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Some Identities Involving Certain Hardy Sums and General Kloosterman Sums

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Abstract: Using the properties of Gauss sums, the orthogonality relation of character sum and the mean value of Dirichlet L-function, we obtain some exact computational formulas for the hybrid mean value involving general Kloosterman sums $K(r,l,\lambda;p)$ and certain Hardy sums $S_1(h,q)$ $\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,n,\lambda;p)K(s,t,\lambda;p)S_1(2m\bar{s},p)$, $\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2 S_1(2m\bar{s},p)$. Our results not only cover the previous results, but also contain something quite new. Actually the previous authors just consider the case of the principal character λ modulo p, while we consider all the cases.

Keywords: certain Hardy sums; general Kloosterman sums; hybrid mean value; exact computational formulas

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1. Introduction

Let k be a positive integer and h be arbitrary integer with $(h,k)=1$, the classical Dedekind sums $S(h,k)$ are defined as

$$S(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

$S(h,k)$ also can be written as

$$S(h,k)=-\frac{1}{4k}\sum_{a=1}^{k-1}\cot\frac{\pi a}{k}\cot\frac{\pi ah}{k},$$

which belong to the family of "cotangent sums".

These sums have a wide range of applications and in some cases relations to some major open problems, such as Riemann Hypothesis. Refs. [1–4] introduced the cotangent sums

$$C\left(\frac{h}{k}\right)=\sum_{a=1}^{k-1}\frac{a}{k}\cot\frac{\pi ah}{k},$$

and the Vasyunin sums

$$V\left(\frac{h}{k}\right)=\sum_{a=1}^{k-1}\left\{\frac{ah}{k}\right\}\cot\frac{\pi ah}{k}=C\left(\frac{\bar{h}}{k}\right),$$

where $\{u\} = u - \lfloor u \rfloor, h\bar{h} \equiv 1 \bmod k$. Actually the Vasyunin sums are associated to the study of the Riemann Hypothesis through the following equation (see [1–4]):

$$\frac{1}{2\pi\sqrt{hk}} \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2}+it\right) \right|^2 \left(\frac{h}{k}\right)^{it} \frac{dt}{\frac{1}{4}+t^2}$$
$$= \frac{\log 2\pi - \gamma}{2} \left(\frac{1}{h}+\frac{1}{k}\right) + \frac{k-h}{2hk} \log\frac{h}{k} - \frac{\pi}{2hk}\left(V\left(\frac{h}{k}\right)+V\left(\frac{k}{h}\right)\right).$$

So the cotangent sums arise in connection with the Nyman-Beurling approach to the Riemann Hypothesis.

Dedekind sums also play an important role in the transformation theory of the Dedekind η function. Dedekind sums have many interesting properties. For example, L. Carlitz [5] obtained the reciprocity theorem of $S(h,k)$ as

$$S(h,k) + S(k,h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right).$$

W. P. Zhang [6] established the relationship between Dedekind sums and Dirichlet L-function:

$$S(a,k) = \frac{1}{\pi^2 k} \sum_{d|k} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1,\chi)|^2, \tag{1}$$

where $k > 2$ is an integer, a is an integer with $(a,k) = 1$, $\varphi(d)$ denotes the Euler function and $L(1,\chi)$ denotes the Dirichlet L-function corresponding to character χ modulo d.

Some scholars also studied Dedekind type sums and obtained interesting results. For example, by considering Dedekind type DC(Daehee-Changhee) sums

$$T_p(h,k) = 2\sum_{u=1}^{k-1} (-1)^{u-1} \frac{u}{k} \overline{E}_p\left(\frac{hu}{k}\right), (h \in Z_+),$$

where $\overline{E}_p(x)$ are the p-th Euler functions, T. Kim [7] proved reciprocity law:

$$k^p T_p(h,k) + h^p T_p(k,h)$$
$$= 2 \sum_{\substack{u=0 \\ u-[\frac{hu}{k}] \equiv 1 \bmod 2}}^{k-1} \left(kh\left(E+\frac{u}{k}\right) + k\left(E+h-\left[\frac{hu}{k}\right]\right)\right)^p + (hE+kE)^p + (p+2)E_p,$$

where $(h,k) = 1$, $[x]$ is the largest integer $\leq x$, E_n are the n-th Euler numbers. Later T. Kim [8] defined p-adic Dedekind-type DC sums as follows:

$$S_{p,q}(s:h,k:q^k) = \sum_{M=1}^{k-1} \left(\frac{1-q^M}{1-q}\right) (-1)^{M-1} T_q(s,hM,k:q^k),$$

where $(h,k) = (p,k) = 1, s \in Z_p$, and $T_q(m,a,N : q^N)$ is a continuous p-adic extension of $(\frac{1-q^N}{1-q})^m E_{m,q^N}(\frac{a}{N})$, then he got a continuous function $S_{p,q}(m:h,k:q^k)$ on Z_p, which satisfies

$$S_{p,q}(m:h,k:q^k) = \left(\frac{1-q^k}{1-q}\right)^{m+1} S_{m,q}(h,k:q^k) - \left(\frac{1-q^k}{1-q}\right)^{m+1} \left(\frac{1-q^{kp}}{1-q^k}\right)^m S_{m,q}((p^{-1}h)_k,k:q^{pk}),$$

where

$$S_{m,q}(h,k:q^l) = \sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^M}{1-q^k}\right)\int_{Z_p} q^{-lx}\left(\frac{1-q^{l(x+\{\frac{hM}{k}\})}}{1-q^l}\right)^m d\mu_{q^l}(x).$$

The other sums analogous to Dedekind sums are defined as

$$S_1(h,k) = \sum_{j=1}^{k-1}(-1)^{j+1+[\frac{hj}{k}]},$$

where h and k are integers with $k > 0$. The sums $S_1(h,k)$ are sometimes called Hardy sums. Some authors studied the properties of $S_1(h,k)$ and related sums, and obtained some interesting results, see [9–12]. A relation between certain Hardy sums $S_1(h,k)$ and classical Dedekind sums $S(h,k)$ can be obtained in [12] that if $(h,k) = 1$, then

$$S_1(h,k) = -8S(h+k,2k) + 4S(h,k).$$

Other scholars showed their interests to the hybrid mean value involving Hardy sums and other famous sums, see [13–15]. For example, W. P. Zhang [15] studied the hybrid mean value involving certain Hardy sums $S_1(h,k)$ and Kloosterman sums

$$K(r,q) = \sum_{a=1}^{q}{}' e\left(\frac{ra+\bar{a}}{q}\right),$$

where $\sum_{a=1}^{q}{}'$ denotes the summation over all a with $(a,q) = 1$, $e(y) = e^{2\pi iy}$, and obtained some exact computational formulas for

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,p)K(s,p)S_1(2m\bar{s},p),$$

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,p)|^2|K(s,p)|^2 S_1(2m\bar{s},p).$$

Actually the transforming formula (1) and

$$S_1(2h,p) = -20S(2h,p) + 8S(4h,p) + 8S(h,p)$$

are used, where p is an odd prime and $0 < h < p$. Therefore,

$$\begin{aligned} S_1(2m\bar{s},p) \quad = \quad & -\frac{20p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2m\bar{s})\,|L(1,\chi)|^2 \\ & +\frac{8p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4m\bar{s})\,|L(1,\chi)|^2 \\ & +\frac{8p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(m\bar{s})\,|L(1,\chi)|^2. \end{aligned} \tag{2}$$

However, as far as we know, it seems that nobody has yet studied the hybrid mean value involving certain Hardy sums $S_1(h,k)$ and general Kloosterman sums $K(r,l,\lambda;p)$, which are defined as follows:

$$K(r,l,\lambda;q) = \sum_{a=1}^{q}{}' \lambda(a) e\left(\frac{ra + l\bar{a}}{q}\right),$$

where λ is a Dirichlet character modulo q. If $\lambda = \lambda_0$ is the principal character modulo q and $l = 1$, then $K(r,1,\lambda_0;q) = K(r,q)$.

Inspired by [15], we will study the hybrid mean value

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,n,\lambda;p)K(s,t,\lambda;p)S_1(2m\bar{s},p),$$

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2 S_1(2m\bar{s},p).$$

Since they are more comprehensive than those in [15], we turn to some novel methods and finally obtain several explicit formulas. Our results not only cover those of [15], but also contain something quite new.

Theorem 1. *Let p be an odd prime. Then for any character λ $\bmod$ p and any integers n, t with $(n,p) = (t,p) = 1$, if $p \equiv 1 \bmod 4$, we have the identity*

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,n,\lambda;p)K(s,t,\lambda;p)S_1(2m\bar{s},p) = 0.$$

If $p \equiv 3 \bmod 4$, we have

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,n,\lambda;p)K(s,t,\lambda;p)S_1(2m\bar{s},p) = \begin{cases} 2p, & \text{if } \overline{\lambda}\chi = \chi_0; \\ 2p^2, & \text{if } \overline{\lambda}\chi \neq \chi_0, \end{cases}$$

where χ is any odd character modulo p and χ_0 is the principal character modulo p.

Note: The emergences of χ in Theorem 1 are due to the application of (2) in the proof.

Theorem 2. *Let p be an odd prime. Then for any character λ $\bmod$ p and any integers n, t with $(n,p) = (t,p) = 1$, if $p \equiv 1 \bmod 4$, we have the identity*

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2 \, |K(s,t,\lambda;p)|^2 S_1(2m\bar{s},p) = 0.$$

If $p \equiv 3 \bmod 8$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2 S_1(2m\bar{s},p) \\
&= \begin{cases} 2p^3 - 36p^2h_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi \neq \chi_0; \\ 2p^2 - 36ph_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi = \chi_0; \\ 2p^2[p^2 - p - 54h_p^2 + 1 + 2(p-1-18h_p^2)Re\,\tau(\overline{\chi}^2)], & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi = \chi_0; \\ 2p^2[2p^2 - 2p - 18ph_p^2 + 1 - 2(p-1-18h_p^2)Re\,\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi) & \\ +36h_p^2 Re\,\tau(\overline{\lambda}\chi)], & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi \neq \chi_0. \end{cases}
\end{aligned}$$

If $p \equiv 7 \bmod 8$, we have

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2 S_1(2m\bar{s},p)$$
$$= \begin{cases} 2p^3+4p^2h_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \bar{\lambda}\chi \neq \chi_0; \\ 2p^2+4ph_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \bar{\lambda}\chi = \chi_0; \\ 2p^2[p^2-p+6h_p^2+1+2(p-1+2h_p^2)Re\ \tau(\overline{\chi}^2)], & \text{if } \overline{\lambda\chi} = \chi_0,\ \bar{\lambda}\chi = \chi_0; \\ 2p^2[2p^2-2p+2ph_p^2+1-2(p-1+2h_p^2)Re\ \tau(\overline{\chi}^2)\tau(\bar{\lambda}\chi) & \\ +4h_p^2 Re\ \tau(\bar{\lambda}\chi)], & \text{if } \overline{\lambda\chi} = \chi_0,\ \bar{\lambda}\chi \neq \chi_0, \end{cases}$$

where h_p denotes the class number of the quadratic field $Q(\sqrt{-p})$, χ is any odd character modulo p, and $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums.

Note: We know that $|\tau(\chi)| = \sqrt{p}$ if χ is a primitive character modulo p. For the case of $\overline{\lambda\chi} = \chi_0$, $\bar{\lambda}\chi = \chi_0$, taking $\sqrt{p}$ as the upper bound estimate of $Re\ \tau(\overline{\chi}^2)$, we can get

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2 S_1(2m\bar{s},p) = \begin{cases} 2p^4+O(p^{\frac{7}{2}}), & \text{if } p \equiv 3 \bmod 8; \\ 2p^4+O(p^{\frac{7}{2}}), & \text{if } p \equiv 7 \bmod 8. \end{cases}$$

Let $n = t = 1$, $\lambda = \lambda_0$ in Theorems, we may immediately obtain Theorems 1 and 2 of [15] as the following:

Corollary 1. *Let p be an odd prime, then we have*

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,p)K(s,p)S_1(2m\bar{s},p) = \begin{cases} 0, & \text{if } p \equiv 1 \bmod 4; \\ 2p^2, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Corollary 2. *Let p be an odd prime, then we have*

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,p)|^2|K(s,p)|^2 S_1(2m\bar{s},p) = \begin{cases} 0, & \text{if } p \equiv 1 \bmod 4; \\ 2p^3-36p^2h_p^2, & \text{if } p \equiv 3 \bmod 8; \\ 2p^3+4p^2h_p^2, & \text{if } p \equiv 7 \bmod 8. \end{cases}$$

It should be pointed out that we only consider the prime modulus case. The question of whether there exist some exact computational formulas for the general modulus q remains open.

2. Some Lemmas

In this section, we will give several simple lemmas, which are necessary in the proof of our theorems. Hereinafter, we shall use some knowledge of elementary number theory, the orthogonality relation of character sum and the properties of Gauss sums, which all can be found in [16], here we only list a few. For example,

$$\sum_{a=1}^{p-1} \chi(a)e\left(\frac{ua}{p}\right) = \tau(\chi)\overline{\chi}(u),$$

if $(u,p) = 1$ or χ is a primitive character modulo p. If χ is a primitive character modulo p, then $|\tau(\chi)| = \sqrt{p}$.

Now we have the following lemmas:

Lemma 1. *Let p be an odd prime, n be any integer with $(n,p)=1$. Then for any non-principal character χ $\bmod p$ and any character λ $\bmod p$, we have*

$$\left|\sum_{m=1}^{p-1}\chi(m)|K(m,n,\lambda;p)|^2\right| = \begin{cases} p|\tau(\overline{\chi}^2)|, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi\neq\chi_0; \\ p^{\frac{1}{2}}|\tau(\overline{\chi}^2)|, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi=\chi_0; \\ p|\tau(\overline{\chi}^2)+(p-1)|, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi=\chi_0; \\ p|-\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi)+(p-1))|, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi\neq\chi_0. \end{cases}$$

Proof. From the properties of Gauss sums and the reduced residue system modulo p, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\chi(m)|K(m,n,\lambda;p)|^2 \\
&= \sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma+n\overline{a}}{p}\right)\right|^2 \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\lambda(a\overline{b})\sum_{m=1}^{p-1}\chi(m)e\left(\frac{m(a-b)+n(\overline{a}-\overline{b})}{p}\right) \\
&= \sum_{a=1}^{p-1}\lambda(a)\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\chi(m)e\left(\frac{mb(a-1)+n\overline{b}(\overline{a}-1)}{p}\right) \\
&= \sum_{a=1}^{p-1}\lambda(a)\sum_{b=1}^{p-1}e\left(\frac{n\overline{b}(\overline{a}-1)}{p}\right)\sum_{m=1}^{p-1}\chi(m)e\left(\frac{mb(a-1)}{p}\right) \\
&= \tau(\chi)\sum_{a=1}^{p-1}\lambda(a)\overline{\chi}(a-1)\sum_{b=1}^{p-1}\chi(b)e\left(\frac{nb(\overline{a}-1)}{p}\right) \\
&= \tau^2(\chi)\overline{\chi}(n)\sum_{a=1}^{p-1}\lambda(a)\overline{\chi}(a-1)\overline{\chi}(\overline{a}-1) \\
&= \tau^2(\chi)\overline{\chi}(n)\sum_{a=1}^{p-1}\lambda(a)\overline{\chi}(2-a-\overline{a}) \\
&= \tau^2(\chi)\overline{\chi}(n)\sum_{a=1}^{p-1}\lambda(a)\overline{\chi}(\overline{a})\overline{\chi}(2a-a^2-1) \\
&= \tau^2(\chi)\overline{\chi}(-n)\sum_{a=1}^{p-1}\lambda\chi(a)\overline{\chi}((a-1)^2) \\
&= \tau^2(\chi)\overline{\chi}(-n)\sum_{a=1}^{p-2}\lambda\chi(a+1)\overline{\chi}^2(a).
\end{aligned} \tag{3}$$

Using the properties of Gauss sums and the periodicity of Dirichlet character, we have

$$\begin{aligned}
&\sum_{a=1}^{p-2}\lambda\chi(a+1)\overline{\chi}^2(a) \\
&= \frac{1}{\tau(\overline{\lambda\chi})}\sum_{b=1}^{p-1}\overline{\lambda\chi}(b)\sum_{a=1}^{p-2}\overline{\chi}^2(a)e\left(\frac{b(a+1)}{p}\right) \\
&= \frac{1}{\tau(\overline{\lambda\chi})}\sum_{b=1}^{p-1}\overline{\lambda\chi}(b)e\left(\frac{b}{p}\right)\sum_{a=1}^{p-2}\overline{\chi}^2(a)e\left(\frac{ab}{p}\right) \\
&= \frac{1}{\tau(\overline{\lambda\chi})}\sum_{b=1}^{p-1}\overline{\lambda\chi}(b)e\left(\frac{b}{p}\right)\left(\sum_{a=1}^{p-1}\overline{\chi}^2(a)e\left(\frac{ab}{p}\right)-\overline{\chi}^2(p-1)e\left(\frac{pb-b}{p}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\tau(\overline{\lambda\chi})}\sum_{b=1}^{p-1}\overline{\lambda\chi}(b)e\left(\frac{b}{p}\right)\left(\sum_{a=1}^{p-1}\overline{\chi}^2(a)e\left(\frac{ab}{p}\right)-e\left(\frac{-b}{p}\right)\right)\\
&= \frac{\tau(\overline{\chi}^2)}{\tau(\overline{\lambda\chi})}\sum_{b=1}^{p-1}\overline{\lambda}\chi(b)e\left(\frac{b}{p}\right)-\frac{1}{\tau(\overline{\lambda\chi})}\sum_{b=1}^{p-1}\overline{\lambda\chi}(b)\\
&= \frac{\tau(\overline{\lambda}\chi)\tau(\overline{\chi}^2)}{\tau(\overline{\lambda\chi})}-\frac{1}{\tau(\overline{\lambda\chi})}\sum_{b=1}^{p-1}\overline{\lambda\chi}(b).
\end{aligned}
$$

Applying the orthogonality relation for character modulo p, it is clear that

$$
\sum_{b=1}^{p-1}\overline{\lambda\chi}(b)=\begin{cases}0, & \text{if } \overline{\lambda\chi}\neq\chi_0;\\ p-1, & \text{if } \overline{\lambda\chi}=\chi_0.\end{cases}
$$

And note that $\tau(\chi_0)=-1$, so we can get

$$
\begin{aligned}
&\sum_{a=1}^{p-2}\lambda\chi(a+1)\overline{\chi}^2(a)\\
&=\begin{cases}\dfrac{\tau(\overline{\lambda}\chi)\tau(\overline{\chi}^2)}{\tau(\overline{\lambda\chi})}, & \text{if } \overline{\lambda\chi}\neq\chi_0;\\ -\tau(\overline{\lambda}\chi)\tau(\overline{\chi}^2)+(p-1), & \text{if } \overline{\lambda\chi}=\chi_0.\end{cases}\\
&=\begin{cases}\dfrac{\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi)}{\tau(\overline{\lambda\chi})}, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi\neq\chi_0;\\ \dfrac{-\tau(\overline{\chi}^2)}{\tau(\overline{\lambda\chi})}, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ \tau(\overline{\chi}^2)+(p-1), & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ -\tau(\overline{\lambda}\chi)\tau(\overline{\chi}^2)+(p-1), & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi\neq\chi_0.\end{cases}
\end{aligned}
\tag{4}
$$

Note that $|\overline{\chi}(-n)|=1$ and $|\tau(\overline{\lambda\chi})|=|\tau(\overline{\lambda}\chi)|=|\tau(\chi)|=\sqrt{p}$ if $\overline{\lambda\chi}$ and $\overline{\lambda}\chi$ are all non-principal characters modulo p. Thus, from (3) and (4) we may immediately deduce that

$$
\begin{aligned}
&\left|\sum_{m=1}^{p-1}\chi(m)|K(m,n,\lambda;p)|^2\right|\\
&=\begin{cases}p\left|\tau(\overline{\chi}^2)\right|, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi\neq\chi_0;\\ p^{\frac{1}{2}}\left|\tau(\overline{\chi}^2)\right|, & \text{if } \overline{\lambda\chi}\neq\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ p\left|(\tau(\overline{\chi}^2)+(p-1))\right|, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ p\left|(-\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi)+(p-1))\right|, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi\neq\chi_0.\end{cases}
\end{aligned}
$$

This proves Lemma 1. □

Lemma 2. *Let p be an odd prime, n be any integer with $(n,p)=1$. Then for any non-principal character χ $\bmod p$ and any character $\lambda \bmod p$, we have*

$$
\left|\sum_{m=1}^{p-1}\chi(m)K(m,n,\lambda;p)\right|=\begin{cases}p^{\frac{1}{2}}, & \text{if } \overline{\lambda}\chi=\chi_0;\\ p, & \text{if } \overline{\lambda}\chi\neq\chi_0.\end{cases}
$$

Proof. From the definition of Kloosterman sums and the properties of Gauss sums we have

$$\begin{aligned}
&\left|\sum_{m=1}^{p-1}\chi(m)K(m,n,\lambda;p)\right| \\
&= \left|\sum_{m=1}^{p-1}\chi(m)\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma+n\bar{a}}{p}\right)\right| \\
&= \left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{n\bar{a}}{p}\right)\sum_{m=1}^{p-1}\chi(m)e\left(\frac{ma}{p}\right)\right| \\
&= \left|\tau(\chi)\sum_{a=1}^{p-1}\lambda(a)\overline{\chi}(a)e\left(\frac{n\bar{a}}{p}\right)\right| \\
&= \left|\tau(\chi)\sum_{a=1}^{p-1}\overline{\lambda}\chi(a)e\left(\frac{na}{p}\right)\right| \\
&= \left|\tau(\chi)\tau(\overline{\lambda}\chi)\lambda\overline{\chi}(n)\right|.
\end{aligned}$$

Note that $|\tau(\chi)| = \sqrt{p}$, $|\lambda\overline{\chi}(n)| = 1$, $\tau(\chi_0) = -1$, then we can immediately deduce that

$$\left|\sum_{m=1}^{p-1}\chi(m)K(m,n,\lambda;p)\right| = \begin{cases} p^{\frac{1}{2}}, & \text{if } \overline{\lambda}\chi = \chi_0; \\ p, & \text{if } \overline{\lambda}\chi \neq \chi_0. \end{cases}$$

This proves Lemma 2. □

Lemma 3. *Let p be an odd prime, then we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{(p-1)^2(p-2)}{p^2},$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2 = \frac{\pi^2}{24}\frac{(p-1)^2(p-5)}{p^2},$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2 = \begin{cases} \frac{\pi^2}{48}\frac{(p-1)^2(p-17)}{p^2}, & p \equiv 1 \bmod 4; \\ \frac{\pi^2}{48}\frac{(p-1)(p^2-6p+17)}{p^2}, & p \equiv 3 \bmod 4. \end{cases}$$

Proof. See Lemma 5 of [15]. □

3. Proof of the Theorems

In this section, we shall use the above lemmas to complete the proof of Theorems. First we prove Theorem 1 From the identity (2), we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,n,\lambda;p)K(s,t,\lambda;p)S_1(2m\bar{s},p) \\
&= -\frac{20p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| \sum_{m=1}^{p-1} \chi(m)K(m,n,\lambda;p) \right|^2 |L(1,\chi)|^2 \\
&\quad + \frac{8p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4) \left| \sum_{m=1}^{p-1} \chi(m)K(m,n,\lambda;p) \right|^2 |L(1,\chi)|^2 \\
&\quad + \frac{8p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left| \sum_{m=1}^{p-1} \chi(m)K(m,n,\lambda;p) \right|^2 |L(1,\chi)|^2 .
\end{aligned}$$

Firstly, we consider the case of $p \equiv 1 \bmod 4$. If $\overline{\lambda}\chi = \chi_0$, from Lemmas 2 and 3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,n,\lambda;p)K(s,t,\lambda;p)S_1(2m\bar{s},p) \\
&= -\frac{20p^2}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2 + \frac{8p^2}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2 \\
&\quad + \frac{8p^2}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 \\
&= -\frac{5}{6}(p-1)(p-5) + \frac{2}{3}(p-1)(p-2) + \frac{1}{6}(p-1)(p-17) \\
&= 0.
\end{aligned}$$

If $\overline{\lambda}\chi \neq \chi_0$, then from Lemmas 2 and 3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,n,\lambda;p)K(s,t,\lambda;p)S_1(2m\bar{s},p) \\
&= -\frac{20p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2 + \frac{8p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2 \\
&\quad + \frac{8p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 \\
&= -\frac{5}{6}p(p-1)(p-5) + \frac{2}{3}p(p-1)(p-2) + \frac{1}{6}p(p-1)(p-17) \\
&= 0.
\end{aligned}$$

For the case of $p \equiv 3 \bmod 4$, similarly we have

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} K(m,n,\lambda;p)K(s,t,\lambda;p)S_1(2m\bar{s},p) = \begin{cases} 2p, & \text{if } \overline{\lambda}\chi = \chi_0; \\ 2p^2, & \text{if } \overline{\lambda}\chi \neq \chi_0. \end{cases}$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. From the identity (2), it is clear that

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1}|K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2S_1(2m\bar{s},p)\\
&=-\frac{20p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left|\sum_{m=1}^{p-1}\chi(m)|K(m,n,\lambda;p)|^2\right|^2|L(1,\chi)|^2\\
&\quad+\frac{8p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)\left|\sum_{m=1}^{p-1}\chi(m)|K(m,n,\lambda;p)|^2\right|^2|L(1,\chi)|^2\\
&\quad+\frac{8p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{m=1}^{p-1}\chi(m)|K(m,n,\lambda;p)|^2\right|^2|L(1,\chi)|^2.
\end{aligned}$$

Firstly, we consider the case of $p\equiv 1 \bmod 4$. Note that $|\tau(\overline{\chi}^2)|=\sqrt{p}$. If $\overline{\lambda\chi}\neq\chi_0$, $\overline{\lambda}\chi\neq\chi_0$, from Lemmas 1 and 3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1}|K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2S_1(2m\bar{s},p)\\
&=-\frac{20p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)|L(1,\chi)|^2+\frac{8p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)|L(1,\chi)|^2\\
&\quad+\frac{8p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2\\
&=-\frac{5}{6}p^2(p-1)(p-5)+\frac{1}{6}p^2(p-1)(p-17)+\frac{2}{3}p^2(p-1)(p-2)\\
&=0.
\end{aligned}$$

If $\overline{\lambda\chi}\neq\chi_0$, $\overline{\lambda}\chi=\chi_0$, from Lemmas 1 and 3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1}|K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2S_1(2m\bar{s},p)\\
&=-\frac{20p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)|L(1,\chi)|^2+\frac{8p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)|L(1,\chi)|^2\\
&\quad+\frac{8p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2\\
&=-\frac{5}{6}p(p-1)(p-5)+\frac{1}{6}p(p-1)(p-17)+\frac{2}{3}p(p-1)(p-2)\\
&=0.
\end{aligned}$$

Similarly, we have

$$\sum_{m=1}^{p-1}\sum_{s=1}^{p-1}|K(m,n,\lambda;p)|^2|K(s,t,\lambda;p)|^2S_1(2m\bar{s},p)=\begin{cases}0, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi=\chi_0;\\ 0, & \text{if } \overline{\lambda\chi}=\chi_0,\ \overline{\lambda}\chi\neq\chi_0.\end{cases}$$

Now, we consider the case of $p \equiv 3 \bmod 4$. In this case, we note that

$$\left(\frac{-1}{p}\right) = \chi_2(-1) = -1,\ L(1,\chi_2) = \frac{\pi h_p}{\sqrt{p}},\ \tau(\overline{\chi_2}^2) = -1,\ \left(\frac{4}{p}\right) = \left(\frac{2^2}{p}\right) = 1.$$

So, if $\overline{\lambda\chi} \neq \chi_0$, $\overline{\lambda}\chi \neq \chi_0$, from Lemmas 1 and 3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2 |K(s,t,\lambda;p)|^2 S_1(2m\overline{s},p)\\
&= -\frac{20p^4}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2 + \frac{8p^4}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2\\
&\quad + \frac{8p^4}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 + \frac{20p^4}{\pi^2(p-1)}\chi_2(2)|L(1,\chi_2)|^2\\
&\quad - \frac{20p^3}{\pi^2(p-1)}\chi_2(2)|L(1,\chi_2)|^2 - \frac{8p^4}{\pi^2(p-1)}\chi_2(4)|L(1,\chi_2)|^2 + \frac{8p^3}{\pi^2(p-1)}\chi_2(4)|L(1,\chi_2)|^2\\
&\quad - \frac{8p^4}{\pi^2(p-1)}|L(1,\chi_2)|^2 + \frac{8p^3}{\pi^2(p-1)}|L(1,\chi_2)|^2\\
&= -\frac{5}{6}p^2(p-1)(p-5) + \frac{1}{6}p^2(p^2-6p+17) + \frac{2}{3}p^2(p-1)(p-2)\\
&\quad + \frac{20p^3}{\pi^2}\chi_2(2)|L(1,\chi_2)|^2 - \frac{16p^3}{\pi^2}|L(1,\chi_2)|^2\\
&= 2p^3 - 16p^2h_p^2 + 20p^2h_p^2\left(\frac{2}{p}\right).
\end{aligned}$$

If $\overline{\lambda\chi} \neq \chi_0$, $\overline{\lambda}\chi = \chi_0$, from Lemmas 1 and 3, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2 |K(s,t,\lambda;p)|^2 S_1(2m\overline{s},p)\\
&= -\frac{20p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2 + \frac{8p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2\\
&\quad + \frac{8p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 + \frac{20p^3}{\pi^2(p-1)}\chi_2(2)|L(1,\chi_2)|^2\\
&\quad - \frac{20p^2}{\pi^2(p-1)}\chi_2(2)|L(1,\chi_2)|^2 - \frac{8p^3}{\pi^2(p-1)}\chi_2(4)|L(1,\chi_2)|^2 + \frac{8p^2}{\pi^2(p-1)}\chi_2(4)|L(1,\chi_2)|^2\\
&\quad - \frac{8p^3}{\pi^2(p-1)}|L(1,\chi_2)|^2 + \frac{8p^2}{\pi^2(p-1)}|L(1,\chi_2)|^2\\
&= -\frac{5}{6}p(p-1)(p-5) + \frac{1}{6}p(p^2-6p+17) + \frac{2}{3}p(p-1)(p-2)\\
&\quad + \frac{20p^2}{\pi^2}\chi_2(2)|L(1,\chi_2)|^2 - \frac{16p^2}{\pi^2}|L(1,\chi_2)|^2\\
&= 2p^2 - 16ph_p^2 + 20ph_p^2\left(\frac{2}{p}\right).
\end{aligned}$$

Similarly, if $\overline{\lambda\chi} = \chi_0$, $\overline{\lambda}\chi = \chi_0$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2 |K(s,t,\lambda;p)|^2 S_1(2m\overline{s},p) \\
&= 2p^2[p + (p-1)^2 + 2(p-1)Re\tau(\overline{\chi}^2)] \\
&\quad +20p^2h_p^2(3+2Re\tau(\overline{\chi}^2))\left(\frac{2}{p}\right) - 16p^2h_p^2(3+2Re\tau(\overline{\chi}^2)).
\end{aligned}$$

If $\overline{\lambda\chi} = \chi_0$, $\overline{\lambda}\chi \neq \chi_0$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2 |K(s,t,\lambda;p)|^2 S_1(2m\overline{s},p) \\
&= 2p^2[p^2 + (p-1)^2 - 2(p-1)Re\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi)] \\
&\quad +20p^2h_p^2(p - 2(Re\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi) + Re\tau(\overline{\lambda}\chi)))\left(\frac{2}{p}\right) \\
&\quad -16p^2h_p^2(p - 2(Re\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi) + Re\tau(\overline{\lambda}\chi))).
\end{aligned}$$

Then if $p \equiv 3 \bmod 8$, note that $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = -1$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2 |K(s,t,\lambda;p)|^2 S_1(2m\overline{s},p) \\
&= \begin{cases}
2p^3 - 36p^2h_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi \neq \chi_0; \\
2p^2 - 36ph_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi = \chi_0; \\
2p^2[p^2 - p - 54h_p^2 + 1 + 2(p-1-18h_p^2)Re\,\tau(\overline{\chi}^2)], & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi = \chi_0; \\
2p^2[2p^2 - 2p - 18ph_p^2 + 1 - 2(p-1-18h_p^2)Re\,\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi) & \\
+36h_p^2 Re\,\tau(\overline{\lambda}\chi)], & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi \neq \chi_0.
\end{cases}
\end{aligned}$$

If $p \equiv 7 \bmod 8$, note that $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = 1$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{s=1}^{p-1} |K(m,n,\lambda;p)|^2 |K(s,t,\lambda;p)|^2 S_1(2m\overline{s},p) \\
&= \begin{cases}
2p^3 + 4p^2h_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi \neq \chi_0; \\
2p^2 + 4ph_p^2, & \text{if } \overline{\lambda\chi} \neq \chi_0,\ \overline{\lambda}\chi = \chi_0; \\
2p^2[p^2 - p + 6h_p^2 + 1 + 2(p-1+2h_p^2)Re\,\tau(\overline{\chi}^2)], & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi = \chi_0; \\
2p^2[2p^2 - 2p + 2ph_p^2 + 1 - 2(p-1+2h_p^2)Re\,\tau(\overline{\chi}^2)\tau(\overline{\lambda}\chi) & \\
+4h_p^2 Re\,\tau(\overline{\lambda}\chi)], & \text{if } \overline{\lambda\chi} = \chi_0,\ \overline{\lambda}\chi \neq \chi_0.
\end{cases}
\end{aligned}$$

This completes the proof of Theorem 2.

4. Conclusions

In this paper, we obtain some exact computational formulas for the hybrid mean value involving general Kloosterman sums and certain Hardy sums. We also prove some identities in the second part by using the properties of Gauss sums, the orthogonality relation of character sum and the mean value of Dirichlet L-function, which are necessary for the proof of our Theorems. We only consider the prime modulus case. The question of whether there exist some exact computational formulas for the general modulus q remains open.

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References

1. Bettin, S.; Conrey, J.B. A reciprocity formula for a cotangent sum. *Int. Math. Res. Not. IMRN* **2013**, *24*, 5709–5726. [CrossRef]
2. Bettin, S.; Conrey, J.B. Period functions and cotangent sums. *Algebra Number Theory* **2013**, *7*, 215–242. [CrossRef]
3. Bettin, S.; Drappeau, S. Partial sums of the cotangent function. *arXiv* **2019**, arXiv:1905.01954.
4. Maier, H.; Rassias, M.T. Distribution of a cotangent sum related to the Nyman-Beurling criterion for the Riemann Hypothesis. *Appl. Math. Comput.* **2019**, *363*, 124589. [CrossRef]
5. Carlitz, L. The reciprocity theorem of Dedekind sums. *Pac. J. Math.* **1953**, *3*, 513–522. [CrossRef]
6. Zhang, W.P. On the mean values of Dedekind sums. *J. Théor. Nombres Bordx.* **1996**, *8*, 429–442. [CrossRef]
7. Kim, T. Note on Dedekind type DC sums. *Adv. Stud. Contemp. Math.* **2009**, *18*, 249–260.
8. Kim, T. Note on q-Dedekind-type sums related to q-Euler polynomial. *Glasg. Math. J.* **2012**, *54*, 121–125. [CrossRef]
9. Apostol, T.M. *Modular Functions and Dirichlet Series in Number Theory*; Springer: New York, NY, USA, 1976.
10. Cetin, E. A note on Hardy type sums and Dedekind sums. *Filomat* **2016**, *30*, 977–983. [CrossRef]
11. Conrey, J.B.; Fransen, E.; Klein, R.; Scott, C. Mean values of Dedekind sums. *J. Number Theory* **1996**, *56*, 214–226. [CrossRef]
12. Sitaramachandrarao, R. Dedekind and Hardy sums. *Acta Arith.* **1987**, *48*, 325–340. [CrossRef]
13. Peng, W.; Zhang, T.P. Some identities involving certain Hardy sum and Kloosterman sum. *J. Number Theory* **2016**, *165*, 355–362. [CrossRef]
14. Guo, X.Y.; Zhang, W.P. A hybrid mean value related to certain Hardy sums and Kloosterman sums. *Czech. Math. J.* **2011**, *61*, 759–769. [CrossRef]
15. Zhang, H.; Zhang, W.P. On the identity involving certain Hardy sums and Kloosterman sums. *J. Inequal. Appl.* **2014**, *52*, 1–9. [CrossRef]
16. Apostol, T.M. *Introduction to Analytic Number Theory*; Springer: New York, NY, USA, 1976.