \setminus \{e\}$	$S \setminus \{e\}$
g	$S \setminus \{g\}$	$S \setminus \{g\}$	$S \setminus \{g\}$	$S \setminus \{g\}$	$S \setminus \{g\}$	$S \setminus \{g\}$

Then (S, f) is an n -ary semihypergroup and it has no proper j -hyperideal of S for all $1 \leq j \leq n$.

3. j -(0-)Simple n -ary Semihypergroups

In this section, we first introduce the notion of j -(0-)simple n -ary semihypergroups and study the properties of j -hyperideals, for all positive integers $1 \leq j \leq n$.

Let A, B be two nonempty subsets of S . For each positive integer $1 \leq j \leq n$, let

$$\mathcal{W}_{A(j)B} := \bigcup_{k \geq 1} \mathcal{W}_k^{A(j)B}$$

where $\mathcal{W}_1^{A(j)B} := f(B^{j-1}, A, B^{n-j})$ and $\mathcal{W}_{k+1}^{A(j)B} := f(B^{j-1}, \mathcal{W}_k^{A(j)B}, B^{n-j})$.

Clearly, $\mathcal{W}_{A(j)B} = f(B^{j-1}, A, B^{n-j}) \cup \underbrace{f(B^{j-1}, f(B^{j-1}, A, B^{n-j}), B^{n-j})}_{f \text{ appears 2 times}} \cup \dots \cup$

$\underbrace{f(B^{j-1}, f(\dots, f(B^{j-1}, A, B^{n-j}), \dots), B^{n-j})}_{f \text{ appears } k \text{ times}} \cup \dots$

For case $A = \{a\}$, we write $\mathcal{W}_{a(j)B} (\mathcal{W}_k^{a(j)B})$ instead of $\mathcal{W}_{\{a\}(j)B} (\mathcal{W}_k^{\{a\}(j)B})$.

Example 5. Let A, B be two nonempty subsets of a ternary semihypergroup (T, f) .

For $j = 1$, we have $\mathcal{W}_{A(1)B} = f(A, B^2) \cup f(f(A, B^2), B^2) \cup f(f(f(A, B^2), B^2), B^2) \cup \dots$. By associativity, $f(f(A, B^2), B^2) = f(A, B, f(B^3)) \subseteq f(A, B, B) = f(A, B^2)$. For each $k \geq 3$, by associativity, we have

$$\begin{aligned} \underbrace{f(f(\dots f(A, B^2), \dots), B^2)}_{f \text{ appears } k \text{ times}} &= f(A, B, \underbrace{f(B^2, f(\dots, f(B^2, f(B^3)) \dots))}_{f \text{ appears } k-1 \text{ times}})) \\ &\subseteq f(A, B, \underbrace{f(B^2, f(\dots, f(B^2, B) \dots))}_{f \text{ appears } k-2 \text{ times}})) \\ &= f(A, B, \underbrace{f(B^2, f(\dots, f(B^3) \dots))}_{f \text{ appears } k-2 \text{ times}})) \\ &\subseteq \dots \\ &\subseteq f(A, B, B) = f(A, B^2). \end{aligned}$$

Thus, $\mathcal{W}_{A(1)B} = f(A, B^2)$.

For $j = 2$, we have $\mathcal{W}_{A(2)B} = f(B, A, B) \cup f(B, f(B, A, B), B) \cup f(B, f(B, f(B, A, B), B), B) \cup \dots$. Clearly, $f(B, f(B, f(B, A, B), B), B) = f(f(B^3), A, f(B^3)) \subseteq f(B, A, B)$ and $f(B, f(B, f(B, f(B, A, B), B), B), B) = f(f(B^3), f(B, A, B), f(B^3)) \subseteq f(B, f(B, A, B), B)$. Similarly, for each $k \geq 5$, we have

$$\underbrace{f(B, f(\dots, f(B, A, B), \dots), B)}_{f \text{ appears } k \text{ times}} \subseteq \begin{cases} f(B, A, B) & \text{if } k \text{ is odd,} \\ f(B, f(B, A, B), B) & \text{if } k \text{ is even.} \end{cases}$$

Thus, $\mathcal{W}_{A(2)B} = f(B, A, B) \cup f(B, f(B, A, B), B)$.

For $j = 3$, we have $\mathcal{W}_{A(3)B} = f(B^2, A) \cup f(B^2, f(B^2, A)) \cup f(B^2, f(B^2, f(B^2, A))) \cup \dots$. Similar to case $j = 1$, we have $\mathcal{W}_{A(3)B} = f(B^2, A)$.

Lemma 1. If J is a j -hyperideal for each positive integer $1 \leq j \leq n$ of S , and H is a nonempty subset of S , then $\mathcal{W}_{x(j)H} \subseteq J$ for all $x \in J$.

Proof. Let $x \in J$. Then $\mathcal{W}_1^{x(j)H} = f(H^{j-1}, x, H^{n-j}) \subseteq f(S^{j-1}, J, S^{n-j}) \subseteq J$. Since J is a j -hyperideal of S , we have $\mathcal{W}_2^{x(j)H} = f(H^{j-1}, \mathcal{W}_1^{x(j)H}, H^{n-j}) \subseteq f(S^{j-1}, J, S^{n-j}) \subseteq J$. It follows that $\mathcal{W}_{k+1}^{x(j)H} = f(H^{j-1}, \mathcal{W}_k^{x(j)H}, H^{n-j}) \subseteq f(S^{j-1}, J, S^{n-j}) \subseteq J$ for all $k \geq 2$. Thus, $\mathcal{W}_{x(j)H} = \bigcup_{k \geq 1} \mathcal{W}_k^{x(j)H} \subseteq J$. $\square$

Lemma 2. Let A be a nonempty subset of S . For each positive integer $1 \leq j \leq n$, $A \cup \mathcal{W}_{A(j)S}$ is the smallest j -hyperideal of S containing A .

Proof. Now, we fix a positive integer j satisfying $1 \leq j \leq n$. First, we show that $A \cup \mathcal{W}_{A(j)S}$ is a j -hyperideal of S . Let $x \in A \cup \mathcal{W}_{A(j)S}$ and $y_1^{j-1}, y_{j+1}^n \in S$. Then we consider into two cases.

Case 1: $x \in A$. Then $f(y_1^{j-1}, x, y_{j+1}^n) \subseteq f(S^{j-1}, x, S^{n-j}) \subseteq f(S^{j-1}, A, S^{n-j}) = \mathcal{W}_1^{A(j)S} \subseteq \mathcal{W}_{A(j)S}$.

Case 2: $x \in \mathcal{W}_{A(j)S}$. Then $x \in \mathcal{W}_k^{A(j)S}$ for some positive integers k . Thus, $f(y_1^{j-1}, x, y_{j+1}^n) \subseteq f(y_1^{j-1}, \mathcal{W}_k^{A(j)S}, y_{j+1}^n) \subseteq f(S^{j-1}, \mathcal{W}_k^{A(j)S}, S^{n-j}) = \mathcal{W}_{k+1}^{A(j)S} \subseteq \mathcal{W}_{A(j)S}$. Hence $A \cup \mathcal{W}_{A(j)S}$ is a j -hyperideal of S . To show that $A \cup \mathcal{W}_{A(j)S}$ is the smallest j -hyperideal of S containing A remains. Let J be a j -hyperideal of S such that $A \subseteq J$. Let $z \in A \cup \mathcal{W}_{A(j)S}$. If $z \in A$, then $z \in J$. If $z \in \mathcal{W}_{A(j)S}$, then, by Lemma 1, $z \in \mathcal{W}_{A(j)S} \subseteq \mathcal{W}_{J(j)S} = \bigcup_{x \in J} \mathcal{W}_{x(j)S} \subseteq J$. Therefore, $A \cup \mathcal{W}_{A(j)S} \subseteq J$ and the proof is complete. $\square$

By the same computation as in Example 5, we get $\mathcal{W}_{A(1)B} = f(A, B^{n-1})$ and $\mathcal{W}_{A(n)B} = f(B^{n-1}, A)$ for all nonempty subsets A, B of S . Indeed, $\mathcal{W}_1^{A(1)B} = f(A, B^{n-1})$. By associativity, $\mathcal{W}_2^{A(1)B} = f(\mathcal{W}_1^{A(1)B}, B^{n-1}) = f(f(A, B^{n-1}), B^{n-1}) = f(A, B^{n-2}, f(B^n)) \subseteq f(A, B^{n-2}, B) = f(A, B^{n-1}) = \mathcal{W}_1^{A(1)B}$. For $k \geq 2$, we have $\mathcal{W}_{k+1}^{A(1)B} = f(\mathcal{W}_k^{A(1)B}, B^{n-1}) = f(f(\mathcal{W}_{k-1}^{A(1)B}, B^{n-1}), B^{n-1}) = f(\mathcal{W}_{k-1}^{A(1)B}, B^{n-2}, f(B^n)) \subseteq f(\mathcal{W}_{k-1}^{A(1)B}, B^{n-2}, B) = f(\mathcal{W}_{k-1}^{A(1)B}, B^{n-1}) = \mathcal{W}_k^{A(1)B}$. Since $\mathcal{W}_1^{A(j)B} \subseteq \mathcal{W}_{A(1)B} = \bigcup_{k \geq 1} \mathcal{W}_k^{A(j)B} \subseteq \mathcal{W}_1^{A(j)B}$, it follows that $\mathcal{W}_{A(1)B} = f(A, B^{n-1})$. Similarly, we have $\mathcal{W}_{A(n)B} = f(B^{n-1}, A)$. The following results is easy to verify.

Remark 1. Let A be a nonempty subset of S . Then,

- (i) $M^j(A) = \mathcal{W}_{A(j)S} \cup A$; for all positive integers $1 \leq j \leq n$;
- (ii) $R(A) = M^1(A) = f(A, S^{n-1}) \cup A$;
- (iii) $L(A) = M^n(A) = f(S^{n-1}, A) \cup A$.

Clearly, some results of lateral hyperideals ($j = 2$) in a ternary semihypergroup ($n = 3$) considered in [16] can be also obtained as applications of Example 5 and Remark 1. Indeed, for any nonempty subset A of a ternary semihypergroup (T, f) , the smallest lateral hyperideal of T containing A is $M^2(A) = \mathcal{W}_{A(2)T} \cup A = f(T, A, T) \cup f(T, f(T, A, T), T) \cup A$.

Lemma 3. For any nonempty subset A of S , $f(S^{n-1}, A)$, $f(A, S^{n-1})$ and $\mathcal{W}_{A(j)S}$ are a left hyperideal, a right hyperideal and a j -hyperideal (for each $1 \leq j \leq n$) of S , respectively.

Lemma 4. For each positive integer $1 \leq j \leq n$, let $\{J_k : k \in \Lambda\}$ be the family of j -hyperideals of S . Then $\bigcup_{k \in \Lambda} J_k$ is a j -hyperideal of S and $\bigcap_{k \in \Lambda} J_k$ is also a j -hyperideal of S if $\bigcap_{k \in \Lambda} J_k \neq \emptyset$.

Definition 1. Let S be an n -ary semihypergroup without zero. For each positive integer $1 \leq j \leq n$, S is called j -simple if it has no proper j -hyperideal.

Definition 2. Let S be an n -ary semihypergroup with zero. For each positive integer $1 \leq j \leq n$, S is called j -0-simple if it has no nonzero proper j -hyperideal and $f(S^n) \neq \{0\}$.

Clearly, the n -ary semihypergroup defined in Example 4 is j -simple. The following lemmas extend and generalize the results of left, lateral (0-)simple ternary semigroups [8,19], lateral (0-)simple ternary semihypergroups [16] and n -(0-)simple n -ary semigroups [18].

Lemma 5. Let S have no zero element. For each positive integer $1 \leq j \leq n$, the following assertions are equivalent.

- (i) S is j -simple.
- (ii) $S = \mathcal{W}_{x(j)S}$ for all $x \in S$.
- (iii) $M^j(x) = S$ for all $x \in S$.

Proof. First, we fix a positive integer j satisfying $1 \leq j \leq n$.

- (i) $\implies$ (ii) By Lemma 3, we have $\mathcal{W}_{x(j)S}$ is a j -hyperideal of S for all $x \in S$. Since S is j -simple, $S = \mathcal{W}_{x(j)S}$ for all $x \in S$.
- (ii) $\implies$ (iii) By Remark 1, $M^j(x) = \mathcal{W}_{x(j)S} \cup \{x\} = S \cup \{x\} = S$ for all $x \in S$.
- (iii) $\implies$ (i) Let J be a j -hyperideal of S and $x \in J$. By Lemma 1, $\mathcal{W}_{x(j)S} \subseteq J$. By (iii), $S = M^j(x) = \mathcal{W}_{x(j)S} \cup \{x\} \subseteq J \subseteq S$. Therefore, $S = J$ and S is j -simple. $\square$

Lemma 6. Let S have a zero element. For each positive integer $1 \leq j \leq n$, the following assertions hold.

- (i) If S is j -0-simple, then $M^j(x) = S$ for all $x \in S \setminus \{0\}$.
- (ii) If $M^j(x) = S$ for all $x \in S \setminus \{0\}$, then either $f(S^n) = \{0\}$ or S is j -0-simple.

Proof. Now, we fix a positive integer $1 \leq j \leq n$.

- (i) Assume that S is j -0-simple and let $x \in S \setminus \{0\}$. By Remark 1, $M^j(x)$ is a nonzero j -hyperideal of S and so $M^j(x) = S$.
- (ii) Assume that $M^j(x) = S$ for all $x \in S \setminus \{0\}$ and $f(S^n) \neq \{0\}$. To show that S is j -0-simple, let J be a nonzero j -hyperideal of S and $x \in J \setminus \{0\}$. By Lemma 1, we have $S = M^j(x) = \mathcal{W}_{x(j)S} \cup \{x\} \subseteq J \subseteq S$. Therefore, $J = S$ and S is j -0-simple. $\square$

Example 6. Let $S = \{1, 2, 3, 4\}$. Define $f : S^n \rightarrow \mathcal{P}^*(S)$ by $f(x_1^n) = ((\dots((x_1 * x_2) * \dots) * x_n),$ for all $x_1^n \in S$, where $*$ is defined by the following table.

$*$	1	2	3	4
1	$\{1\}$	$\{1, 2\}$	$\{1, 3\}$	S
2	$\{2\}$	$\{2\}$	$\{2, 4\}$	$\{2, 4\}$
3	$\{3\}$	$\{3, 4\}$	$\{3\}$	$\{3, 4\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$

Then (S, f) is an n -ary semihypergroup without zero (see [20]). Clearly, for each $j = 2, \dots, n$, we have $f(S^{j-1}, x, S^{n-j}) = S$ for all $x \in S$. It follows that $S = \mathcal{W}_1^{x(j)S} \subseteq \mathcal{W}_{x(j)S} \subseteq S$ —that is, $S = \mathcal{W}_{x(j)S}$, for all

$x \in S$. By Lemma 5 ((ii) $\Rightarrow$ (i)), we can conclude that S is a j -simple n -ary semihypergroup for all $j = 2, \dots, n$. Next, we consider case $j = 1$. Then there exists $2 \in S$ such that $M^1(2) = \{2\} \cup f(2, S^{n-1}) = \{2\} \cup \{2, 4\} = \{2, 4\} \neq S$. By Lemma 5 ((iii) $\Rightarrow$ (i)), we can conclude that S is not 1-simple. In fact, it is easy to check that the 1-hyperideals of S are $\{4\}$, $\{2, 4\}$, $\{3, 4\}$ and $\{2, 3, 4\}$ and S .

Definition 3. Let S be an n -ary semihypergroup without zero. S is called simple if it has no proper hyperideal.

Definition 4. Let S be an n -ary semihypergroup with zero. S is called 0-simple if it has no nonzero proper hyperideal and $f(S^n) \neq \{0\}$.

Example 7. We know that the n -ary semihypergroup S defined in Example 4 has no proper j -hyperideal of S . It follows that S has no proper hyperideal of S . Therefore, S is a simple n -ary semihypergroup.

From the idea of Theorem 2.3 in [6], we obtain the following lemmas.

Lemma 7. Let S have no zero element. For $n \geq 3$, S is simple if and only if $\bigcup_{1 \leq i \leq n} \mathcal{W}_1^{x(i)S} \cup f(S^{n-1}, f(x, S^{n-1})) = S$ for all $x \in S$.

Proof. ($\Rightarrow$) For convenience, we give a set $\mathcal{R} = \bigcup_{1 \leq i \leq n} \mathcal{W}_1^{x(i)S} \cup f(S^{n-1}, f(x, S^{n-1}))$. We will show that $\mathcal{R}$ is a hyperideal of S ; that is, $\mathcal{R}$ is a j -hyperideal of S for all $j = 1, \dots, n$. Let $z \in \mathcal{R}$ and $y_1^{j-1}, y_{j+1}^n \in S$ for all positive integers $j = 1, \dots, n$. Then we consider the following two cases.

Case 1: $z \in f(S^{n-1}, f(x, S^{n-1}))$. Then

$$\begin{aligned} f(y_1^{j-1}, z, y_{j+1}^n) &\subseteq f(y_1^{j-1}, f(S^{n-1}, f(x, S^{n-1})), y_{j+1}^n) \\ &\subseteq f(S^{j-1}, f(S^{n-1}, f(x, S^{n-1})), S^{n-j}). \end{aligned}$$

Case 1.1: $j = 1$. We obtain

$$\begin{aligned} f(y_1^{j-1}, z, y_{j+1}^n) &\subseteq f(f(S^{n-1}, f(x, S^{n-1})), S^{n-1}) \\ &= f(f(S^{n-1}, x), S^{n-2}, f(S^n)) \\ &\subseteq f(f(S^{n-1}, x), S^{n-2}, S) \\ &= f(S^{n-1}, f(x, S^{n-1})) \subseteq \mathcal{R}. \end{aligned}$$

Case 1.2: $2 \leq j \leq n - 1$. We obtain

$$\begin{aligned} f(y_1^{j-1}, z, y_{j+1}^n) &\subseteq f(S^{j-2}, f(S^n), x, f(S^n), S^{n-j-1}) \\ &\subseteq f(S^{j-2}, S, x, S, S^{n-j-1}) \\ &= f(S^{j-1}, x, S^{n-j}) = \mathcal{W}_1^{x(j)S} \subseteq \mathcal{R}. \end{aligned}$$

Case 1.3: $j = n$. The proof is similar to Case 1.1.

Case 2: $z \in \mathcal{W}_1^{x(i)S} = f(S^{i-1}, x, S^{n-i})$ for some $i = 1, \dots, n$. Then

$$\begin{aligned} f(y_1^{j-1}, z, y_{j+1}^n) &\subseteq f(y_1^{j-1}, f(S^{i-1}, x, S^{n-i}), y_{j+1}^n) \\ &\subseteq f(S^{j-1}, f(S^{i-1}, x, S^{n-i}), S^{n-j}). \end{aligned}$$

Case 2.1: $j = 1$. We obtain

$$\begin{aligned} f(y_1^{j-1}, z, y_{j+1}^n) &\subseteq f(f(S^{i-1}, x, S^{n-i}), S^{n-1}) \\ &= f(S^{i-1}, x, S^{n-i-1}, f(S^n)) \\ &\subseteq f(S^{i-1}, x, S^{n-i}) = \mathcal{W}_1^{x(i)S} \subseteq \mathcal{R}. \end{aligned}$$

Case 2.2: $2 \leq j \leq n-1$. Since $1 \leq i \leq n$, we have $3 \leq i+j \leq 2n-1$. Consider the following three subcases.

Case 2.2.1: $3 \leq i+j \leq n$. We obtain $f(y_1^{j-1}, z, y_{j+1}^n) \subseteq f(S^{i+j-2}, x, S^{n-i-j+1}) = \mathcal{W}_1^{x(i+j-1)S} \subseteq \mathcal{R}$.

Case 2.2.2: $i+j = n+1$. We obtain $f(y_1^{j-1}, z, y_{j+1}^n) \subseteq f(S^{n-1}, f(x, S^{n-1})) \subseteq \mathcal{R}$.

Case 2.2.3: $n+2 \leq i+j \leq 2n-1$. We obtain $f(y_1^{j-1}, z, y_{j+1}^n) \subseteq f(S^{i+j-n-1}, x, S^{2n-i-j}) = \mathcal{W}_1^{x(i+j-n)S} \subseteq \mathcal{R}$.

Case 2.3: $j = n$. The proof is similar to Case 2.1. Thus, $\mathcal{R}$ is a hyperideal of S . Since S is simple, we have $\mathcal{R} = S$.

($\Leftarrow$) To show that S is simple, let B be a hyperideal of S and $x \in B$. Let $y \in S = \bigcup_{1 \leq i \leq n} \mathcal{W}_1^{x(i)S} \cup f(S^{n-1}, f(x, S^{n-1}))$. If $y \in f(S^{n-1}, f(x, S^{n-1}))$, then $y \in f(S^{n-1}, f(x, S^{n-1})) \subseteq f(S^{n-1}, f(B, S^{n-1})) \subseteq f(S^{n-1}, B) \subseteq B$ because B is a 1-hyperideal and n -hyperideal of S . If $y \in \mathcal{W}_1^{x(i)S}$, for some $i = 1, \dots, n$, then $y \in \mathcal{W}_1^{x(i)S} = f(S^{i-1}, x, S^{n-i}) \subseteq f(S^{i-1}, B, S^{n-i}) \subseteq B$ because B is a i -hyperideal of S . Therefore, $S \subseteq B$ and the proof is complete. $\square$

Applying the proof of Lemma 6 and 7, we obtain Lemma 8.

Lemma 8. Let S have a zero element. For $n \geq 3$, the following assertions hold.

- (i) If S is 0-simple, then $\bigcup_{1 \leq i \leq n} \mathcal{W}_1^{x(i)S} \cup f(S^{n-1}, f(x, S^{n-1})) = S$ for all $x \in S \setminus \{0\}$.
- (ii) If $\bigcup_{1 \leq i \leq n} \mathcal{W}_1^{x(i)S} \cup f(S^{n-1}, f(x, S^{n-1})) = S$ for all $x \in S \setminus \{0\}$, then either $f(S^n) = \{0\}$ or S is 0-simple.

Example 8. Let $S = \{1, 2, 3, 4\}$. Define $f : S^n \rightarrow \mathcal{P}^*(S)$ by $f(x_1^n) = (((...(x_1 * x_2) * ...) * x_n))$, for all $x_1^n \in S$, where $*$ is defined by the following table.

$*$	1	2	3	4
1	{1}	{2}	{3}	{4}
2	{2}	{1, 3}	{2, 3}	{4}
3	{3}	{2, 3}	{1, 2}	{4}
4	{4}	{4}	{4}	S

Then (S, f) is an n -ary semihypergroup without zero (see [11], page 93). For case $i = 1$, we have $\mathcal{W}_1^{x(1)S} = f(x, S^{n-1}) = S$ for all $x \in S$. This follows that $S = \mathcal{W}_1^{x(1)S} \subseteq \bigcup_{i \geq 1} \mathcal{W}_1^{x(i)S} \cup f(S^{n-1}, f(x, S^{n-1})) \subseteq S$, for all $x \in S$. By Lemma 7, we can conclude that S is simple (one can also check it independently).

In order to characterize the (0-)minimal j -hyperideals of n -ary semihypergroups, we first give the following lemma.

Lemma 9. Let J be a j -hyperideal for each positive integer $1 \leq j \leq n$ of S , and N an n -ary subsemihypergroup of S . Then the following assertions hold.

- (i) If N is j -simple such that $N \cap J \neq \emptyset$, then $N \subseteq J$.
- (ii) If N is j -0-simple such that $(N \setminus \{0\}) \cap J \neq \emptyset$, then $N \subseteq J$.

Proof. (i) Assume that N is j -simple such that $N \cap J \neq \emptyset$. Let $x \in N \cap J$. By Lemma 1, $\mathcal{W}_{x(j)N} \subseteq J$. Then $\mathcal{W}_{x(j)N} \cap N \subseteq J \cap N \subseteq N$. Clearly, $\mathcal{W}_{x(j)N} \cap N$ is a j -hyperideal of N . Since N is j -simple, we have $\mathcal{W}_{x(j)N} \cap N = N$. Thus, $N \subseteq \mathcal{W}_{x(j)N} \subseteq \mathcal{W}_{x(j)S} \subseteq J$ by Lemma 1.

(ii) Assume that N is j -0-simple such that $(N \setminus \{0\}) \cap J \neq \emptyset$. Let $x \in (N \setminus \{0\}) \cap J$. By Lemma 1, Lemma 6 and Remark 1(iii), we have $N = M_N^j(x) = (\mathcal{W}_{x(j)N} \cup \{x\}) \cap N \subseteq \mathcal{W}_{x(j)N} \cup \{x\} \subseteq J \cup J = J$. $\square$

Example 9. Let $S = \{a, b, c, d\}$. Define $f : S^4 \rightarrow \mathcal{P}^*(S)$ by $f(x_1^4) = (((x_1 * x_2) * x_3) * x_4)$, for all $x_1^4 \in S$, where $*$ is defined by the following table.

$*$	a	b	c	d
a	$\{a, d\}$	$\{a, d\}$	$\{a, c, d\}$	$\{a\}$
b	$\{a, d\}$	$\{b\}$	$\{a, c, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{a, c, d\}$	$\{a, d\}$
d	$\{a\}$	$\{a, d\}$	$\{c\}$	$\{d\}$

Then (S, f) is an 4-ary semihypergroup without zero (see [21]). One can check that the 2-hyperideals of S are the set $\{a, c, d\}$ and S . Clearly, the set $H_1 := \{a, d\}$ and $H_2 := \{a, b, d\}$ are 4-ary subsemihypergroups of S and they are not 2-hyperideals of S . It is easy to prove that H_1 is a 2-simple 4-ary semihypergroup and $H_1 \subseteq \{a, c, d\}$, which is also a consequence of Lemma 9.

4. (0-)Minimal j -Hyperideals

In this section, we characterize the (0-)minimal j -hyperideals and the j -(0-)simple n -ary semihypergroups, for all positive integers $1 \leq j \leq n$, and investigate the connections between them.

Definition 5. Let S be an n -ary semihypergroup without zero. For each positive integer $1 \leq j \leq n$, a j -hyperideal J of S is called minimal if there is no j -hyperideal A of S such that $A \subset J$. Equivalently, if A is a j -hyperideal of S such that $A \subseteq J$, then $A = J$.

Definition 6. Let S be an n -ary semihypergroup with zero. For each positive integer $1 \leq j \leq n$, a nonzero j -hyperideal J of S is called 0-minimal if there is no nonzero j -hyperideal A of S such that $A \subset J$. Equivalently, if A is a j -hyperideal of S such that $A \subseteq J$, then $A = \{0\}$ or $A = J$.

Example 10. By Example 2, we have A_2 is a minimal j -hyperideal of S . By Example 3, B_1 is a 0-minimal j -hyperideal of S .

The following theorem extends Theorem 3.6 [16] and generalizes Theorem 1 [18].

Theorem 1. Let S have no zero element and J be a j -hyperideal of S with $j = 1$ ($j = n$). Then the following assertions hold.

- (i) J is a minimal j -hyperideal without zero (i.e., J has no zero element) of S if and only if J is j -simple.
- (ii) If J is a minimal j -hyperideal with zero (i.e., J has zero element) of S , then J is j -0-simple.

Proof. (i) For case $j = 1$, assume that J is a minimal j -hyperideal without zero of S . To show that J is j -simple, let K be a j -hyperideal of J . By Lemma 1, $\emptyset \neq f(K, J^{n-1}) = \mathcal{W}_{K(j)J} \subseteq K \subseteq J$. First, we will

show that $f(K, J^{n-1})$ is a j -hyperideal of S . Let $z_1^{n-1} \in S$ and $y \in f(K, J^{n-1})$. Then $y \in f(a, b_1^{n-1})$ for some $a \in K$ and some $b_1^{n-1} \in J$. By associativity, we have

$$\begin{aligned} f(y, z_1^{n-1}) &\subseteq f(f(a, b_1^{n-1}), z_1^{n-1}) \\ &= f(a, b_1^{n-2}, f(b_{n-1}, z_1^{n-1})) \\ &\subseteq f(a, b_1^{n-2}, J), \text{ since } J \text{ is } 1\text{-hyperideal of } S, \\ &\subseteq f(K, J^{n-2}, J) = f(K, J^{n-1}). \end{aligned}$$

Thus, $f(K, J^{n-1})$ is a j -hyperideal of S . Since J is a minimal j -hyperideal of S and $f(K, J^{n-1}) \subseteq K \subseteq J$, we obtain $f(K, J^{n-1}) = J$ and so $K = J$. Therefore, J is j -simple. Conversely, assume that J is j -simple. Let P be a j -hyperideal of S such that $P \subseteq J$. By Lemma 9(i), we have $J \subseteq P$ and so $P = J$. Therefore, J is a minimal j -hyperideal of S .

For case $j = n$, we first replace the set $f(K, J^{n-1})$ by $f(J^{n-1}, K)$ and then we will use the same approach as in case $j = 1$, so we omit its proof.

(ii) The proof is similar to (i). $\square$

By the proof of Theorem 1, we immediately obtain the following corollary.

Corollary 1. If K is a nonempty subset of a j -hyperideal J of S with $j = 1$ ($j = n$), then $\mathcal{W}_{K(1)J} = f(K, J^{n-1})$ ($\mathcal{W}_{K(n)J} = f(J^{n-1}, K)$) is a 1-(n -)hyperideal of S .

Theorem 2. Let S have a zero element and J be a nonzero j -hyperideal of S with $j = 1$ ($j = n$). Then the following assertions hold.

- (i) If J is a 0-minimal j -hyperideal of S , then either there exists a nonzero j -hyperideal K of J such that $\mathcal{W}_{K(1)J} = \{0\}$ ($\mathcal{W}_{K(n)J} = \{0\}$) or J is j -0-simple.
- (ii) If J is j -0-simple, then J is a 0-minimal j -hyperideal of S .

Proof. The proof is similar to the proof of Theorem 1 by applying Corollary 1 and Lemma 5(ii). $\square$

Theorem 3. Let S have no zero element and J be a j -hyperideal of S with a positive integer $1 < j < n$ and $n \geq 3$. Then the following assertions hold.

- (i) If J is a minimal j -hyperideal without zero of S , then either there exists a j -hyperideal K of J such that $\mathcal{W}_{n-1}^{K(j)J} \neq f(J^{j-1}, K, J^{n-j})$ or J is j -simple.
- (ii) If J is a minimal j -hyperideal with zero of S , then either there exists a nonzero j -hyperideal K of J such that $\mathcal{W}_{n-1}^{K(j)J} \neq f(J^{j-1}, K, J^{n-j})$ or J is j -0-simple.
- (iii) If J is j -simple, then J is a minimal j -hyperideal of S .

Proof. First, we fix a positive integer j satisfying $1 \leq j \leq n$.

(i) Let J be a minimal j -hyperideal of S and J have no zero element. Assume that $\mathcal{W}_{n-1}^{K(j)J} = f(J^{j-1}, K, J^{n-j})$ for all j -hyperideals K of J . To show that J is j -simple, let K be a j -hyperideal of J . Then $f(J^{j-1}, K, J^{n-j}) \subseteq K \subseteq J$ and

$$\begin{aligned} f(J^{j-1}, K, J^{n-j}) &= \mathcal{W}_{n-1}^{K(j)J} \\ &= f(J^{j-1}, \mathcal{W}_{n-2}^{K(j)J}, J^{n-j}) \\ &= \dots \\ &= \underbrace{f(J^{j-1}, f(\dots, f}_{f \text{ appears } n-1 \text{ times}}(J^{j-1}, K, J^{n-j}), \dots), J^{n-j}). \end{aligned}$$

First, we will show that $f(J^{j-1}, K, J^{n-j})$ is a j -hyperideal of S . Let $z_1^{j-1}, z_{j+1}^n \in S$ and $y \in f(J^{j-1}, K, J^{n-j})$. Then

$$y \in \underbrace{f(u_1^{j-1}, f(\dots, f(u_{(j-1)(n-2)+1}^{(n-1)(j-1)}, x, v_1^{n-j}), \dots), v_{(n-2)n-(n-2)j+1}^{(n-1)n-(n-1)j})}_{f \text{ appears } n-1 \text{ times}}$$

for some $u_1^{(n-1)(j-1)}, v_1^{(n-1)n-(n-1)j} \in J$ and some $x \in K$. By associativity, we have

$$\begin{aligned} f(z_1^{j-1}, y, z_{j+1}^n) &\subseteq f(z_1^{j-1}, \underbrace{f(u_1^{j-1}, f(\dots, f(u_{(j-1)(n-2)+1}^{(n-1)(j-1)}, x, v_1^{n-j}), \dots), v_{(n-2)n-(n-2)j+1}^{(n-1)n-(n-1)j})}_{f \text{ appears } n-1 \text{ times}}, z_{j+1}^n) \\ &= \underbrace{f(f(f(\dots f(f(z_1^{j-1}, u_1^{n-j+1}), u_{n-j+2}^{2n-j}), \dots), u_{(n-1)(j-1)-j-2}^{(n-1)(j-1)-j-2}), \dots)}_{f \text{ appears } j-1 \text{ times}}, \\ &\quad u_{(n-1)(j-1)-j+3}^{(n-1)(j-1)}, x, v_1^{n-j-1}, \\ &\quad \underbrace{f(v_{n-j}^{2n-j-2}, f(\dots, f(v_{(n-1)n-(n-1)j-j+2}^{(n-1)n-(n-1)j-j}, f(v_{(n-1)n-(n-1)j-j+1}^{(n-1)n-(n-1)j-j}, z_{j+1}^n)) \dots))}_{f \text{ appears } n-j \text{ times}})) \\ &= f(A, u_{(n-1)(j-1)-j+3}^{(n-1)(j-1)}, x, v_1^{n-j-1}, B), \text{ the position of } x \text{ is } j^{\text{th}}, \end{aligned}$$

where $A = f(f(\dots f(f(z_1^{j-1}, u_1^{n-j+1}), u_{n-j+2}^{2n-j}), \dots), u_{(n-1)(j-1)-j-2}^{(n-1)(j-1)-j-2})$ and

$B = f(v_{n-j}^{2n-j-2}, f(\dots, f(v_{(n-1)n-(n-1)j-j+2}^{(n-1)n-(n-1)j-j}, f(v_{(n-1)n-(n-1)j-j+1}^{(n-1)n-(n-1)j-j}, z_{j+1}^n)) \dots))$.

Since $z_1^{j-1} \in S$, $u_1 \in J$ and J is a j -hyperideal of S , we have $f(z_1^{j-1}, u_1^{n-j+1}) \subseteq J$. Similarly, since $z_{j+1}^n \in S$, $v_{(n-1)n-(n-1)j} \in J$ and J is a j -hyperideal of S , we have $f(v_{(n-1)n-(n-1)j-j+1}^{(n-1)n-(n-1)j-j}, z_{j+1}^n) \subseteq J$. Then $A, B \subseteq J$. Since $x \in K$ and K is a j -hyperideal of J , we get $f(A, u_{(n-1)(j-1)-j+3}^{(n-1)(j-1)}, x, v_1^{n-j-1}, B) \subseteq J$. Thus, $f(z_1^{j-1}, y, z_{j+1}^n) \subseteq J$ and $f(J^{j-1}, K, J^{n-j})$ is a j -hyperideal of S . Since $f(J^{j-1}, K, J^{n-j}) \subseteq K \subseteq J$ and J is a minimal j -hyperideal of S , we obtain $f(J^{j-1}, K, J^{n-j}) = J$ and so $K = J$. Therefore, J is j -simple.

The proof of the statement (ii) is similar to (i). The proof of (iii) is similar to the proof of the converse of Theorem 1(i) by applying Lemma 9. $\square$

Corollary 2. If K is a nonempty subset of a j -hyperideal J of S with a positive integer $1 < j < n$ and $n \geq 3$ such that $\mathcal{W}_{n-1}^{K(j)J} = f(J^{j-1}, K, J^{n-j})$, then $f(J^{j-1}, K, J^{n-j})$ is a j -hyperideal of S .

Proof. It is straightforward from Theorem 3. $\square$

By applying the proof of Theorem 3, we obtain the following theorem.

Theorem 4. Let S have a zero element and J be a nonzero j -hyperideal of S with a positive integer $1 < j < n$ and $n \geq 3$. Then the following assertions hold.

- (i) If J is a 0-minimal j -hyperideal of S , then either there exists a nonzero j -hyperideal K of J such that $\mathcal{W}_{n-1}^{K(j)J} \neq f(J^{j-1}, K, J^{n-j}) = \{0\}$ or J is j -simple.
- (ii) If J is j -0-simple, then J is a 0-minimal j -hyperideal of S .

Remark 2. Since any n -ary semigroup can be considered as an n -ary semihypergroup, the open problem given in [18] is answered by Theorems 3 and 4.

Example 11. Consider an 4 ary-semihypergroup (S, f) that has been given in Example 9. We know that $J := \{a, c, d\}$ is a proper 2-hyperideal of S . Looking at the table, we immediately see that there is no a proper

2-hyperideal K of J such that $\mathcal{W}_3^{K(2)J} \neq f(J, K, J, J)$. By Theorem 3(i), we can conclude that $\{a, c, d\}$ is a 2-simple 4-ary semihypergroup (one can also check it independently).

Theorem 5. Let S have no zero element. For each positive integer $1 \leq j \leq n$, let S have proper j -hyperideals. Then every proper j -hyperideal of S is minimal if and only if one of the following assertions holds.

- (i) S has only one proper j -hyperideal.
- (ii) S has only two proper j -hyperideals J_1, J_2 such that $J_1 \cup J_2 = S$ and $J_1 \cap J_2 = \emptyset$.

Proof. First, we fix a positive integer j such that $1 \leq j \leq n$.

($\Rightarrow$) Suppose that every proper j -hyperideal of S is minimal. Let J be a proper j -hyperideal of S . By hypothesis, J is minimal and $S \setminus J \neq \emptyset$, where $S \setminus J$ is the complement of J in S . We consider the following two cases.

Case 1: $S = M^j(x)$ for all $x \in S \setminus J$.

In this case, we will show that J is the unique proper j -hyperideal of S . Assume that there exists a proper j -hyperideal A of S such that $A \neq J$. If $A \cap J = \emptyset$, then $S = M^j(x)$ for all $x \in A \subseteq S \setminus J$. By Lemma 1 and Remark 1, $S = M^j(x) = \mathcal{W}_{x(j)S} \cup \{x\} \subseteq A \cup A = A \subseteq S$. Thus, $A = S$. This is a contradiction. Thus, $A \cap J \neq \emptyset$, and we have the following two cases.

Case 1.1: $A \setminus J = \emptyset$. Then $A \subseteq J$. Since J is minimal, $A = J$. This is a contradiction.

Case 1.2: $A \setminus J \neq \emptyset$. Then there exists $x \in A \setminus J \subseteq S \setminus J$. By Lemma 1 and Remark 1, we obtain $S = M^j(x) = \mathcal{W}_{x(j)S} \cup \{x\} \subseteq A \cup (A \setminus J) = A \subseteq S$. Thus, $A = S$. It is impossible. Thus, J is the unique proper j -hyperideal of S and the assertion (i) holds.

Case 2: $S \neq M^j(x)$ for some $x \in S \setminus J$.

Then $M^j(x)$ is a proper j -hyperideal of S . By hypothesis, $M^j(x)$ is a minimal j -hyperideal of S and $M^j(x) \neq J$ (if $M^j(x) = J$; then $x \in M^j(x) = J$, which leads to a contradiction with $x \in S \setminus J$). By Lemma 4, $M^j(x) \cup J$ is a j -hyperideal of S . Assume that $M^j(x) \cup J \neq S$. Then $M^j(x) \cup J$ is a proper j -hyperideal of S . By hypothesis, $M^j(x) \cup J$ is minimal. It is impossible because $J \subset M^j(x) \cup J$. So $M^j(x) \cup J = S$. Next, assume that $M^j(x) \cap J \neq \emptyset$. By Lemma 4, $M^j(x) \cap J$ is a proper j -hyperideal of S and $M^j(x) \cap J \subset J$. It is impossible because J is minimal. Thus, $M^j(x) \cap J = \emptyset$. Finally, we will show that J and $M^j(x)$ are only two proper j -hyperideals of S such that $J \cup M^j(x) = S$ and $J \cap M^j(x) = \emptyset$. Let A be a proper j -hyperideal of S . By hypothesis, A is minimal. Then $A = A \cap S = A \cap (M^j(x) \cup J) = (A \cap M^j(x)) \cup (A \cap J)$. Consider the following two cases.

Case 2.1: $A \cap J \neq \emptyset$. By Lemma 4, $A \cap J$ is a j -hyperideal of S . Since $A \cap J \subseteq J$ and J is minimal, we gain $A \cap J = J$ and so $J \subseteq A$. Since A is minimal, we have $A = J$.

Case 2.2: $A \cap J = \emptyset$. Then $A = A \cap M^j(x)$ and so $A \subseteq M^j(x)$. Since $M^j(x)$ is minimal, we have $A = M^j(x)$. Thus, the assertion (ii) is proved.

($\Leftarrow$) If S has only one proper j -hyperideal J , then J is minimal. Next, suppose that the assertion (ii) holds. Then S has only two proper j -hyperideals J_1 and J_2 such that $S = J_1 \cup J_2$ and $J_1 \cap J_2 = \emptyset$. Thus, $J_1 \not\subseteq J_2$ and $J_2 \not\subseteq J_1$. This implies that J_1 and J_2 are minimal j -hyperideals of S and the proof is complete. $\square$

Theorem 6. Let S have a zero element. For each positive integer $1 \leq j \leq n$, let S have nonzero proper j -hyperideals. Then every nonzero proper j -hyperideal of S is 0-minimal if and only if one of the following assertions holds.

- (i) S has only one nonzero proper j -hyperideal.
- (ii) S has only two nonzero proper j -hyperideals J_1, J_2 such that $J_1 \cup J_2 = S$ and $J_1 \cap J_2 = \{0\}$.

Proof. The proof is similar to Theorem 5. $\square$

Example 12. Let $S = \{1, 2, 3, 4, 5\}$. Define $f : S^n \rightarrow \mathcal{P}^*(S)$ by $f(x_1^n) = ((\dots((x_1 * x_2) * \dots) * x_n))$, for all $x_1^n \in S$, where $*$ is defined by the following table.

$*$	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1, 2\}$	$\{1\}$	$\{1, 4\}$	$\{1\}$
3	$\{1\}$	$\{1, 5\}$	$\{1, 3\}$	$\{1, 3\}$	$\{1, 5\}$
4	$\{1\}$	$\{1, 2\}$	$\{1, 4\}$	$\{1, 4\}$	$\{1, 2\}$
5	$\{1\}$	$\{1, 5\}$	$\{1\}$	$\{1, 3\}$	$\{1\}$

Then (S, f) is an n -ary semihypergroup with a zero element 1 (see [22]). The nonzero proper n -hyperideals of S are $\{1, 2, 5\}$ and $\{1, 3, 4\}$. Clearly, S has exactly two nonzero proper n -hyperideal $\{1, 2, 5\}$ and $\{1, 3, 4\}$ such that $\{1, 2, 5\} \cap \{1, 3, 4\} = \{1\}$ and $\{1, 2, 5\} \cup \{1, 3, 4\} = \{1, 2, 3, 4, 5\} = S$. By Theorem 6 (ii), we can conclude that the nonzero proper n -hyperideals $\{1, 2, 5\}$ and $\{1, 3, 4\}$ are 0-minimal. Furthermore, it is easy to check that there exists a nonzero n -hyperideal $K := \{1, 5\}$ of $J := \{1, 2, 5\}$ such that $\mathcal{W}_{K(n)J} = f(J^{n-1}, K) = f(J^{n-1}, 1) \cup f(J^{n-1}, 5) = \{1\}$. By Theorem 2, we can conclude that $\{1, 2, 5\}$ is not n -0-simple.

5. Maximal j -Hyperideals

In this section, we give the characterization of the maximality of j -hyperideals, for each positive integer $1 \leq j \leq n$, in n -ary semihypergroups and study their interesting properties.

Definition 7. For any positive integer $1 \leq j \leq n$, a proper j -hyperideal J of S is called maximal if for every j -hyperideal A of S such that $J \subset A$, then $A = S$.

Equivalently, if for every proper j -hyperideal A of S is such that $J \subseteq A$, then $A = J$.

Example 13. Consider Examples 2 and 3, we have that A_1 and B_2 are maximal j -hyperideals of S , respectively.

The following results are generalizations of the results of maximal lateral hyperideals in ternary semihypergroups [16].

Theorem 7. For each positive integer $1 \leq j \leq n$, a proper j -hyperideal J of S is maximal if and only if one of the following assertions holds.

- (i) $S \setminus J = \{x\}$ and $f(S^{j-1}, x, S^{n-j}) \subseteq J$ for some $x \in S$.
- (ii) $S \setminus J \subseteq \mathcal{W}_{x(j)S}$ for all $x \in S \setminus J$.

Proof. Fix a positive integer j satisfying $1 \leq j \leq n$.

($\Rightarrow$) Let J be a maximal j -hyperideal of S . We consider two cases.

Case 1: There exists $x \in S \setminus J$ such that $\mathcal{W}_{x(j)S} \subseteq J$.

In this case, we will prove the assertion (i). Clearly, $f(S^{j-1}, x, S^{n-j}) = \mathcal{W}_1^{x(j)S} \subseteq \bigcup_{k \geq 1} \mathcal{W}_k^{x(j)S} = \mathcal{W}_{x(j)S} \subseteq J$. By Remark 1, $J \cup \{x\} = (J \cup \mathcal{W}_{x(j)S}) \cup \{x\} = J \cup (\mathcal{W}_{x(j)S} \cup \{x\}) = J \cup M^j(x)$. By Lemma 4, $J \cup \{x\} = J \cup M^j(x)$ is a j -hyperideal of S . Since J is maximal, $J \cup \{x\} = S$. Thus, $S \setminus J = \{x\}$.

Case 2: $\mathcal{W}_{x(j)S} \not\subseteq J$ for all $x \in S \setminus J$.

In this case, we will prove the assertion (ii). Let $x \in S \setminus J$. By Lemma 3 and 4, $J \cup \mathcal{W}_{x(j)S}$ is a j -hyperideal of S . Since J is maximal, $J \cup \mathcal{W}_{x(j)S} = S$. Since $x \notin J$, $x \in \mathcal{W}_{x(j)S}$. Thus, $S \setminus J \subseteq \mathcal{W}_{x(j)S}$.

($\Leftarrow$) Let K be a j -hyperideal of S such that $J \subset K$. Then $K \setminus J \neq \emptyset$. If there exists $x \in S$ such that $S \setminus J = \{x\}$ and $f(S^{j-1}, x, S^{n-j}) \subseteq J$, then $K \setminus J \subseteq S \setminus J = \{x\}$. Thus, $K \setminus J = \{x\}$ and $K = J \cup \{x\} = S$. Thus, J is a maximal j -hyperideal of S . Next, if $S \setminus J \subseteq \mathcal{W}_{x(j)S}$ for all $x \in S \setminus J$, then, by Lemma 1, $S \setminus J \subseteq \mathcal{W}_{x(j)S} \subseteq K$ for all $x \in K \setminus J \subseteq S \setminus J$. Hence $S = (S \setminus J) \cup J \subseteq K \cup K = K \subseteq S$. Therefore, $K = S$ and J is a maximal j -hyperideal of S . $\square$

For each positive integer $1 \leq j \leq n$, let $\mathcal{U}^j$ denote the union of all proper j -hyperideals of S without zero. Clearly, by Lemma 4, $\mathcal{U}^j$ is a j -hyperideal of S . Similarly, the union of all nonzero proper j -hyperideals of S with zero is denoted by $\mathcal{U}_0^j$. The following lemmas are easy to verify.

Lemma 10. For each positive integer $1 \leq j \leq n$, $\mathcal{U}^j = S$ if and only if $M^j(x) \neq S$ for all $x \in S$.

Lemma 11. For each positive integer $1 \leq j \leq n$, if $\mathcal{U}^j \neq S$, then $\mathcal{U}^j$ is the unique maximal j -hyperideal of S .

Remark 3. The above two lemmas are still true if we replace the set $\mathcal{U}^j$ by $\mathcal{U}_0^j$.

As a consequence of Theorem 7 and Lemma 10–11, we obtain some interesting properties of n -ary semihypergroups as follows.

Theorem 8. Let S have no zero element. For each positive integer $1 \leq j \leq n$, one of the following assertions holds.

- (i) S is j -simple.
- (ii) $M^j(x) \neq S$ for all $x \in S$.
- (iii) There exists $x \in S$ such that $M^j(x) = S$, $f(S^{j-1}, x, S^{n-j}) \subseteq \mathcal{U}^j = S \setminus \{x\}$ and $x \notin \mathcal{W}_{x(j)S}$ and $\mathcal{U}^j$ is the unique maximal j -hyperideal of S .
- (iv) $S \setminus \mathcal{U}^j = \{y \in S : \mathcal{W}_{y(j)S} = S\}$ and $\mathcal{U}^j$ is the unique maximal j -hyperideal of S .

Proof. Now, we fix a positive integer j satisfying $1 \leq j \leq n$ and assume that S is not j -simple. Thus, there is a proper j -hyperideal of S in $\mathcal{U}^j$ and $\mathcal{U}^j \neq \emptyset$. We consider the following two cases.

Case 1: $\mathcal{U}^j = S$. Then, by Lemma 10, $M^j(x) \neq S$ for all $x \in S$. The assertion (ii) holds.

Case 2: $\mathcal{U}^j \neq S$. In this case, we will show that the assertions (iii) and (iv) hold. By Lemma 11, $\mathcal{U}^j$ is the unique maximal j -hyperideal of S . Theorem 7 implies that

2.1. $S \setminus \mathcal{U}^j = \{x\}$ and $f(S^{j-1}, x, S^{n-j}) \subseteq \mathcal{U}^j$ for some $x \in S$ or

2.2. $S \setminus \mathcal{U}^j \subseteq \mathcal{W}_{x(j)S}$ for all $x \in S \setminus \mathcal{U}^j$.

Case 2.1: We will show that the assertion (iii) holds. Suppose that $S \setminus \mathcal{U}^j = \{x\}$ and $f(S^{j-1}, x, S^{n-j}) \subseteq \mathcal{U}^j$ for some $x \in S$. Then $f(S^{j-1}, x, S^{n-j}) \subseteq \mathcal{U}^j = S \setminus \{x\}$. Next, if $M^j(x) \neq S$, then $M^j(x)$ is a proper j -hyperideal of S and so $x \in M^j(x) \subseteq \mathcal{U}^j$. This is a contradiction with $x \notin \mathcal{U}^j$. Thus, $M^j(x) = S$. Finally, assume that $x \in \mathcal{W}_{x(j)S}$. Then $S = M^j(x) = \mathcal{W}_{x(j)S} \cup \{x\} = \mathcal{W}_{x(j)S}$. Since $f(S^{j-1}, x, S^{n-j}) \subseteq \mathcal{U}^j$ and $\mathcal{U}^j$ is a j -hyperideal of S , we gain $\mathcal{W}_2^{x(j)S} = f(S^{j-1}, f(S^{j-1}, x, S^{n-j}), S^{n-j}) \subseteq f(S^{j-1}, \mathcal{U}^j, S^{n-j}) \subseteq \mathcal{U}^j$. Similarly, for any positive integer $k \geq 3$, we have

$$\mathcal{W}_k^{x(j)S} = \underbrace{f(S^{j-1}, f(\dots, f(S^{j-1}, x, S^{n-j}), \dots), S^{n-j})}_{f \text{ appears } k \text{ times}} \subseteq \mathcal{U}^j.$$

Then $S = \mathcal{W}_{x(j)S} = \bigcup_{k \geq 1} \mathcal{W}_k^{x(j)S} \subseteq \mathcal{U}^j \subseteq S$ and so $S = \mathcal{U}^j$. It is impossible because $\mathcal{U}^j \neq S$. Therefore, $x \notin \mathcal{W}_{x(j)S}$ and the assertion (iii) is satisfied.

Case 2.2: Suppose that $S \setminus \mathcal{U}^j \subseteq \mathcal{W}_{x(j)S}$ for all $x \in S \setminus \mathcal{U}^j$. We will show that $S \setminus \mathcal{U}^j = \{y \in S : \mathcal{W}_{y(j)S} = S\}$. Let $y \in S \setminus \mathcal{U}^j$. Then $y \in \mathcal{W}_{y(j)S}$. By Remark 1, $M^j(y) = \mathcal{W}_{y(j)S} \cup \{y\} = \mathcal{W}_{y(j)S}$. Since $\mathcal{U}^j \neq S$, by Lemma 10, we obtain $S = M^j(y) = \mathcal{W}_{y(j)S}$. Conversely, let $y \in S$ such that $\mathcal{W}_{y(j)S} = S$. Then, by Remark 1, we get $M^j(y) = \mathcal{W}_{y(j)S} \cup \{y\} = S \cup \{y\} = S$. To show that $y \in S \setminus \mathcal{U}^j$, assume that $y \in \mathcal{U}^j$. By Lemma 1, we obtain $S = M^j(y) = \mathcal{W}_{y(j)S} \cup \{y\} \subseteq \mathcal{U}^j \cup \mathcal{U}^j = \mathcal{U}^j$. It is impossible because $S \neq \mathcal{U}^j$. Thus, $y \in S \setminus \mathcal{U}^j$ and so $S \setminus \mathcal{U}^j = \{y \in S : \mathcal{W}_{y(j)S} = S\}$. The assertion (iv) holds. $\square$

Theorem 9. Let S have a zero element and $f(S^n) \neq \{0\}$. For each positive integer $1 \leq j \leq n$, one of the following assertions hold.

- (i) S is j -0-simple.
- (ii) $M^j(x) \neq S$ for all $x \in S$.
- (iii) There exists $x \in S$ such that $M^j(x) = S$, $f(S^{j-1}, x, S^{n-j}) \subseteq \mathcal{U}_0^j = S \setminus \{x\}$, $x \notin \mathcal{W}_{x(j)S}$ and $\mathcal{U}_0^j$ is the unique maximal j -hyperideal of S .
- (iv) $S \setminus \mathcal{U}_0^j = \{y \in S : \mathcal{W}_{y(j)S} = S\}$ and $\mathcal{U}_0^j$ is the unique maximal j -hyperideal of S .

Proof. The proof is similar to Theorem 8. $\square$

Theorem 10. Let S have no zero element. For each positive integer $1 \leq j \leq n$, let S have proper j -hyperideals. Then every proper j -hyperideal of S is maximal if and only if one of the following assertions holds.

- (i) S has only one proper j -hyperideal.
- (ii) S has only two proper j -hyperideals J_1, J_2 such that $J_1 \cup J_2 = S$ and $J_1 \cap J_2 = \emptyset$.

Proof. First, we fix a positive integer j satisfying $1 \leq j \leq n$.

($\Rightarrow$) Let J be a proper j -hyperideal of S . By hypothesis, J is maximal. We consider into two cases.

Case 1: $S = M^j(x)$ for all $x \in S \setminus J$.

In this case, we will show that J is the unique j -hyperideal of S . Now, let A be a proper j -hyperideal of S such that $A \neq J$. By hypothesis, A is a maximal j -hyperideal of S . Since $A \neq J$, we can divide into 3 cases, i.e., (1.1) $A \cap J = \emptyset$, (1.2) $A \cap J \neq \emptyset$ and $A \setminus J = \emptyset$ and (1.3) $A \cap J \neq \emptyset$ and $A \setminus J \neq \emptyset$.

Case 1.1 $A \cap J = \emptyset$. Then $S = M^j(x)$ for all $x \in A \subseteq S \setminus J$. Then $S = M^j(x) = \mathcal{W}_{x(j)S} \cup \{x\} \subseteq A \cup A = A \subseteq S$. This contradicts with A is a proper j -hyperideal of S .

Case 1.2 $A \cap J \neq \emptyset$ and $A \setminus J = \emptyset$. Then $A \subseteq J$. Since A is maximal, $A = J$. This is a contradiction.

Case 1.3 $A \cap J \neq \emptyset$ and $A \setminus J \neq \emptyset$. Then there exists $x \in A \setminus J$ and $S = M^j(x) = \mathcal{W}_{x(j)S} \cup \{x\} \subseteq A \cup A \setminus J = A \subseteq S$. This is impossible because $A \neq S$. The assertions (i) holds.

Case 2: $S \neq M^j(x)$ for some $x \in S \setminus J$.

By hypothesis, $M^j(x)$ is a maximal j -hyperideal of S and $J \neq M^j(x)$. By Lemma 4, $M^j(x) \cup J$ is a j -hyperideal of S . Since J is maximal, $M^j(x) \cup J = S$. Since $M^j(x) \cap J \subset J$ and by hypothesis, it follows that $M^j(x) \cap J = \emptyset$. Next, we will confirm that $M^j(x)$ and J are only two proper j -hyperideals of S such that $M^j(x) \cup J = S$ and $M^j(x) \cap J = \emptyset$. Let A be a proper j -hyperideal of S . By hypothesis, A is maximal. Then $A = A \cap S = A \cap (M^j(x) \cup J) = (A \cap M^j(x)) \cup (A \cap J)$. By the same manner as in Theorem 5, we obtain either $A = J$ or $A = M^j(x)$. Therefore, the assertion (ii) holds.

($\Leftarrow$) The proof is similar to Theorem 5. $\square$

Theorem 11. Let S have a zero element. For each positive integer $1 \leq j \leq n$, let S have nonzero proper j -hyperideals. Then every nonzero proper j -hyperideal of S is maximal if and only if one of the following assertions holds.

- (i) S has only one nonzero proper j -hyperideal.
- (ii) S has only two nonzero proper j -hyperideals J_1, J_2 such that $J_1 \cup J_2 = S$ and $J_1 \cap J_2 = \{0\}$.

Proof. The proof is similar to Theorem 10. $\square$

Next, we shall apply the theorems of this section to the following example.

Example 14. Consider an n -ary semihypergroup (S, f) with a zero element 1 has been given in Example 12.

- (1) Clearly, the 1-hyperideals of S are $\{1\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$ and S . It follows that the nonzero proper 1-hyperideals of S are $\{1, 2, 4\}$ and $\{1, 3, 5\}$. Since $\{1, 2, 4\} \cap \{1, 3, 5\} = \{1\}$ and $\{1, 2, 4\} \cup \{1, 3, 5\} =$

$\{1, 2, 3, 4, 5\} = S$, by Theorem 11, we can conclude that the nonzero proper 1-hyperideals $\{1, 2, 4\}$ and $\{1, 3, 5\}$ are maximal.

- (2) As an application of (1), let $H = \{1, 3, 5\}$. By Example 12, we have an n -ary semihypergroup (H, f) with a zero element 1, where an n -ary hyperoperation f is defined by $f(x_1^n) = ((\dots((x_1 * x_2) * \dots) * x_n))$, for all $x_i^n \in S$, and $*$ is defined by the following table. Put $A := \{1, 5\}$. Clearly, A is only one nonzero proper

$*$	1	3	5
1	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1, 3\}$	$\{1, 5\}$
5	$\{1\}$	$\{1\}$	$\{1\}$

2-hyperideal of H . We have $H \setminus A = \{3\}$ and $\mathcal{W}_1^{3(2)H} = f(H, 3, H^{n-2}) = H$. Thus, $H \setminus A \subseteq \mathcal{W}_1^{x(2)H} \subseteq \mathcal{W}_{x(2)H}$ for all $x \in H \setminus A$. By Theorem 7 (ii), we can conclude that A is a maximal 2-hyperideal of H .

- (3) We shall give an example that satisfies Theorem 8 (9). Put $B := \{1, 3\}$. Clearly, A, B are exactly two nonzero proper n -hyperideals of H . Thus, H is not n -simple (that is, the condition (i) in Theorem 9 is not satisfied). Note that B is also not a j -hyperideal of H for all $j = 1, 2, \dots, n-1$. Since $\mathcal{U}_0^n = A \cup B = \{1, 3, 5\} = H$ is not a proper n -hyperideal of H , it follows that $\mathcal{U}_0^n$ is not a maximal n -hyperideal of H (that is, the conditions (iii) and (iv) in Theorem 9 are not satisfied). As we know, $M^n(x) = f(H^{n-1}, x)$ for all $x \in H$. We have $M^n(1) = f(H^{n-1}, 1) = \{1\}$, $M^n(3) = f(H^{n-1}, 3) = \{1, 3\}$ and $M^n(5) = f(H^{n-1}, 5) = \{1, 5\}$. Thus, $M^n(x) \neq H$ for all $x \in H$. Therefore, the n -ary semihypergroup (H, f) satisfies the condition (ii) in Theorem 9. Furthermore, this example is also satisfied Lemma 10.

6. Conclusions

Similarly to the theory of semigroups, hyperideal theory in semihypergroups plays an important role in the studying the algebraic properties and the structural properties of semihypergroups. As we know, a semihypergroup represents a natural extension of a semigroup. Moreover, the structure of an n -ary semihypergroup [12] can also be considered as a generalization of semigroups, ternary semigroups and n -ary semigroups, and an analogous structure for semihypergroups. This paper is a contribution to the study of j -hyperideal [14], for all $1 \leq j \leq n$, in n -ary semihypergroups where $n \geq 2$. We introduced the concept of j -(0)-simple and (0)-simple n -ary semihypergroups. We gave the properties of j -hyperideals in n -ary semihypergroups and characterized the minimality and maximality of j -hyperideals in n -ary semihypergroups. We established the relationships between (0)-minimal j -hyperideals and the j -(0)-simple n -ary semihypergroups. Furthermore, the connections between maximal j -hyperideals and j -(0)-simple n -ary semihypergroups were presented. Our obtained results were gotten to extend and generalize some similar results in semigroups, ternary semigroups [8,19] and ternary semihypergroups [16]. Moreover, we gave a complete answer of the open problem given by Petchkaew and Chinram in [18].

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References

1. Kasner, E. An extension of the group concept (reported by L.G. Weld). *Bull. Amer. Math. Soc.* **1904**, *10*, 290–291.
2. Dörnte, W. Untersuchungen über einen verallgemeinerten Gruppenbegriff. *Math. Zeit.* **1928**, *29*, 1–19. [CrossRef]
3. Sioson, F.M. On regular algebraic systems. *Proc. Japan. Acad.* **1963**, *39*, 283–286. [CrossRef]

4. Dudek, W.A. On divisibility in n -semigroups. *Demonstrati. Math.* **1980**, *13*, 355–367.
5. Dudek, W.A.; Groździńska, I. On ideals in regular n -semigroups. *Mat. Bilten (Skopje)* **1980**, *4*, 35–44.
6. Pop, M.S. On simple n -semigroups. *Bul. Ştiinţ. Univ. Baia Mare Ser. B* **1999**, *15*, 33–44.
7. Dixit, V.N.; Dewan, S. A note on quasi and bi-ideals in ternary semigroups. *Int. J. Math. Math. Sci.* **1995**, *18*, 501–508. [[CrossRef](#)]
8. Iampan, A. Lateral ideals of ternary semigroup. *Ukrainian Mathe. Bull.* **2007**, *4*, 525–534.
9. Marty, F. Sur une generalization de la notion de group. In Proceedings of the 8th Congress on Mathematics Scandenaves, Stockholm, Sweden, 14–18 August 1934; pp. 45–49.
10. Corsini, P. Hypergraphs and Hypergroups. *Algebra Univers.* **1996**, *35*, 548–555. [[CrossRef](#)]
11. Davvaz, B. *Semihypergroup Theory*; Elsevier/Academic Press: London, UK, 2016.
12. Davvaz, B.; Dudek, W.A.; Vougiouklis, T. A generalization of n -ary algebraic systems. *Commun. Algebra* **2009**, *37*, 1248–1263. [[CrossRef](#)]
13. Davvaz, B.; Dudek, W.A.; Mirvakili, S. Neutral elements, fundamental relations and n -ary hypersemigroups. *Internat. J. Algebra Comput.* **2009**, *19*, 567–583. [[CrossRef](#)]
14. Hila, K.; Davvaz, B.; Naka, K.; Dine, J. Regularity in term of hyperideals. *Chinese J. Math.* **2013**, *2013*, 4. [[CrossRef](#)]
15. Davvaz, B.; Leoreanu-Fotea, V. Binary relations on ternary semihypergroups. *Commun. Algebra* **2010**, *38*, 3621–3636. [[CrossRef](#)]
16. Naka, K.; Hila, K. Some properties of hyperideals in ternary semihypergroups. *Math. Slovaca* **2013**, *63*, 449–468. [[CrossRef](#)]
17. Naka, K.; Hila, K. Regularity of ternary semihypergroups. *Quasigroups Relat. Syst.* **2017**, *25*, 291–306.
18. Petchkaew, P.; Chinram, R. The minimality and maximality of n -ideal in n -ary semigroup. *Eur. J. Pure Appl. Math.* **2018**, *11*, 762–773. [[CrossRef](#)]
19. Iampan, A. The minimality and maximality of left(right) ideals of ternary semigroups. *Int. J. Contemp. Math. Sciences* **2010**, *49*, 2409–2417.
20. Tang, J.; Feng, X.; Davvaz, B.; Xie, X.Y. A further study on ordered regular equivalence relations in ordered semihypergroups. *Open Math.* **2018**, *16*, 168–184. [[CrossRef](#)]
21. Tang, J.; Davvaz, B. Study on Green's relations in ordered semihypergroups. *Soft Comput.* **2020**, *24*, 11189–11197. [[CrossRef](#)]
22. Ardekani, L.K.; Davvaz, B. Ordered semihypergroup constructions. *Bol. Mat.* **2018**, *25*, 77–99.

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