



Article Certain Properties of Vague Graphs with a Novel Application

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Abstract: Fuzzy graph models enjoy the ubiquity of being present in nature and man-made structures, such as the dynamic processes in physical, biological, and social systems. As a result of inconsistent and indeterminate information inherent in real-life problems that are often uncertain, for an expert, it is highly difficult to demonstrate those problems through a fuzzy graph. Resolving the uncertainty associated with the inconsistent and indeterminate information of any real-world problem can be done using a vague graph (VG), with which the fuzzy graphs may not generate satisfactory results. The limitations of past definitions in fuzzy graphs have led us to present new definitions in VGs. The objective of this paper is to present certain types of vague graphs (VGs), including strongly irregular (SI), strongly totally irregular (STI), neighborly edge irregular (NEI), and neighborly edge totally irregular vague graphs (NETIVGs), which are introduced for the first time here. Some remarkable properties associated with these new VGs were investigated, and necessary and sufficient conditions under which strongly irregular vague graphs (SIVGs) and highly irregular vague graphs (HIVGs) are equivalent were obtained. The relation among strongly, highly, and neighborly irregular vague graphs was established. A comparative study between NEI and NETIVGs was performed. Different examples are provided to evaluate the validity of the new definitions. A new definition of energy called the Laplacian energy (LE) is presented, and its calculation is shown with some examples. Likewise, we introduce the notions of the adjacency matrix (AM), degree matrix (DM), and Laplacian matrix (LM) of VGs. The lower and upper bounds for the Laplacian energy of a VG are derived. Furthermore, this study discusses the VG energy concept by providing a real-time example. Finally, an application of the proposed concepts is presented to find the most effective person in a hospital.

Keywords: vague set (VS); vague graph (VG); strongly irregular (SI); highly irregular (HI); neighborly irregular (NI); dominating set (DS); spectrum

1. Introduction

Graph theory serves as an exceptionally beneficial tool in solving combinatorial problems in various fields, such as geometry, algebra, number theory, topology, and social systems. A graph basically holds a model of relations, and it is used to depict the real-life problems encompassing the relationships among objects. To represent the objects and the relations between objects, the graph vertices and edges are employed, respectively. In numerous optimization problems, the existing information is inexact or imprecise for various reasons, such as the loss of information, a lack of evidence, imperfect statistical data, and insufficient information. Generally, the uncertainty in real-life problems may be available in the information that outlines the problem. Fuzzy graph (FG) models are helpful mathematical tools in order to address the combinatorial problems in various fields incorporating research, optimization, algebra, computing, environmental science, and topology. Due to

the natural existence of vagueness and ambiguity, fuzzy graphical models are noticeably better than graphical models. Originally, fuzzy set theory was required to deal with many multifaceted issues, which are replete with incomplete information. In 1965 [1], fuzzy set theory was first suggested by Zadeh. Fuzzy set theory is a highly powerful mathematical tool for solving approximate reasoning related problems. These notions meritoriously illustrate complicated phenomena, which are not precisely described using classical mathematics. In 1993, Gau and Buehrer [2] organized the fuzzy set theory by presenting the VS notion by changing the value of an element in a set with a sub-interval of [0, 1]. Specifically, a true membership function of $t_{\nu}(x)$ and a false membership function of $f_{\nu}(x)$ are used to define the boundaries of the membership degree. The VSs describe more possibilities than fuzzy sets. A VS is more initiative and helpful due to the existence of the false membership degree. The first description of FGs was proposed by Kafmann [3] in 1993, taken from Zadeh's fuzzy relations [4,5]. However, Rosenfeld [6] described another detailed definition, including fuzzy vertex and fuzzy edges and various fuzzy analogs of graphical theoretical concepts, including paths, cycles, connectedness, etc. Ramakrishna [7] recommended the vague graph (VG) notion and evaluated some of its features. Akram et al. [8–14] presented new definitions of FGs. Borzooei and Rashmanlou [15–18] investigated different concepts on VG. Gani and Radha [19] recommended RFGsand TRFGs. Samanta et al. [20–23] introduced fuzzy competition graphs and some properties of irregular bipolar fuzzy graphs. Shao et al. [24] presented some results for intuitionistic fuzzy graphs with an application to water supplier systems. Harish Garg et al. [25,26] introduced the vertex rough graph, and novel concepts in interval-valued fuzzy graphs and neutrosophic graphs. Kumaravel and Radha described the concepts of the edge degree and the total edge degree in RFGs [27]. Mathew et al. [28] studied the energy of an FG. Rashmanlou et al. [29–32] introduced some properties of FGs. Sunitha et al. [33–35] presented new concepts for fuzzy graphs. Ghosh et al. [19,36] investigated new concepts of the signed product and the total signed product on complex graphs. In 2008, Gani and Latha [37] investigated the properties of NIFGs and HIFGs.

Irregularity definitions serve as highly significant tools in the study of network heterogeneity, having implications for networks that are present across biology, ecology, technology, and the economy. Several graph statistics have been proposed so far, many of which are based on the number of vertices in a graph and their degrees. The irregularity concepts play a crucial role in both graph theory and application in the vague environment. The highly irregular graph's characterization has also been applied to the heterogeneity question, yet all of these fail to shed enough light on real-world situations. Efforts are being made to find appropriate ways to quantify network heterogeneity. The graph energy is one of the emerging concepts within graph theory. This concept acts as a frontier between chemistry and mathematics. Vague sets are considered as higher-order fuzzy sets. The higher-order fuzzy set's application complicates the solution procedure, but better results can be obtained if the complexity regarding computation time, computation volume, and memory space is not a matter of concern. There are some remarkable features for handling vague data that are exclusive to vague sets. For example, vague sets allow a more intuitive graphical representation of vague data, significantly facilitating better analysis in data relationships, incompleteness, and similarity measures. The vague model is more adaptable and usable than fuzzy and intuitionistic fuzzy models. In many applications, including urban traffic planning, telecommunication message routing, telemarketing operator scheduling, VLSI chip optimal pipelining, texture mapping, etc., VGs emerge as mathematical models of the observed real world systems. Hence, in this paper, we describe new concepts, such as the energy and Laplacian energy of VGs. We explain these notions with real examples and evaluated parts of their qualities. Likewise, we present and explore the key properties of strongly irregular (SI), strongly totally irregular (STI), neighborly edge irregular (NEI), and neighborly edge totally irregular (NETI) in VGs. The concept of being totally irregular is very important in FG theory and it can be useful in social networking and communication; however, it has not been much investigated as it should be. Therefore, in this paper we define certain types of totally irregular vague graphs and explain them with suitable examples. Dominating sets enjoy real-world interest in several areas. In wireless networking,

dominating sets are applied to find efficient routes within ad-hoc mobile networks. Thus, a new kind of domination set in irregular vague graphs (IVGs) was introduced and its properties were studied.

The construction of the paper is as follows: In Section 2 we propose the concepts of highly irregular (HI), neighborly irregular (NI), strongly irregular (SI), highly totally irregular (HTI), neighborly totally irregular (NTI), and STIVG in VGs and study their properties. Section 3 deals with the investigation of the concepts of energy; spectrum; and out-degree matrix OM, degree matrix (DM), LM, and Laplacian energy (LE) in VGs, and descriptions of their properties in detail. Section 4 introduces the energy of a VG through an example. Section 5 highlights the concepts that we used to find the most effective person in an organization, and finally, we provide the conclusion in Section 6.

2. Preliminaries

In this section, aimed at facilitating the next sections, a brief review of VGs is presented.

By a graph, we mean a pair of $G^* = (V, E)$, where *V* is the set and *E* is a relation on *V*. The elements of *V* are vertices of G^* and the elements of *E* are edges of G^* .

A fuzzy subset ψ on a set of *X* is a map ψ : $X \to [0,1]$. A fuzzy relation on *X* is a fuzzy subset of ϕ : $X \times X \to [0,1]$ on $X \times X$.

A fuzzy graph $\zeta = (V, \psi, \phi)$ is an algebraic structure of a non-empty set *V* together with a pair of mappings $\psi : V \to [0, 1]$ and $\phi : V \times V \to [0, 1]$ and is defined as $\phi(mn) \leq \psi(m) \land \psi(n), \forall m, n \in V$, and ϕ is a symmetric fuzzy relation on ψ and \wedge denotes the minimum. Here, $\psi(m)$ and $\phi(mn)$ represent the membership values of the vertex *m* and of the edge (m, n) in ζ , respectively.

The underlying crisp graph G = (V', E) of FG $\zeta = (V, \psi, \phi)$ is such that $V' = \{m \in V | \psi(m) > 0\}$ and $E = \{(m, n) | \phi(m, n) > 0\}$.

A vague set (VS) *A* is a pair of (t_A, f_A) on set *V* where t_A and f_A are taken as real valued functions which can be defined on $V \rightarrow [0, 1]$, so that $t_A(m) + f_A(m) \leq 1$ for all *m* belongs to *V*. The interval $[t_A(m), 1 - f_A(m)]$ is known as the vague value of *m* in *A*.

 $t_A(m)$, in this definition, is taken for the membership degree as the lower bound when *m* in *A* and $f_A(m)$ is the lower bound for negative membership of *m* in *A*. Thus, the interval given by $[t_A(m), 1 - f_A(m)]$ determines the degree of membership of *m* in the VS *A*.

Suppose that *X* and *Y* are ordinary, finite, non-empty sets. A vague relation (VR) is a vague subset of $X \times Y$, that is, an expression *R* described by

$$R = \{ \langle (m,n), t_R(m,n), f_R(m,n) \rangle | m \in X, n \in Y \}$$

where $t_R : X \times Y \to [0,1]$, $f_R : X \times Y \to [0,1]$, which holds the condition $0 \le t_R(m,n) + f_R(m,n) \le 1$, $\forall (m,n) \in X \times Y$.

 G^* will be a crisp graph (V, E) and G a VG (A, B) throughout this paper. As an edge of $xy \in E$ is identified with an ordered pair of $(x, y) \in V \times V$, a VR on E can be identified with a vague set on E, giving the possibility of defining a VG as a pair of vague sets.

Definition 1 ([7]). A VR B on a set V is a VR from V to V. If A is a VS on a set V, then a VR B on A is a VR which holds that $t_B(mn) \le \min(t_A(m), t_A(n))$ and $f_B(mn) \ge \max(f_A(m), f_A(n))$, for all m, n in V.

Definition 2 ([7]). A pair G = (A, B) with V as the set of nodes is said to be a VG, where $A = (t_A, f_A)$ is a VS of V and $B = (t_B, f_B)$ is a VR on V (see Figure 1).

A VG G is called strong if $t_B(mn) = \min(t_A(m), t_A(n))$ and

 $f_B(mn) = \max(f_A(m), f_A(n))$, for each edge $mn \in E$. A VG G is called complete if $t_B(mn) = \min(t_A(m), t_A(n))$ and $f_B(mn) = \max(f_A(m), f_A(n))$, $\forall m, n \in V$.

Definition 3 ([17]). Suppose that G = (A, B) is a VG. Then,

(i) The degree of a node m is defined as $d_G(m) = (d_t(m), d_f(m))$, where $d_t(m) = \sum_{m \neq n} t_B(mn)$ and $d_f(m) = \sum_{m \neq n} f_B(mn)$.

(*ii*) The total degree of a node *m* is defined by $td_G(m) = (td_t(m), td_f(m))$, where $td_t(m) = \sum_{m \neq n} t_B(mn) + t_A(m)$ and $td_f(m) = \sum_{m \neq n} f_B(mn) + f_A(m)$.

Definition 4 ([18]). *The complement of a VG G* = (*A*, *B*) *is a VG* $\overline{G} = (\overline{A}, \overline{B})$ *where* $\overline{A} = A = (\overline{t}_A, \overline{f}_A)$ *and* $\overline{B} = (\overline{t}_B, \overline{f}_B)$ *are defined by:*

$$\begin{array}{l} (i) \ \overline{V} = V \\ (ii) \ \overline{t}_A(m) = t_A(m), \ \overline{f}_A(m) = f_A(m), \ \forall m \in V, \\ (iii) \ \overline{t}_B(mn) = \left\{ \begin{array}{l} 0 & if \ t_B(mn) > 0, \\ \min(t_A(m), t_A(n)) & if \ t_B(mn) = 0, \end{array} \right. \\ \overline{f}_B(mn) = \left\{ \begin{array}{l} 0 & if \ f_B(mn) > 0, \\ \max(f_A(m), f_A(n)) & if \ f_B(mn) = 0. \end{array} \right. \end{array} \right.$$

Definition 5 ([17]). Let G = (A, B) be a VG.

(i) G is irregular if there is a node neighboring the nodes with distinct degrees.

(*ii*) *G* is (TI), *if there is a node neighboring the nodes with distinct total degrees.*

All the basic notation is shown in Table 1.

Notation	Meaning				
ζ	Fuzzy Graph				
VG	Vague Graph				
SI	Strongly Irregular				
HI	Highly Irregular				
TI	Totally Irregular				
STI	Strongly Total Irregular				
NEI	Neighborly Edge Irregular				
NETI	Neighborly Edge Totally Irregular				
NIVG	Neighborly Irregular Vague Graph				
HIVG	Highly Irregular Vague Graph				
SIVG	Strongly Irregular Vague Graph				
NTIVG	Neighborly Totally Irregular Vague Graph				
LE	Laplacian Energy				
AM	Adjacency Matrix				
DM	Degree Matrix				
LM	Laplacian Matrix				
OM	Out-Degree Matrix				
DS	Dominating Set				
\triangle -DS	riangle-Dominating Set				
VDG	Vague digraph G				
LS	Laplacian Spectrum				
IG	Influence Graph				

Table 1. Some basic notation.

3. New Concepts of IVGs

The irregularity concepts play an important role in both the graph theory application and theory in the vague environment. The characterizations of highly irregular and neighborly irregular graphs have also been applied to the question of heterogeneity. One of the most broadly studied classes of FGs is IFGs. They are being applied in many contexts, for example, the r-irregular FGs, with connectivity and edge-connectivity equal to r, play a key role in designing reliable communication networks. This idea inspires us to present different types of IGVs, such as HI, NI, SI, HTI, NTI, and STIVG, and their related theorems.

Definition 6 ([17]). Let G = (A, B) be a connected vague graph (CVG).

(*i*) *G* is said to be a highly irregular vague graph (HIVG) if every node of G is a neighbor to vertices with distinct neighborhood degrees.

(*ii*) *G* is assumed to be a neighborly irregular vague graph (NIVG) if every two neighbor nodes of G have distinct degrees.

(iii) G is said to be a strongly irregular vague graph (SIVG) if every pair of nodes in G has distinct degrees.

This definition is supported with the help of the following example.

Example 1. Consider the vague graph G as shown in Figure 1.



Figure 1. Vague Graph (VG) (G).

By a simple calculation we have:

$$d_G(m) = (0.3, 0.8), \ d_G(n) = (0.2, 0.7), \ d_G(x) = (0.3, 0.9).$$

Hence, G is a HIVG.

Corollary 1. If G = (A, B) is a SIVG, then it is both HIVG and NIVG.

Proof. It is obvious. \Box

Example 2. We have considered an example of a SIVG-G, presented in Figure 2, that is both HIVG and NIVG.



Figure 2. Strongly Irregular Vague Graph (SIVG) (G).

It is easy to show that $d_G(m) = (0.4, 1.2)$, $d_G(n) = (0.3, 1.1)$, $d_G(z) = (0.6, 1.1)$, and $d_G(x) = (0.7, 1.2)$. Clearly G is both a HIVG and a NIVG according to Definition 6.

Theorem 1. A HIVG or NIVG G = (A, B) does not need to be a SIVG.

Proof. Assume that *m* and *n* are any two nodes of *G*, which are neither neighbors nor incidental on the same node and may happen to have the same degrees. Such a condition contradicts the SIVG definition. \Box

Example 3. Consider Highly irregular vague graph-graph (HIVG-G) in Figure 3.



Figure 3. Highly irregular vague graph (HIVG) (G).

From Figure 3, $d_G(m) = (0.2, 0.5)$, $d_G(n) = (0.5, 1.2)$, and $d_G(z) = (0.5, 1.2)$, $d_G(x) = (0.2, 0.5)$. Clearly, G is a HIVG but it is not a SIVG.

Theorem 2. Let G = (A, B) be a HI and a NIVG. If each pair of nodes in G is either neighbor or incident on the same vertex, then G is a SIVG.

Proof. It is clear. \Box

Next, a critical theorem that describes a necessary and sufficient condition for a vague graph to be SIVG is provided.

Theorem 3. A VG G where G^* is a cycle with nodes 3 is SIVG if and only if the weights of the edges between each pair of nodes are all distinct.

Proof. If the weights of any edges are identical, then it violates the SIVGs definition. Conversely, the weights of edges between each pair of nodes are all distinct. Let *m*, *n*, and *w* be the nodes of *G*. Suppose $d_f(m) = d_f(n)$. Hence, $f_B(mn) + f_B(mw) = f_B(mn) + f_B(nw)$. This, $f_B(mw) = f_B(nw)$, a contradiction. Similarly, a true membership can be proven. Therefore, *G* is a SIVG. \Box

Example 4. *In this example (Figure 4) G is a NIVG.*



Figure 4. Neighborly Irregular Vague Graph (NIVG) (G).

In the following theorem, a strong condition for the complement of an VG to be a SIVG is presented.

Theorem 4. Let G = (A, B) be a VG where G^* is regular; t_A and f_A are constant functions; and $\forall m$, $n \in V(G)$, $t_B(mn) \leq t_A(m) \wedge t_A(n)$, and $f_B(mn) \geq \max(f_A(m), f_A(n))$. Then, G is a SIVG iff \overline{G} is a SIVG.

Proof. Let *G* be a SIVG, $f_A(m) = c$, and $t_A(m) = d$, $\forall m \in V(G)$. Since *G* is a SIVG, by Definition 6 we have:

 $d_f(m) \neq d_f(n), \forall m, n \in V(G)$

If and only if $\sum_{i=1}^{n} f_B(mx_i) \neq \sum_{j=1}^{n} f_B(ny_j)$, for all x_i adjoining the m and for all y_j adjoining the n. If and only if $\sum_{i=1}^{n} [c - f_B(mx_i)] \neq \sum_{j=1}^{n} [c - f_B(ny_j)]$, since G^* is regular.

If and only if $\sum_{i=1}^{n} [f_A(m) \lor f_A(x_i) - f_B(mx_i)] \neq \sum_{j=1}^{n} [f_A(n) \lor f_A(y_j) - f_B(ny_j)]$, for all x_i adjoining the *m* and for all y_j adjoining the *n*.

If and only if $\sum_{i=1}^{n} \overline{f_B}(mx_i) \neq \sum_{j=1}^{n} \overline{f_B}(ny_j), \forall m, n \in V(\overline{G}).$ Similarly, we can prove that for the true degree. Hence, \overline{G} is an SIVG. \Box

Remark 1. Let G = (A, B) be a SIVG; then the partial subgraph H = (A', B') need not be a SIVG.

Theorem 5. The vague subgraph H = (A', B') of a SIVG G = (A, B) is a SIVG too.

Proof. Let *G* be a SIVG. Then $d_t(m) \neq d_t(n)$, $\forall m, n \in V(G)$. Thus, $\sum t_B(mx_i) \neq \sum t_B(ny_j)$, for all x_i adjoining the *m* and for all y_j adjoining the *n*. Hence, $\sum t_B(mx_i) \neq \sum t_B(ny_j)$, for all x_i adjoining the *m* and for all y_j adjoining the *n*. Therefore, $d_t(m) \neq d_t(n)$, $\forall m, n \in V(H)$. \Box

Example 5. Consider the vague graph of G and vague subgraph of H as shown in Figure 5.



Figure 5. VG (*G*) and vague subgraph (*H*).

By a simple calculation it is easy to show that G and H are both SIVGs too.

The first definition of neighborly irregular in fuzzy graph was introduced in [37]. The neighborly irregular has not been much discussed, although the totally irregular concept is very important and it can be useful in computer science and for the problem of finding the shortest path in a computer network. Therefore, the following definition is provided to highlight the issue.

Definition 7. Let G = (A, B) be a CVG. Then

(*i*) *G* is called a HTIVG if each node of *G* is a neighbor to vertices with different neighborhood total degrees.
(*ii*) *G* is said to be a neighborly totally irregular vague graph if each two neighbor nodes of *G* have distinct total degrees.

(iii) *G* is said to be a STIVG if each pair of node in *G* has distinct total degrees.

Example 6. Consider the VG G as in Figure 6.



Figure 6. Neighborly Totally Irregular Vague Graph (NTIVG)(*G*).

By direct calculations:

 $td_G(m) = (0.8, 1.6), \ td_G(z) = (0.4, 1.8), \ td_G(n) = (0.5, 1.4), \ and \ td_G(x) = (0.7, 1.6).$

It is clear from the above calculations that G is a neighborly totally irregular vague graph (NTIVG).

Example 7. *In Example 4, we have:*

td(m) = (0.7, 2.1), td(n) = (0.6, 1.4), td(x) = (0.7, 1.8), and td(w) = (0.7, 1.4). Thus, G is a STIVG.

Theorem 6. Let G = (A, B) be a VG and t_A and f_A be constant functions. Then, G is a SIVG if and only if G is a STIVG.

Proof. Let *G* be a VG; t_A and f_A be constant functions; $m_1, m_2, ..., m_n$ be the nodes of *G*; and $d_t(m_i) = k_i^t$, $d_f(m_i) = k_i^f$, $d_t(m_j) = k_j^t$, and $d_f(m_j) = k_j^f$, where $k_i \neq k_j$. Additionally, $t_A(m_i) = c_2$, $f_A(m_i) = c_1$, and $\forall m_i \in V(G)$ where c_1 and c_2 are constant functions and $c_i \in [0, 1]$. Suppose that *G* is a SIVG: If and only if $d_t(m_i) \neq d_t(m_j)$, $d_f(m_i) \neq d_f(m_j)$, $\forall m_i, m_j \in V(G)$; If and only if $[k_i^t, k_i^f] \neq [k_j^t, k_j^f]$; If and only if $[k_i^t + c_2, k_i^f + c_1] \neq [k_j^t + c_2, k_j^f + c_1]$; If and only if $td(m_i) \neq td(m_j)$, $\forall m_i, m_j \in V(G)$; If and only if $td(m_i) \neq td(m_j)$, $\forall m_i, m_j \in V(G)$; If and only if c is STIVG. \Box

Theorem 7. If G = (A, B) is a STIVG then it is both a HTI and a NTIVG.

Proof. It is clear. \Box

Example 8. Consider STIVG-G, as shown in Figure 7.



Figure 7. Strongly Total Irregular Vague Graph (STIVG) (G).

Since $td_G(m) = (0.4, 0.8)$, $td_G(n) = (0.6, 1.6)$, $td_G(z) = (0.8, 1.7)$, and $td_G(x) = (0.7, 1.3)$, G is both a HTI and a NTIVG.

Theorem 8. Let G = (A, B) be a VG, where G^* is regular; t_A , f_A are constant functions; and $t_B(mn) < t_A(m) \land t_A(n)$ and $f_B(mn) > f_A(m) \lor f_A(n)$, $\forall m, n \in V(G)$. Then, G is a STIVG if and only if \overline{G} is a STIVG.

Proof. Let *G* be a STIVG, $f_A(m) = c$, and $t_A(m) = d$, $\forall m \in V(G)$. Since *G* is STIVG, by Definition 7 we have:

 $td_f(m) \neq td_f(n), \forall m, n \in V(G)$

If and only if $f_A(m) + \sum_{i=1}^n f_B(mx_i) \neq f_A(n) + \sum_{j=1}^n f_B(ny_j)$, for all x_i adjoining the *m* and for all y_i adjoining the *n*.

If and only if $f_A(m) + \sum_{i=1}^n [c - f_B(mx_i)] \neq f_A(n) + \sum_{j=1}^n [c - f_B(ny_j)]$, since G^* is regular.

If and only if $f_A(m) + \sum_{i=1}^n [f_A(m) \lor f_A(x_i) - f_B(mx_i)] \neq f_A(n) + \sum_{j=1}^n [f_A(n) \lor f_A(y_j) - f_B(ny_i)]$, for all x_i adjoining the *m* and for all y_i adjoining the *n*.

If and only if $f_A(m) + \sum_{i=1}^n \overline{f_B}(mx_i) \neq f_A(n) + \sum_{j=1}^n \overline{f_B}(ny_j)$, $\forall m, n \in V(\overline{G})$. In the same way, we can prove that for the true degree. \Box

Dominating sets have practical demand in many subjects. In networking, they are being employed to find efficient routes within ad-hoc mobile networks. Likewise, they have been involved in designing secure systems for electrical grids. For the first time, the concept of domination in VGs was defined in [16]. Now, a new definition of domination set in IGVs is being introduced and its properties are investigated here.

Definition 8. A set $S \subseteq V$ in an IGV G = (A, B) is called a \triangle – dominating set $(\triangle - DS)$, if for each $m \in V - S \exists n \in S$ so that m and n are neighbors in G and $d(n) = \triangle(G) = [\triangle^-, \triangle^+]$. Note that \triangle^- and \triangle^+ are considered for the true membership function and false membership function, respectively.

Example 9. Consider IGV G as follows in Figure 8.



Figure 8. Irregular Vague Graph (IVG)(G).

Here,
$$d(m) = d(n) = [0.3, 1.5] = [\triangle^-, \triangle^+]$$
. *Clearly*, $S = \{m, n\}$ *is a* $(\triangle - DS)$.

Theorem 9. If G is a NIVG and S is a $(\triangle - DS)$ of G, then V - S is not a dominating set.

Proof. Let *S* be a dominating set (DS) of *G*; *u* and *n* are neighbors in *G*; $m \in S$, $n \in V - S$, and $d(m) = \triangle(G)$. Suppose V - S is a DS; then $d(n) = \triangle(G)$, $n \in V - S$ which contradicts the definition of NI. \Box

Theorem 10. *If G is an IGV and S is a* $(\triangle - DS)$ *of G with* |S| > 1*, then G is not SI.*

Proof. Suppose that *S* is a $(\triangle - DS)$ of a VG *G* and |S| > 1. Then, there exists at least two nodes in *S* which dominate all the nodes in V - S and $d(m_i) = \triangle$, $\forall m_i \in S$. That is, the number of nodes whose degrees are equal to \triangle contradicts the definition of SI which exceeds one. \Box

Theorem 11. Let *G* be a SIVG with n + 1 nodes and $S \subseteq V$ is a \triangle -DS. Then, $K_{1,n}$ is an induced subgraph of G^* .

Proof. Let *G* be a SIVG with n + 1 nodes and *S* is a $(\triangle - DS)$. Then, |S| = 1 and |V - S| = n. Let $m \in S$; then *u* dominates all the *n* vertices of V - S; that is, *m* is a neighbor to all the *n* nodes of V - S. Hence, $K_{1,n}$ is the induced subgraph of G^* . \Box

Next, different kinds of edge irregularity in VGs are introduced and demonstrated using two examples.

Definition 9. Suppose G = (A, B) is a CVG on $G^* = (V, E)$. Then G is said to be: (*i*) A NEIVG if each pair of neighbor edges has distinct degrees. (*ii*) A neighborly edge totally irregular vague graph (NETIVG) if each pair of neighbor edges has distinct total degrees.

Example 10. Consider the VG-G as in Figure 9. Accordingly, it is easy to show that each neighbor edges pair has distinct degrees. Hence, G is a NEI-VG.



Figure 9. Neighborly Edge Irregular Vague Graph (NEIVG) (G).

 $d_G(mn) = (0.1, 0.7), d_G(nx) = (0.5, 1.1), d_G(zx) = (0.1, 0.7).$

Accordingly, the degree of an edge in IGVs that help us to find the most effective person in terms of influence in a social group or an organization is introduced.

Definition 10. Let *G* be a *VG* on $G^* = (V, E)$. (*i*) The degree of an edge mn is defined as $d_G(mn) = (d_t(mn), d_f(mn))$ where $d_t(mn) = d_t(m) + d_t(n) - 2t_B(mn)$ and $d_f(mn) = d_f(m) + d_f(n) - 2f_B(mn)$. (*ii*) The total degree of an edge mn is defined as $td_G(mn) = (td_t(mn), td_f(mn))$ where $td_t(mn) = d_t(m) + d_t(n) - t_B(mn) = d_t(mn) + t_B(mn)$ and $td_f(mn) = d_f(m) + d_f(n) - f_B(mn) = d_f(mn) + f_B(mn)$.

Example 11. In this example we consider $G^* = (V, E)$, as shown in Figure 10, in which $V = \{m, n, x, w\}$ and $E = \{mn, nw, wx, xm\}$.



Figure 10. Both Neighborly Edge Irregular (NEI) and Neighborly Edge Totally Irregular Vague Graph (NETIVG.

Considering the Figure 10, in which $d_G(m) = (0.3, 1.3)$, $d_G(n) = (0.3, 1.3)$, $d_G(w) = (0.3, 1.3)$, and $d_G(x) = (0.3, 1.3)$. The calculation of degrees of the edges is as follows:

 $d_G(mn) = (0.4, 1.4), d_G(nw) = (0.2, 1.2), d_G(wx) = (0.4, 1.4), and d_G(xm) = (0.2, 1.2).$ It is readily seen that every pair of neighboring edges has distinct degrees. Thus, G is a NEIVG. Total degrees of the edges are calculated as follows:

 $td_G(mn) = (0.5, 0.2), td_G(nw) = (0.4, 1.9), td_G(wx) = (0.5, 2), td_G(xm) = (0.4, 1.9).$ It is observed that each pair of neighboring edges has different total degrees. Hence, G is a NETIVG. Therefore, G is both a NEI and a NETIVG.

Theorem 12. Let G = (A, B) be a CVG on $G^* = (V, E)$ and (t_B, f_B) be constant functions. Then, G is a NEIVG iff G is a NETIVG.

Proof. Suppose that (t_B, f_B) is a constant function, let $t_B(mn) = c_1$ and $f_B(mn) = c_2$, $\forall mn \in E$ where $c = (c_1, c_2)$. Let mn and nw be a pair of neighboring edges in E, then we have $d_G(mn) \neq d_G(nw)$ If and only if $d_G(mn) + c \neq d_G(nw) + c$;

If and only if $(d_t(mn), d_f(mn)) + (c_1, c_2) \neq (d_t(nw), d_f(nw)) + (c_1, c_2);$ If and only if $(d_t(mn) + c_1, d_f(mn) + c_2) \neq (d_t(nw) + c_1, d_f(nw) + c_2);$ If and only if $(d_t(mn) + t_B(mn), d_f(mn) + f_B(mn)) \neq (d_t(nw) + t_B(nw), d_f(nw) + f_B(nw));$ If and only if $(td_t(mn), td_f(mn)) \neq (td_t(nw), td_f(nw));$

If and only if $td_G(mn) \neq td_G(nw)$. Hence, every pair of neighboring edges has distinct degrees if and only if they have distinct total degrees. Thus, *G* is a NEIVG iff *G* is a NETIVG. \Box

Theorem 13. Let G = (A, B) be a CVG on $G^* = (V, E)$ and (t_B, f_B) be a constant function. If G is a SIVG, then G is a NEIVG.

Proof. Suppose that G = (A, B) is a CVG on $G^* = (V, E)$. Suppose that (t_B, f_B) is a constant function; let $t_B(mn) = c_1$ and $f_B(mn) = c_2$, $\forall mn \in E$ where $C = (c_1, c_2)$ is constant. Let mn and nw be any two neighboring edges in G. Assume that G is a SIVG. Then, every pair of nodes in G has distinct degrees. Hence, $d_G(m) \neq d_G(w)$.

$$\begin{aligned} \left(d_t(m), d_f(m)\right) &\neq \left(d_t(n), d_f(n)\right) \neq \left(d_t(w), d_f(w)\right) \\ \Rightarrow & \left(d_t(m), d_f(m)\right) + \left(d_t(n), d_f(n)\right) - 2(c_1, c_2) \neq \left(d_t(n), d_f(n)\right) + \left(d_t(w), d_f(w)\right) - 2(c_1, c_2) \\ \Rightarrow & \left(d_t(m) + d_t(n) - 2c_1, d_f(m) + d_f(n) - 2c_2\right) \neq \left(d_t(n) + d_t(w) - 2c_1, d_f(n) + d_f(w) - 2c_2\right) \\ \Rightarrow & \left(d_t(m) + d_t(n) - 2t_B(mn), d_f(m) + d_f(n) - 2f_B(mn)\right) \neq \left(d_t(n) + d_t(w) - 2t_B(nw), d_f(n) + d_f(w) - 2f_B(nw)\right) \\ \Rightarrow & \left(d_t(mn), d_f(mn)\right) \neq \left(d_t(nw), d_f(nw)\right) \\ \Rightarrow & d_G(mn) \neq d_G(nw). \end{aligned}$$

Therefore, every pair of neighbor edges has distinct degrees. Hence, *G* is a NEIVG. \Box

4. Laplacian Energy of VGs

This section deals with the investigation of the concepts of energy, spectrum, OM, DM, LM, and LM in VGs and descriptions of their properties in detail.

Definition 11. The energy of VG G is defined as

$$E(G) = (E(t_B(v_l v_m)), E(f_B(v_l v_m))) = (\sum_{l=1}^n |\lambda_l|, \sum_{l=1}^n |\mu_l|).$$

Note that λ_l and μ_l $(l = 1, 2, \dots, n)$ are the eigenvalues of Laplacian matrix $(LM) = (L(t_B(v_lv_m)), L(f_B(v_lv_m))).$

Definition 12. Let $G = (A, \overrightarrow{B})$ be a vague digraph (VDG) on *n* vertices. The energy of *G* is defined as:

$$E(G) = \left(E(t_{\overrightarrow{B}(v_l v_m)}), E(f_{\overrightarrow{B}(v_l v_m)})\right) = \left(\sum_{l=1}^n |Re(\lambda_l)|, \sum_{l=1}^n |Re(\mu_l)|\right).$$

We now define the LE of a VG and discuss its properties.

Definition 13. Let G = (A, B) be a VG with n nodes and m edges. The DM, $D_G = [d_{lm}]$ of G is a square matrix of order n, where

$$[d_{ml}] = \begin{cases} d_G(v_1) & if \ l = m \\ 0 & otherwise \end{cases}$$

Definition 14. Let G = (A, B) be a VG. The adjacency matrix (AM) of G is an $n \times n$ matrix defined as $AM = [a_{ij}], where \ a_{ij} = (t_B(v_i v_j), f_B(v_i v_j)).$

Definition 15. The LM of a VG is a matrix of the form L(G) = D(G) - A(G), where D(G) is the DM and A(G) is the AM of a VG G = (A, B).

Remark 2. Here, we can write LM of VG in two different matrices: one containing the entries as true membership values and the other containing false membership values, i.e., $L(G) = (L(t_B), L(f_B))$.

Definition 16. The spectrum of LM of VG G is described as (Y_L, Z_L) , where Y_L and Z_L are the sets of eigenvalues of $L(t_B(v_lv_m))$ and $L(f_B(v_lv_m))$, respectively. Note that the MATLAB software has been used to facilitate the calculation of eigenvalues for the LM.

Example 12. Consider VG G = (A, B), as shown in Table 2 and Figure 11.

 $v_6(0.3,0.4)$

Nodes (A) v_1 v_2 v_3 v_4 v_5 v_6 t_A 0.1 0.2 0.3 0.5 0.5 0.3 0.3 0.3 0.4 0.6 0.5 0.4 ţΑ Edges(B) $v_1 v_2$ $v_1 v_3$ $v_1 v_6$ $v_2 v_3$ $v_{3}v_{4}$ $v_4 v_5$ $v_5 v_6$ t_B 0.1 0.1 0.1 0.2 0.3 0.4 0.3 0.40.5 0.6 0.5 0.7 0.7 0.5 fв v2 (0.2,0.3) v3 (0.3,0.4) (0.2, 0.5)(0,3,0,1) (0.1.0.2) (0.1,0.5) v4(0.5,0.6) v1(0.1,0.3) (0.4.0.T (0,1,0,6) (0.3, 0.5)

Table 2. Weights of nodes and edges.

v5(0.4,0.5)

The AM, DM, and LM of VG G shown in Figure 11 are as follows:

$$\begin{split} A(G) = \begin{pmatrix} (0,0) & (0.1,0.4) & (0.1,0.5) & (0,0) & (0,0) & (0.1,0.6) \\ (0.1,0.4) & (0,0) & (0.2,0.5) & (0,0) & (0,0) & (0,0) \\ (0.1,0.5) & (0.2,0.5) & (0,0) & (0.3,0.7) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.3,0.7) & (0,0) & (0.4,0.7) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0.4,0.7) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0.3,0.5) \\ (0,0) & (0.3,0.9) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (-0.1,-0.4) & (0.3,0.9) & (-0.2,0.5) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (-0.3,-0.7) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (-0.4,-0.7) & (0.7,1.2) & (-0.3,-0.5) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (-0.3,-0.5) & (0.4,1.1) \end{pmatrix} \end{split}$$

The Laplacian spectrum (LS) on VG G = (A, B), is given below.

 $LS(t_B(v_lv_m)) = \{0, 0.1824, 0.3222, 0.4772, 0.7869, 1.2313\}$

 $LS(f_B(v_l v_m)) = \{2.4912, 2.2354 + 0i, 0.1457, 0.7902, 1.0688 + 0.1913i, 1.0688 - 0.1913i\}$

 $LS(G) = \{(0, 2.4912), (0.1824, 2.2354), (0.3222, 0.1457), (0.4772, 0.7902), (0.7869, 1.0688 + 0.1913i), (1.2313, 1.0688 - 0.1913i)\}.$

Theorem 14. Let G = (A, B) be a VG and let $L(G) = (L(t_B(v_lv_m)), L(f_B(v_lv_m)))$ be a LM of G. If $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ are the eigenvalues of $L(t_B(v_lv_m))$ and $L(f_B(v_lv_m))$, respectively, then

$$1.\sum_{l=1}^{n} \lambda_{l} = 2\sum_{1 \le l < m \le n} t_{B}(v_{l}v_{m}), \sum_{l=1}^{n} \mu_{l} = 2\sum_{1 \le l < m \le n} f_{B}(v_{l}v_{m})$$

$$2.\sum_{l=1}^{n} \lambda_{l}^{2} = 2\sum_{1 \le l < m \le n} (t_{B}(v_{l}v_{m}))^{2} + \sum_{l=1}^{n} d_{t_{B}(v_{l}v_{m})}^{2}(v_{l}), \sum_{l=1}^{n} \mu_{l}^{2} = 2\sum_{1 \le l < m \le n} (f_{B}(v_{l}v_{m}))^{2} + \sum_{l=1}^{n} d_{f_{B}(v_{l}v_{m})}^{2}(v_{l}), \sum_{l=1}^{n} \mu_{l}^{2} = 2\sum_{1 \le l < m \le n} (f_{B}(v_{l}v_{m}))^{2} + \sum_{l=1}^{n} d_{f_{B}(v_{l}v_{m})}^{2}(v_{l}), \sum_{l=1}^{n} \mu_{l}^{2} = 2\sum_{1 \le l < m \le n} (f_{B}(v_{l}v_{m}))^{2} + \sum_{l=1}^{n} d_{f_{B}(v_{l}v_{m})}^{2}(v_{l})$$

Proof. Being a symmetric matrix L(G) has non-negative Laplacian eigenvalues such that

$$\sum_{l=1}^{n} \lambda_{l} = tr(L(G)) = \sum_{l=1}^{n} d_{t_{B}(v_{l}v_{m})}(v_{l}) = 2 \sum_{1 \le l < m \le n} d_{t_{B}(v_{l}v_{m})}(v_{l}).$$

Similarly, we can show that

$$\sum_{l=1}^n \mu_l = 2 \sum_{1 \le l < m \le n} f_B(v_l v_m).$$

The general LM, can be written in this way

$$L(t_B(v_Iv_m)) = \begin{pmatrix} d_{t_B(v_Iv_m)}(v_1) & -t_B(v_1v_2) & \cdots & -t_B(v_1v_n) \\ -t_B(v_2v_1) & d_{t_B(v_Iv_m)}(v_2) & \cdots & -t_B(v_2v_n) \\ \vdots & & & \\ -t_B(v_nv_1) & -t_B(v_nv_2) & \cdots & d_{t_B(v_Iv_m)}(v_n) \end{pmatrix}$$

Hence, the trace property of a matrix shows that $tr((L(t_B(v_lv_m)))^2) = \sum_{l=1}^n \lambda_l^2$, where

$$tr((L(t_B(v_lv_m)))^2) = d_{t_B(v_lv_m)}^2(v_1) + (t_B(v_1v_2))^2 + \dots + (t_B(v_1v_n))^2 + (t_B(v_2v_1))^2 + d_{t_B(v_lv_m)}^2(v_2) + \dots + (t_B(v_2v_n))^2 + \dots + (t_B(v_nv_1))^2 + (t_B(v_nv_2))^2 + \dots + d_{t_B(v_lv_m)}^2(v_n) = 2\sum_{1 \le l < m \le n} (t_B(v_lv_m))^2 + \sum_{l=1}^n d_{t_B(v_lv_m)}^2(v_l).$$

Therefore,

$$\sum_{l=1}^{n} \lambda_n^2 = 2 \sum_{1 \le l < m \le n} (t_B(v_l v_m))^2 + \sum_{l=1}^{n} d_{t_B(v_l v_m)}^2(v_l).$$

In the same way, it is simple to show that

$$\sum_{l=1}^{n} \mu_n^2 = 2 \sum_{1 \le l < m \le n} (f_B(v_l v_m))^2 + \sum_{l=1}^{n} d_{f_B(v_l v_m)}^2(v_l).$$

Definition 17. *The LM of a VG* G = (A, B) *is defined as*

$$LE(G) = (LE(t_B(v_l v_m)), LE(f_B(v_l v_m))) = (\sum_{l=1}^n |\pi_l|, \sum_{l=1}^n |\nu_l|),$$

where,

$$\pi_{l} = \lambda_{l} - \frac{2\sum_{1 \le l < m \le n} t_{B}(v_{l}v_{m})}{n}, \quad \nu_{l} = \mu_{l} - \frac{2\sum_{1 \le l < m \le n} f_{B}(v_{l}v_{m})}{n}.$$

Example 13. The LM of a VG G = (A, B), shown in Figure 11 is:

$$LE(G) = (LE(t_B(v_lv_m)), LE(f_B(v_lv_m))) = (2.2001, 6.0421).$$

Theorem 15. Assume that G = (A, B) is a VG on n nodes and $L(G) = (L(t_B(v_lv_m)), L(f_B(v_lv_m)))$ is the LM of G. Then,

$$1.LE(t_B(v_lv_m)) \le \sqrt{2n \sum_{1 \le l < m \le n} (t_B(v_lv_m))^2 + n \sum_{l=1}^n \left(d_{t_B(v_lv_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} t_B(v_lv_m)}{n} \right)^2} \\ 2.LE(f_B(v_lv_m)) \le \sqrt{2n \sum_{1 \le l < m \le n} (f_B(v_lv_m))^2 + n \sum_{l=1}^n \left(d_{f_B(v_lv_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} t_B(v_lv_m)}{n} \right)^2}$$

Proof. Using the Cauchy–Schwarz inequality on the *n* numbers $1, 1, \dots, 1$ and $|\pi_1|, |\pi_2|, \dots, |\pi_n|$, we find:

$$\sum_{l=1}^{n} |\pi_l| \le \sqrt{n} \sqrt{\sum_{l=1}^{n} |\pi_l|^2}$$
$$LE(t_B(v_l v_m)) \le \sqrt{n} \sqrt{2k_{t_B}} = \sqrt{2nk_{t_B}}.$$

As

$$k_{t_B} = \sum_{1 \le l < m \le n} (t_B(v_l v_m))^2 + \frac{1}{2} \sum_{l=1}^n \left(d_{t_B(v_l v_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} t_B(v_l v_m)}{n} \right)^2,$$

Thus,

$$LE(t_B(v_lv_m)) \le \sqrt{2n \sum_{1 \le l < m \le n} (t_B(v_lv_m))^2 + n \sum_{l=1}^n \left(d_{t_B(v_lv_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} t_B(v_lv_m)}{n} \right)^2}.$$

Similarly, we can show that

$$LE(f_B(v_lv_m)) \le \sqrt{2n \sum_{1 \le l < m \le n} (f_B(v_lv_m))^2 + n \sum_{l=1}^n \left(d_{f_B(v_lv_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} f_B(v_lv_m)}{n} \right)^2}.$$

Theorem 16. Suppose that G = (A, B) is a VG on n nodes and $L(G) = (L(t_B(v_lv_m)), L(f_B(v_lv_m)))$ is the LM of G. Then,

$$1.LE(t_B(v_lv_m)) \ge 2\sqrt{\sum_{1 \le l < m \le n} (t_B(v_lv_m))^2 + \frac{1}{2}\sum_{l=1}^n \left(d_{t_B(v_lv_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} t_B(v_lv_m)}{n}\right)^2}{2.LE(f_B(v_lv_m)) \ge 2\sqrt{\sum_{1 \le l < m \le n} (f_B(v_lv_m))^2 + \frac{1}{2}\sum_{l=1}^n \left(d_{f_B(v_lv_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} t_B(v_lv_m)}{n}\right)^2}{2}}$$

Proof.

$$\left(\sum_{l=1}^{n} |\pi_{l}|\right)^{2} = \sum_{l=1}^{n} |\pi_{l}|^{2} + 2\sum_{1 \le l < m \le n} |\pi_{l}\pi_{m}| \ge 4k_{t_{B}}$$
$$LE(t_{B}(v_{l}v_{m})) \ge 2\sqrt{k_{t_{B}}}.$$

As

$$k_{t_B} = \sum_{1 \le l < m \le n} (f_B(v_l v_m))^2 + \frac{1}{2} \sum_{l=1}^n \left(d_{f_B(v_l v_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} f_B(v_l v_m)}{n} \right)^2.$$

Thus,

$$LE(t_B(v_lv_m)) \ge 2\sqrt{\sum_{1 \le l < m \le n} (t_B(v_lv_m))^2 + \frac{1}{2}\sum_{l=1}^n \left(d_{t_B(v_lv_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} t_B(v_lv_m)}{n}\right)^2}.$$

Analogously, we can show that

$$LE(f_B(v_l v_m)) \ge 2\sqrt{\sum_{1 \le l < m \le n} (f_B(v_l v_m))^2 + \frac{1}{2} \sum_{l=1}^n \left(d_{f_B(v_l v_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} f_B(v_l v_m)}{n} \right)^2}$$

Theorem 17. Let G = (A, B) be a VG on n nodes and $L(G) = (L(t_B(v_lv_m)), L(f_B(v_lv_m)))$ be the LM of *G*. Then

$$LE(t_B(v_lv_m)) \le |\pi_1| + \sqrt{(n-1)\left(2\sum_{1\le l< m\le n} (t_B(v_lv_m))^2 + \sum_{l=1}^n \left(d_{t_B(v_lv_m)}(v_l) - \frac{2\sum_{1\le l< m\le n} t_B(v_lv_m)}{n}\right)^2 - \pi_1^2\right)},$$

$$LE(f_B(v_lv_m)) \le |v_1| + \sqrt{(n-1)\left(2\sum_{1\le l< m\le n} (f_B(v_lv_m))^2 + \sum_{l=1}^n \left(d_{f_B(v_lv_m)}(v_l) - \frac{2\sum_{1\le l< m\le n} f_B(v_lv_m)}{n}\right)^2 - v_1^2\right)}.$$

Proof. Using Cauchy-Schwarz inequality, we have

$$\begin{split} &\sum_{l=1}^{n} |\pi_{l}| \leq \sqrt{n \sum_{l=1}^{n} |\pi_{l}|^{2}} \\ &\sum_{l=2}^{n} |\pi_{l}| \leq \sqrt{n \sum_{l=2}^{n} |\pi_{l}|^{2}} \\ &LE(t_{B}(v_{l}v_{m})) - |\pi_{1}| \leq \sqrt{(n-1)(2k_{t_{B}} - \pi_{1}^{2})} \\ &LE(t_{B}(v_{l}v_{m})) \leq |\pi_{1}| + \sqrt{(n-1)(2k_{t_{B}} - \pi_{1}^{2})} \end{split}$$

As

$$k_{t_B} = \sum_{1 \le l < m \le n} (t_B(v_l v_m))^2 + \frac{1}{2} \sum_{l=1}^n \left(d_{t_B(v_l v_m)}(v_l) - \frac{2\sum_{1 \le l < m \le n} t_B(v_l v_m)}{n} \right)^2.$$

Thus,

$$LE(t_B(v_lv_m)) \le |\pi_1| + \sqrt{(n-1)\left(2\sum_{1\le l< m\le n} (t_B(v_lv_m))^2 + \sum_{l=1}^n \left(d_{t_B(v_lv_m)}(v_l) - \frac{2\sum_{1\le l< m\le n} t_B(v_lv_m)}{n}\right)^2 - \pi_1^2\right)}, \quad (1)$$

Analogously, it is easy to show that

$$LE(f_B(v_lv_m)) \le |v_1| + \sqrt{(n-1)\left(2\sum_{1\le l< m\le n} (f_B(v_lv_m))^2 + \sum_{l=1}^n \left(d_{f_B(v_lv_m)}(v_l) - \frac{2\sum_{1\le l< m\le n} f_B(v_lv_m)}{n}\right)^2 - v_1^2\right)}$$

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Theorem 18. If the VG G = (A, B) is regular, then

$$1.LE(t_B(v_lv_m)) \le |\pi_1| + \sqrt{(n-1)\left(2\sum_{1\le l < m \le n} (t_B(v_lv_m))^2 - \pi_1^2\right)}$$
$$2.LE(f_B(v_lv_m)) \le |\nu_1| + \sqrt{(n-1)\left(2\sum_{1\le l < m \le n} (f_B(v_lv_m))^2 - \nu_1^2\right)}.$$

Proof. Let *G* be a VG; then

$$d_{t_B(v_l v_m)}(v_l) = \frac{2\sum_{1 \le l < m \le n} t_B(v_l v_m)}{n}$$
(2)

By putting (2) in (1), we get

$$LE(t_B(v_lv_m)) \le |\pi_1| + \sqrt{(n-1)\left(2\sum_{1 \le l < m \le n} (t_B(v_lv_m))^2 - \pi_1^2\right)}$$

Similarly we have

$$LE(f_B(v_l v_m)) \le |v_1| + \sqrt{(n-1)\left(2\sum_{1 \le l < m \le n} (f_B(v_l v_m))^2 - v_1^2\right)}$$

We now discuss the concept of LM of VDGs.

Definition 18. Let $G = (A, \overrightarrow{B})$ be a VDG on nodes. The OM $D^{out}(G) = (D^{out}(t_{\overrightarrow{B}(v_lv_m)}), D^{out}(f_{\overrightarrow{B}(v_lv_m)}))$ of G is a $n \times n$ diagonal matrix defined as

$$[d_{lm}] = \begin{cases} d_G^{xt}(v_l), & if \ l = m \\ 0, & otherwise \end{cases}$$

Definition 19. The LM of a VDG $G = (A, \overrightarrow{B})$ is defined as $L(G) = (L(t_{\overrightarrow{B}}(v_l v_m)), L(f_{\overrightarrow{B}}(v_l v_m))) = D^{out}(G) - A(G)$, where $D^{out}(G)$ is OM and A(D) is an AM of a VDG $G = (A, \overrightarrow{B})$.

Definition 20. The spectrum of a LM of a VDG L(G) is described as $(\widehat{Y}_l, \widehat{Z}_l)$, so that \widehat{Y}_l and \widehat{Z}_l are the sets of Laplacian eigenvalues of $L(t_{\overrightarrow{B}}(v_lv_m))$ and $L(f_{\overrightarrow{B}}(v_lv_m))$, respectively.

Theorem 19. Let $G = (A, \overrightarrow{B})$ be a VDG and let L(G) be the LM of G. If $\widehat{\lambda}_1 \ge \widehat{\lambda}_2 \ge \cdots \ge \widehat{\lambda}_n$ and $\widehat{\mu}_1 \ge \widehat{\mu}_2 \ge \cdots \ge \widehat{\mu}_n$ are the eigenvalues of $L(t_{\overrightarrow{B}(v_1v_m)})$ and $L(f_{\overrightarrow{B}(v_1v_m)})$, respectively, then

$$\sum_{l=1}^{n} Re(\widehat{\lambda}_{l}) = \sum_{1 \le l < m \le n} t_{\overrightarrow{B}(v_{l}v_{m})}, \quad \sum_{l=1}^{n} Re(\widehat{\mu}_{l}) = \sum_{1 \le l < m \le n} f_{\overrightarrow{B}(v_{l}v_{m})}$$

Definition 21. The (LE of a VDG $G = (A, \overrightarrow{B})$ is defined as

$$LE(G) = \left(LE(t_{\overrightarrow{B}(v_l v_m)}), LE(f_{\overrightarrow{B}(v_l v_m)})\right) = \left(\sum_{l=1}^n |\theta_l|, \sum_{l=1}^n |\delta_l|\right)$$

where $\theta_l = \operatorname{Re}(\widehat{\lambda}_l) - \frac{\sum_{1 \leq l < m \leq n} t_{\overrightarrow{B}(v_l v_m)}}{n}$ and $\delta_l = \operatorname{Re}(\widehat{\mu}_l) - \frac{\sum_{1 \leq l < m \leq n} f_{\overrightarrow{B}(v_l v_m)}}{n}$

Example 14. Consider a VDG $G = (A, \overrightarrow{B})$ on $V = \{v_1, v_2, v_3, v_4, v_5\}$ as shown in Figure 12.



Figure 12. VDG (*G*).

The AM, OM, and LM of VDG G shown in Figure 12 are as follows:

$$A(G) = \begin{pmatrix} (0,0) & (0.1,0.5) & (0,0) & (0,0) & (0.2,0.6) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0.1,0.5) \\ (0,0) & (0.1,0.4) & (0,0) & (0,0) & (0,0) \\ (0.2,0.6) & (0,0) & (0.2,0.7) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.2,0.7) & (0.1,0.5) & (0,0) \end{pmatrix}$$

$$D^{out}(G) = \begin{pmatrix} (0.3,1.1) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.1,0.4) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.1,0.4) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0.4,1.3) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0.3,1.2) \end{pmatrix}$$

$$L^{+}(G) = \begin{pmatrix} (0.3,1.1) & (-0.1,-0.5) & (0,0) & (0,0) & (-0.2,-0.6) \\ (0,0) & (0.1,0.5) & (0,0) & (0,0) & (0.3,1.2) \\ (0,0) & (-0.1,-0.4) & (0.1,0.4) & (0,0) & (0,0) \\ (-0.2,-0.6) & (0,0) & (-0.2,-0.7) & (0.4,1.3) & (0,0) \\ (0,0) & (0,0) & (-0.2,-0.7) & (-0.1,-0.5) & (0.3,1.2) \end{pmatrix}$$

$$\begin{aligned} Spec(LE(t_{\overrightarrow{B}(v_{l}v_{m})})) &= \{0.4015 + 0.1234i, 0.4015 - 0.1234i, 0, 0.1985 + 0.733i, 0.1985 - 0.733i\} \\ Spec(LE(f_{\overrightarrow{B}(v_{l}v_{m})})) &= \{0, 1.3297 + 0.4282i, 1.3297 - 0.4282i, 0.9203 + 0.2380i, 0.9203 - 0.2380i\} \\ Spec(L(G)) &= \{(0.4015 + 0.1234i, 0), (0.4015 - 0.1234i, 0, 1.3297 + 0.4282i), (0, 1.3297 - 0.4282i), (0.1985 + 0.733i, 0.9203 + 0.2380i), (0.1985 - 0.733i, 0.9203 - 0.2380i)\} \end{aligned}$$

5. Numerical Examples

Here, the theory of energy of a VG is being considered by an actual example. The http://www.pantechsolutions.net website is considered which is referred to as a VG model by following the customer's navigation. The VG in this site is weighed up at different time intervals. For every period, the energy of a VG is measured. Moreover, the energy is given in a bar chart format. Four links, (1) microcontroller-boards, (2) /log-inhtml, (3) / and (4) /projectkits for our calculations are considered. This data being considered are from 7 July 2019 to 8 August 2019 as follows (see Figure 13):



For this graph we have

Spec $(t_B(v_iv_j)) = \{0.2818, -0.2355, 0.0221, -0.0683\},\$ Spec $(f_B(v_iv_j)) = \{-1.0041, -0.6776, 0.0228, 1.6528\}.\$ $E(t_B(v_iv_j)) = 0.2818 + 0.2355 + 0.0221 + 0.0683 = 0.6077,\$ $E(f_B(v_iv_j)) = 1.0041 + 0.6776 + 0.0228 + 1.6528 = 3.3633.$

Now, we consider the period 9 August 2019 to 8 September 2019 (see Figure 14).



Figure 14. VG (G₂).

Here, we have

Spec $(t_B(v_iv_j)) = \{-0.2425, 0.3758, -0.1410, 0.0078\},\$ Spec $(f_B(v_iv_j)) = \{-1.1496, -0.6710, 0.0184, 1.8022\}.\$ $E(t_B(v_iv_j)) = 0.2425 + 0.3758 + 0.1410 + 0.0078 = 0.7671,\$ $E(f_B(v_iv_j)) = 1.1496 + 0.6710 + 0.0184 + 1.8022 = 3.4612.\$

For the interval between 9 September 2019 and 8 October 2019 we have (see Figure 15):



Spec $(t_B(v_iv_j)) = \{-0.2541, -0.1618, 0.0618, 0.3541\},\$ Spec $(f_B(v_iv_j)) = \{-1.1656, -0.8618, 0.0121, 1.9653\}.\$ $E(t_B(v_iv_j)) = 0.2541 + 0.1618 + 0.0618 + 0.3541 = 0.8318,\$ $E(f_B(v_iv_j)) = 1.1656 + 0.8618 + 0.0121 + 1.9653 = 4.0048.$

At the end, for the period of 9 October 2019 to 8 November 2019 we have (see Figure 16):



Figure 16. VG (*G*₄).

Spec $(t_B(v_iv_j)) = \{-0.2919, -0.1300, 0.2310, 0.4218\},\$ Spec $(f_B(v_iv_j)) = \{-1.2058, -0.0007, -0.6024, 1.8102\}.\$ $E(t_B(v_iv_j)) = 0.2919 + 0.1300 + 0.2310 + 0.4218 = 1.0747,\$ $E(f_B(v_iv_j)) = 1.2058 + 0.0007 + 0.6024 + 1.8102 = 3.6218.$

Consider the following bar diagrams for the above four periods as follows (see Figures 17 and 18).



Figure 17. Energy of true membership values.



Figure 18. Energy of false membership values.

As can be seen in the bar chart above, the true membership energy from October to November is soaring compared to the other periods. Correspondingly, the false membership energy from September to October is high. Since the energy amount for the true membership revealed itself to be higher in the period from October to November, it is concluded that the sales for this store were more than the sales in previous periods. In a similar way, since the amount of energy for false membership was higher in September–October than the other periods, it is easy to conclude that the store experienced the lowest sales in 2019 during this period.

6. Application VG to Find the Most Dominant Person in a Hospital

For this section, a VG model to detect the most important person in a hospital is being considered, being referred to as the influence graph (abbreviated as IG). In an IG, each node represents an employee and the edges show each employee's influence on another employee in a hospital. These kinds of graphs are used in the social structure communication and distributed computing modeling. Table 3 shows a module of a hospital with employees and their roles. The employees' set of $E = \{H, D, S, A, F, L, PH\}$ is considered for this hospital. As mentioned, an IG can be improved, but these graphs cannot exactly characterize the employees' power in an organization and the employees' degrees of influence on each other. As the powers and influence do not have clear-cut boundaries, using a fuzzy set for their representation would be an appropriate step. The fuzzy digraph illustrates the employees' influences on each other. However, there exists a small chance of the non-null hesitation share at each moment of influence measurement. The VS idea is being applied which properly points out the influence and contracts among the employees. The employees' vague set is given as follows. The influence in the VDG is represented by an edge. The result of VDG is represented in Figure 19, and the related AM is shown in Table 4.

Being interpreted as percentage, the employee and his/her power in terms of the membership and non-membership degrees are provided by the VDG nodes in Figure 19; for example, *D* retains 90% of power in the organization (Table 5). Correspondingly, a VDG edge signifies the impact of one person on another person, that is, the edges end node. The positive and negative influence percentages can be attributed to the membership and non-membership degrees; for example, 40% of the time, *D* acts based on *A*'s attitude, but 30% of the time, *D* does not conform to *A*'s ideas. In Figure 19, it can be observed that *A* has impacts on both *D* and *H*. As the membership degree in both cases is 0.4, *A* can equally have impacts on both of them; that is, 40%. However, considering *H* as the hesitation degree is 0.4; that is to say, ($\pi = 1 - 0.4 - 0.2$), and for *D*, it is 0.3—that is, ($\pi = 1 - 0.4 - 0.3$), indicating that the hesitation for *D* is more than that of *H*. However, it is crystal clear that *A* is the most dominant employee in the organization. Additionally, take in the fact that no other employees have impacts on both *D* and *H*, enjoying 90% of power within the organization.

Name Designation			
Н	Head of the hospital		
D	Doctor		
S	Supervisor		
А	Administrative Staff		
F	Financial Manager		
L	Laboratory		
PH	Pharmacy Manager		

Table 3. Employees' names in a hospital and their designations.

Table 4. AM corresponding to Figure 19.

	D	Н	S	Α	F	L	РН
D	(0.0,1.0)	(0.7.0.2)	(0.0,1.0)	(0.4,0.3)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)
Н	(0.0,1.0)	(0.0,1.0)	(0.5,0.3)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.1,0.4)
S	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.2,0.3)
А	(0.4,0.3)	(0.4,0.2)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.2,0.3)
F	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.3,0.4)	(0.2,0.4)
L	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.2,0.4)
PH	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)	(0.0,1.0)

Table 5. Power based on membership and non-membership degrees.

	Н	D	S	Α	F	L	PH
t_A	0.9	0.9	0.6	0.5	0.4	0.4	0.3
f_A	0.0	0.0	0.2	0.2	0.3	0.3	0.2



Figure 19. IVDG.

7. Conclusions

Graph theory has numerous applications in solving several domains problems, containing networking, communication, data mining, clustering, image capturing, image segmentation, planning, and scheduling. However, in some situations, certain aspects related to the graph-theoretical system may be uncertain. Applying the fuzzy-graphical methods in meeting the ambiguity and vague demands is very natural. Fuzzy-graph theory has wide-ranging applications in modeling various real-time systems in which the inherent information levels in the systems vary with different precision levels. A vague set model is considered appropriate for modeling problems with uncertainty, indeterminacy, and inconsistent information in which human knowledge is required and human evaluation is needed. Vague models provide more precision, flexibility, and compatibility to the system as compared to the classical, fuzzy, and intuitionistic fuzzy models. A VG can comprehensively describe all kinds of networks' uncertainties. The main contribution of this manuscript was to introduce the idea of irregularity in vague graph theory. In this paper, we represented the notions of the Laplacian energy (LE), adjacency matrix (AM), degree matrix (DM), and Laplacian matrix (LM) of VGs. Some different types of vague graphs, such as the highly irregular, neighborly irregular, strongly irregular, neighborly edge irregular, neighborly edge totally irregular, and highly totally irregular vague graphs were introduced here. The concept of energy of a VG through a real time example was given. Finally, an application of the proposed concepts was presented to find the most effective person in a hospital. Throughout this article, all those terms which were not previously defined well were clearly defined. We are planing to extend our research work to: (1) vague graph structures; (2) simplified vague graph structures; and (3) cubic vague graph structures.

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