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Comultiplications on the Localized Spheres and Moore Spaces

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Abstract: Any nilpotent CW-space can be localized at primes in a similar way to the localization of a ring at a prime number. For a collection $\mathcal{P}$ of prime numbers which may be empty and a localization $X_{\mathcal{P}}$ of a nilpotent CW-space X at $\mathcal{P}$, we let $|C(X)|$ and $|C(X_{\mathcal{P}})|$ be the cardinalities of the sets of all homotopy comultiplications on X and $X_{\mathcal{P}}$, respectively. In this paper, we show that if $|C(X)|$ is finite, then $|C(X)| \geq |C(X_{\mathcal{P}})|$, and if $|C(X)|$ is infinite, then $|C(X)| = |C(X_{\mathcal{P}})|$, where X is the k-fold wedge sum $\bigvee_{i=1}^{k} \mathbb{S}^{n_i}$ or Moore spaces $M(G, n)$. Moreover, we provide examples to concretely determine the cardinality of homotopy comultiplications on the k-fold wedge sum of spheres, Moore spaces, and their localizations.

Keywords: comultiplications; localized spheres; basic Whitehead products; Hilton formula; Moore space

1. Introduction

Homotopy comultiplications, one of the Eckmann–Hilton dual notions of homotopy multiplications, play a fundamental role in classical and rational homotopy theories. One reason for this is that the set of pointed homotopy classes of base point preserving continuous maps from a co-H-space X to a space Y has a canonical binary operation with identity induced by the homotopy comultiplication on a co-H-space. The calculation of homotopy comultiplications is very complicated in that there are usually many homotopy comultiplications on a given co-H-space with many different homotopy properties.

Localization theories in the pointed homotopy category of simply connected (or nilpotent) CW-complexes have been developed in algebraic or topological forms. Moreover, the study of homotopy comultiplications, binary operations for homotopy groups with coefficients and the same n-type structures of co-H-spaces has been carried out by several authors; see [1–8]. For example, homotopy comultiplications on the wedge sum of circles were developed by using methods of group theory in [9]. The techniques of rational homotopy were applied to obtain some rational results for finite 1-connected co-H-spaces in [10,11]. The homotopy comultiplications on the wedge sum of two Moore spaces were investigated using homological algebra in [12]. The cardinality of homotopy comultiplications on a suspension has been determined in [13]. A topological transversality theorem for multivalued maps with continuous, compact selections was developed in [14]. From the equivariant homotopy theoretic point of view, an explicit expression of the behavior of the local cohomology spectral sequence graded on the representation ring was presented with many pictures in [15].

In this paper, we study the cardinality of the set of all homotopy comultiplications on a wedge of (localized) spheres or (localized) Moore spaces, and develop the properties of the homotopy comultiplications of those CW-spaces. Our methods can be used to study homotopy comultiplications on a wedge of any number of spheres.

The paper is organized as follows: In Section 2, we describe the fundamental notions of homotopy comultiplications and introduce the localized version of Hilton's formula. In Section 3, we establish basic facts regarding the types of homotopy comultiplications on the k-fold wedge sum of localized spheres and define certain homotopy comultiplications on the k-fold wedge sum of localized spheres and Moore spaces. In Section 4, we describe the main result of this paper and give an example to illustrate it.

Convention. In this paper, we work in the pointed homotopy category $\mathcal{C}$ of connected nilpotent CW-complexes and homotopy classes of base point preserving continuous maps. We mostly use '$=$' for the homotopy '$\simeq$' unless we emphasize the homotopy. We also do not distinguish notationally between a continuous map and its homotopy class.

2. Preliminaries

Let $\{X_\gamma | \gamma \in \Gamma\}$ be a family of pointed topological spaces whose base points are x_γ for each $\gamma \in \Gamma$. The *wedge sum* $\bigvee_{\gamma\in\Gamma} X_\gamma$ of $\{X_\gamma | \gamma \in \Gamma\}$ is defined as the quotient space

$$\bigvee_{\gamma\in\Gamma} X_\gamma = \coprod_{\gamma\in\Gamma} X_\gamma / \{x_\gamma | \gamma \in \Gamma\},$$

where $\coprod_{\gamma\in\Gamma} X_\gamma$ is the topological sum of $\{X_\gamma | \gamma \in \Gamma\}$.

For any abelian group G and a positive integer $n \geq 2$, there exists a 1-connected CW-complex X such that

$$H_i(X) = \begin{cases} G & \text{if } i = n; \\ 0 & \text{if } i \neq n. \end{cases}$$

The homotopy type of X is uniquely determined up to homotopy and it is said to be a *Moore space* of type (G, n). We denote the Moore space of type (G, n) (or any CW-space which has the same homotopy type to it) by $M(G, n)$.

Localization plays a pivotal role in both algebra and algebraic topology, especially in homotopy theory. For a collection $\mathcal{P}$ of prime numbers which may be empty, a group G is called a $\mathcal{P}$*-local group* if the map

$$q : G \to G$$

given by

$$q(g) = \underbrace{g + g + \ldots + g}_{(q-\text{times})}$$

is bijective for all $q \in \mathcal{P}^c$. A nilpotent CW-complex X is said to be $\mathcal{P}$*-local* if $\pi_n(X)$ is $\mathcal{P}$-local for all $n \geq 1$. Any nilpotent CW-space can be localized at primes in the sense of Sullivan [16] and Bousfield and Kan [17] in a similar way to the localization of a ring at a prime number. As usual, the localization of a nilpotent CW-space X at $\mathcal{P}$ is denoted as $X_{\mathcal{P}}$ which is unique up to homotopy.

A pair (X, ψ) consisting of a space X and a base point preserving continuous map $\psi : X \to X \vee X$ is said to be a *co-Hopf space*, or *co-H-space* for short, if $\pi^1\psi \simeq 1_X$ and $\pi^2\psi \simeq 1_X$, where 1_X is the identity map of X and $\pi^1, \pi^2 : X \vee X \to X$ are the first and second projections, respectively. In this case, the map $\psi : X \to X \vee X$ is said to be a *homotopy comultiplication*, or *comultiplication* for short. Homotopy comultiplication is an Eckmann–Hilton dual notion of homotopy multiplication; see [18–20]. It can be seen that (X, ψ) is a co-H-space if

$$j\psi \simeq \Delta : X \to X \times X,$$

where $j : X \vee X \to X \times X$ is the inclusion map and Δ is the diagonal map.

We now consider the k-fold wedge sum of spheres $X := \mathbb{S}^{n_1} \vee \mathbb{S}^{n_2} \vee \cdots \vee \mathbb{S}^{n_k}$. Let $\xi_i : \mathbb{S}^{n_i} \to X,\ i = 1, 2, \ldots, k$ be the canonical inclusion maps. We now construct the so-called *basic Whitehead products* (see also [21–24]) by using the homotopy classes of those canonical inclusion maps by induction on n. We denote the basic Whitehead products of weight 1 by $\xi_1, \xi_2, \ldots, \xi_k$ that are ordered by the canonical inclusion maps in the way so that $\xi_1 < \xi_2 < \cdots < \xi_k$. Assume that the basic Whitehead products of weight less than or equal to $n-1$ have been already defined and ordered so that if $r < s < n$, all basic Whitehead products of weight r are less than all of the basic Whitehead products of weight s. Then a basic Whitehead product of weight n is a basic Whitehead product $[a, b]$, where a and b are basic Whitehead products of weights p and q, respectively, with $p + q = n$ and $a < b$. If b is a basic Whitehead product $[c, d]$ consisting of basic Whitehead products c and d, then the condition $c \leq a$ is necessary. Then the basic Whitehead products of weight n are greater than any basic Whitehead product of weight less than n and are ordered among themselves arbitrarily.

For a basic Whitehead product of weight n, we associate to it a string of canonical inclusion maps $\xi_1, \xi_2, \ldots, \xi_k$ that appear in the basic Whitehead products. The *height* h_s of the basic Whitehead product w_s is defined as

$$h_s = \sum_i t_i(n_i - 1),$$

where t_i is the number of the canonical inclusion map ξ_i that appears in the basic Whitehead product w_s for $s = 1, 2, 3, \ldots$.

We are interested in the cardinality of the set of homotopy comultiplications on a co-H-space. In the case of a wedge of two spheres, we obtain the following.

Lemma 1. *The cardinality of homotopy comultiplications of* $\mathbb{S}^n \vee \mathbb{S}^m$ *is*

$$\prod_{s=3}^{\infty} |\pi_m(\mathbb{S}^{h_s+1})|,$$

where $2 \leq n < m$ *and* $|X|$ *denotes the cardinality of a set* X.

Proof. See ([22], Corollary 2.9) and ([24], Corollary 2.15) for the more general case. □

One of the most important theorems in algebraic topology is the Hilton's formula [25], which is described as follows.

Theorem 1. *Let* $w_1, w_2, \ldots, w_s, \ldots$ *be the basic Whitehead products of the k-fold wedge sum of spheres*

$$X = \mathbb{S}^{n_1} \vee \mathbb{S}^{n_2} \vee \cdots \vee \mathbb{S}^{n_k}.$$

Then for all m*, we have*

$$\pi_m(X) \cong \bigoplus_{s=1}^{\infty} \pi_m(\mathbb{S}^{h_s+1}).$$

Here,

- h_s *is the height of* w_s*; and*
- *the isomorphism* $\theta : \bigoplus_{s=1}^{\infty} \pi_m(\mathbb{S}^{h_s+1}) \to \pi_m(X)$ *is given by*

$$\theta|\pi_m(\mathbb{S}^{h_s+1}) = w_{s*} : \pi_m(\mathbb{S}^{h_s+1}) \to \pi_m(X).$$

The direct sum is finite for each positive integer m since the height h_s goes to infinity as $s \to \infty$.

3. Comultiplications on a Wedge of (Localized) Spheres and Moore Spaces

In this section, we investigate the cardinality of the set of all homotopy comultiplications on the k-fold wedge sum of localized spheres and Moore spaces. In particular, we are interested in determining inequalities and relationships between the cardinalities of the sets of homotopy comultiplications on the wedge sum of spheres, Moore spaces and their localizations.

3.1. The Localized Version of Hilton's Formula

Let $\mathcal{P}$ be a collection of prime numbers and let $G_{\mathcal{P}}$ denote the $\mathcal{P}$-localization of a nilpotent group G. It is well known that if $h : G \to H$ is any homomorphism in the category of nilpotent groups, then we have a unique map $h_{\mathcal{P}} : G_{\mathcal{P}} \to H_{\mathcal{P}}$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[d, "e"] & H \arrow[d, "e"] \\ G_{\mathcal{P}} \arrow[r, "h_{\mathcal{P}}"] & H_{\mathcal{P}} \end{tikzcd}$$

is strictly commutative, where $e : G \to G_{\mathcal{P}}$ is a $\mathcal{P}$-localizing map; similarly, for the nilpotent CW-spaces.

Lemma 2. *Let $\psi : X \to X \vee X$ be a comultiplication on a nilpotent CW-space X. Then the localization $\psi_{\mathcal{P}} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ at a set $\mathcal{P}$ of primes is a comultiplication.*

Proof. Since the localization of a nilpotent CW-space has the functorial property, all the faces including the bottom triangle colored in blue of the following triangular prism are strictly commutative.

Here (in Figure 1),

- $\pi^1 : X \vee X \to X$ is the first projection;
- $\pi^1_{\mathcal{P}} : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}}$ is the first projection (obtained by π^1 by taking the $\mathcal{P}$-localization);
- 1_Y is the identity map of Y; and
- $e : X \to X_{\mathcal{P}}$ is a $\mathcal{P}$-localizing map.

Similarly, we obtain the same result for the second projection $\pi^2 : X \vee X \to X$. This implies that if $\psi : X \to X \vee X$ is a comultiplication on X, then its localization $\psi_{\mathcal{P}} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ also has the comultiplication structure on the localization $X_{\mathcal{P}}$ of X. $\square$

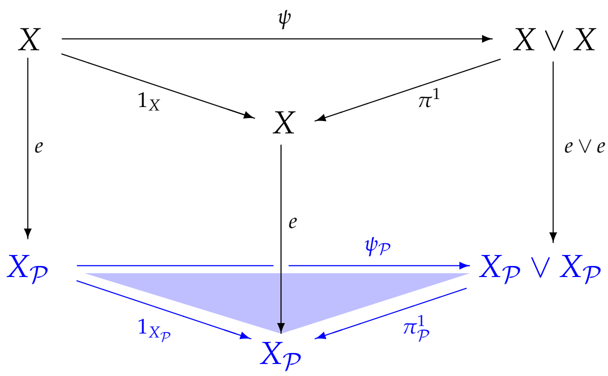

Figure 1. The triangular prism.

We now consider the localized version of the Hilton's formula described in Theorem 1 of Section 2 as follows.

Theorem 2. *Let $X_{\mathcal{P}}$ be the localization of the k-fold wedge sum of spheres $X := \mathbb{S}^{n_1} \vee \mathbb{S}^{n_2} \vee \cdots \vee \mathbb{S}^{n_k}$. Then we have*

$$\pi_m(X_{\mathcal{P}}) \cong \bigoplus_{s=1}^{\infty} \pi_m(\mathbb{S}^{h_s+1}_{\mathcal{P}})$$

for all m.

Proof. We note that localization preserves the Cartesian product, the smash product and the wedge sum in the pointed homotopy category of pointed nilpotent CW-complexes; that is,

- $(X \times Y)_{\mathcal{P}} \simeq X_{\mathcal{P}} \times Y_{\mathcal{P}}$;
- $(X \wedge Y)_{\mathcal{P}} \simeq X_{\mathcal{P}} \wedge Y_{\mathcal{P}}$; and
- $(X \vee Y)_{\mathcal{P}} \simeq X_{\mathcal{P}} \vee Y_{\mathcal{P}}$.

Since the localizations of nilpotent CW-complexes are unique up to homotopy equivalence and the localizations of nilpotent groups commute with finite direct sums, we have

$$\begin{aligned}\pi_m(X_{\mathcal{P}}) &\cong \pi_m(X) \otimes \mathbb{Z}_{\mathcal{P}} \\ &\cong \left(\bigoplus_{s=1}^{\infty} \pi_m(\mathbb{S}^{h_s+1})\right) \otimes \mathbb{Z}_{\mathcal{P}} \\ &\cong \bigoplus_{s=1}^{\infty}\left(\pi_m(\mathbb{S}^{h_s+1}) \otimes \mathbb{Z}_{\mathcal{P}}\right) \\ &\cong \bigoplus_{s=1}^{\infty} \pi_m(\mathbb{S}_{\mathcal{P}}^{h_s+1}),\end{aligned}$$

as required. □

3.2. The 2-Fold Wedge of (Localized) Spheres

We begin with the 2-fold wedge sum of (localized) spheres to illustrate our methods. The results will be applied to the k-fold wedge sum of (localized) spheres for $k \geq 3$.
Notation. Throughout this (sub)section, we will make use of the following notations.

- $X := \mathbb{S}^n \vee \mathbb{S}^m$ is the wedge sum of spheres, where $2 \leq n < m$.
- $k^1 : \mathbb{S}^n \to X$ and $k^2 : \mathbb{S}^m \to X$ are the inclusion maps.
- $i^1, i^2 : \mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$ are the first and second inclusion maps, respectively.
- $p^1, p^2 : \mathbb{S}^n \vee \mathbb{S}^n \to \mathbb{S}^n$ are the first and second projections, respectively.
- $\iota^1, \iota^2 : X \to X \vee X$ are the first and second inclusion maps, respectively.
- $\pi^1, \pi^2 : X \vee X \to X$ are the first and second projections, respectively.
- $\mathcal{P}$ is a collection of prime numbers.
- $f_{\mathcal{P}} : A_{\mathcal{P}} \to B_{\mathcal{P}}$ is the $\mathcal{P}$-localization of a map $f : A \to B$, where A and B are object classes in the pointed homotopy category $\mathcal{C}$ of connected nilpotent CW-spaces.
- $C(X) \subseteq [X; X \vee X]$ is the set of all homotopy classes of comultiplications on a nilpotent CW-space X, e.g., $X := \vee_{i=1}^{k} \mathbb{S}^{n_i}$, Moore spaces, or their localizations.

Lemma 3. *Let $X_{\mathcal{P}} = \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^m_{\mathcal{P}}$ be a wedge sum of localized spheres with $2 \leq n < m$. Then any comultiplication $\psi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ has the following type*

$$\begin{cases} \psi k^1_{\mathcal{P}} &= \iota^1_{\mathcal{P}} k^1_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^1_{\mathcal{P}} \\ \psi k^2_{\mathcal{P}} &= \iota^1_{\mathcal{P}} k^2_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^2_{\mathcal{P}} + Q \end{cases} \tag{1}$$

for some homotopy class $Q : \mathbb{S}^m_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ satisfying $\pi^1_{\mathcal{P}} Q = 0 = \pi^2_{\mathcal{P}} Q$, where the additions are the homotopy additions in the homotopy groups.

We note that if Y is a CW-complex which is filtered by its n-skeletons Y^n for $n \geq 0$, then its localization $Y_{\mathcal{P}}$ of Y at $\mathcal{P}$ is filtered by $Y^n_{\mathcal{P}}$. Moreover, the two CW-spaces $Y^{n+1}_{\mathcal{P}}/Y^n_{\mathcal{P}}$ and $(Y^{n+1}/Y^n)_{\mathcal{P}}$ have the same homotopy type, and each is a wedge sum of the localized spheres $\mathbb{S}^{n+1}_{\mathcal{P}}$.

Proof. We observe that $X_{\mathcal{P}}$ is a localized CW-complex and thus there is a cellular map ([19] p. 77)

$$\psi' : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

such that

$$\psi \simeq \psi' : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}.$$

Therefore, we have

$$\psi'(X^n_{\mathcal{P}}) \subset (X_{\mathcal{P}} \vee X_{\mathcal{P}})^n.$$

We note that $X^n_{\mathcal{P}} = \mathbb{S}^n_{\mathcal{P}}$ and that the localized sphere $\mathbb{S}^n_{\mathcal{P}}$ has a CW-decomposition that consists of one localized zero cell and one localized n-cell. Therefore, we obtain

$$(X_{\mathcal{P}} \vee X_{\mathcal{P}})^n = \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}$$

and observe that ψ' gives a base point preserving continuous map $\eta : \mathbb{S}^n_{\mathcal{P}} \to \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}$ such that the following diagram

$$\begin{tikzcd} \mathbb{S}^n_{\mathcal{P}} \arrow[r, "\eta"] \arrow[d, "k^1_{\mathcal{P}}"] & \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \arrow[r, "I"] \arrow[d, "k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}}"] & \mathbb{S}^n_{\mathcal{P}} \times \mathbb{S}^n_{\mathcal{P}} \arrow[d, "k^1_{\mathcal{P}} \times k^1_{\mathcal{P}}"] \\ X_{\mathcal{P}} \arrow[r, "\psi \simeq \psi'"] & X_{\mathcal{P}} \vee X_{\mathcal{P}} \arrow[r, "J"] & X_{\mathcal{P}} \times X_{\mathcal{P}} \end{tikzcd} \tag{2}$$

is commutative up to homotopy, where I and J are the inclusion maps. We also note that the following triangle

$$\begin{tikzcd} X_{\mathcal{P}} \arrow[r, "\psi"] \arrow[dr, "\Delta"'] & X_{\mathcal{P}} \vee X_{\mathcal{P}} \arrow[d, "J"] \\ & X_{\mathcal{P}} \times X_{\mathcal{P}} \end{tikzcd} \tag{3}$$

is homotopy commutative, where $\Delta : X_{\mathcal{P}} \to X_{\mathcal{P}} \times X_{\mathcal{P}}$ is the diagonal map. From the homotopy commutative diagrams (2) and (3), we obtain

$$(k^1_{\mathcal{P}} \times k^1_{\mathcal{P}})I\eta \simeq J\psi k^1_{\mathcal{P}} \simeq \Delta k^1_{\mathcal{P}} = (k^1_{\mathcal{P}} \times k^1_{\mathcal{P}})\bar{\Delta}, \tag{4}$$

where $\bar{\Delta} : \mathbb{S}^n_{\mathcal{P}} \to \mathbb{S}^n_{\mathcal{P}} \times \mathbb{S}^n_{\mathcal{P}}$ is the diagonal map. Since

$$(k^1_{\mathcal{P}} \times k^1_{\mathcal{P}})_* : [\mathbb{S}^n_{\mathcal{P}}; \mathbb{S}^n_{\mathcal{P}} \times \mathbb{S}^n_{\mathcal{P}}] \to [\mathbb{S}^n_{\mathcal{P}}; X_{\mathcal{P}} \times X_{\mathcal{P}}]$$

is a monomorphism of homotopy groups, from (4) we have

$$I\eta \simeq \bar{\Delta};$$

that is, the following triangle

$$\begin{tikzcd} \mathbb{S}^n_{\mathcal{P}} \arrow[r, "\eta"] \arrow[dr, "\bar{\Delta}"'] & \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \arrow[d, "I"] \\ & \mathbb{S}^n_{\mathcal{P}} \times \mathbb{S}^n_{\mathcal{P}} \end{tikzcd} \tag{5}$$

is also commutative up to homotopy, where $\bar{\Delta} : \mathbb{S}^n_{\mathcal{P}} \to \mathbb{S}^n_{\mathcal{P}} \times \mathbb{S}^n_{\mathcal{P}}$ is the diagonal map. This implies that the map $\eta : \mathbb{S}^n_{\mathcal{P}} \to \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}$ is also a comultiplication. Since the localized sphere $\mathbb{S}^n_{\mathcal{P}}$ is 1-connected for $n \geq 2$, we see that the comultiplication η is unique up to homotopy (see ([1] Proposition 3.1)) and has the form

$$\eta = i^1_{\mathcal{P}} +_{\eta} i^2_{\mathcal{P}},$$

where $+_\eta$ is the addition induced by the comultiplication η. By the uniqueness of the comultiplication

$$\eta : \mathbb{S}^n_{\mathcal{P}} \to \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}$$

as the standard (or suspension) comultiplication, we have

$$i^1_{\mathcal{P}} +_\eta i^2_{\mathcal{P}} = i^1_{\mathcal{P}} + i^2_{\mathcal{P}},$$

where $+$ is the homotopy addition in the homotopy group

$$[\mathbb{S}^n_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}] \cong [\mathbb{S}^n; X_{\mathcal{P}} \vee X_{\mathcal{P}}] \cong \pi_n(X_{\mathcal{P}} \vee X_{\mathcal{P}}).$$

Hence

$$\psi k^1_{\mathcal{P}} = (k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})\eta = (k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})(i^1_{\mathcal{P}} +_\eta i^2_{\mathcal{P}}) = \iota^1_{\mathcal{P}} k^1_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^1_{\mathcal{P}}.$$

We now consider the homotopy class Q in the homotopy group $[\mathbb{S}^n_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ as an abelian group given by

$$Q = \psi k^2_{\mathcal{P}} - \iota^1_{\mathcal{P}} k^2_{\mathcal{P}} - \iota^2_{\mathcal{P}} k^2_{\mathcal{P}};$$

that is,

$$\psi k^2_{\mathcal{P}} = \iota^1_{\mathcal{P}} k^2_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^2_{\mathcal{P}} + Q.$$

Then we obtain

$$\pi^1_{\mathcal{P}} Q = \pi^1_{\mathcal{P}} \psi k^2_{\mathcal{P}} - \pi^1_{\mathcal{P}} \iota^1_{\mathcal{P}} k^2_{\mathcal{P}} - \pi^1_{\mathcal{P}} \iota^2_{\mathcal{P}} k^2_{\mathcal{P}} = k^2_{\mathcal{P}} - k^2_{\mathcal{P}} - 0 = 0$$

and

$$\pi^2_{\mathcal{P}} Q = \pi^2_{\mathcal{P}} \psi k^2_{\mathcal{P}} - \pi^2_{\mathcal{P}} \iota^1_{\mathcal{P}} k^2_{\mathcal{P}} - \pi^2_{\mathcal{P}} \iota^2_{\mathcal{P}} k^2_{\mathcal{P}} = k^2_{\mathcal{P}} - 0 - k^2_{\mathcal{P}} = 0,$$

as required. □

Lemma 4. *If $Q : \mathbb{S}^m_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is any homotopy class in $[\mathbb{S}^m_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ such that*

$$\pi^1_{\mathcal{P}} Q = 0 = \pi^2_{\mathcal{P}} Q,$$

then the map $\psi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ defined in Equation (1) is a comultiplication.

Proof. We have

- $\pi^1_{\mathcal{P}} \psi k^1_{\mathcal{P}} = \pi^1_{\mathcal{P}}(\iota^1_{\mathcal{P}} k^1_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^1_{\mathcal{P}}) = k^1_{\mathcal{P}} + 0 = k^1_{\mathcal{P}};$
- $\pi^1_{\mathcal{P}} \psi k^2_{\mathcal{P}} = \pi^1_{\mathcal{P}}(\iota^1_{\mathcal{P}} k^2_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^2_{\mathcal{P}} + Q) = k^2_{\mathcal{P}} + 0 + \pi^1_{\mathcal{P}} Q = k^2_{\mathcal{P}} + 0 + 0 = k^2_{\mathcal{P}};$
- $\pi^2_{\mathcal{P}} \psi k^1_{\mathcal{P}} = \pi^2_{\mathcal{P}}(\iota^1_{\mathcal{P}} k^1_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^1_{\mathcal{P}}) = 0 + k^1_{\mathcal{P}} = k^1_{\mathcal{P}};$ and
- $\pi^2_{\mathcal{P}} \psi k^2_{\mathcal{P}} = \pi^2_{\mathcal{P}}(\iota^1_{\mathcal{P}} k^2_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^2_{\mathcal{P}} + Q) = 0 + k^2_{\mathcal{P}} + \pi^2_{\mathcal{P}} Q = 0 + k^2_{\mathcal{P}} + 0 = k^2_{\mathcal{P}};$

that is, $\pi^1_{\mathcal{P}} \psi = 1_{X_{\mathcal{P}}}$ and $\pi^2_{\mathcal{P}} \psi = 1_{X_{\mathcal{P}}}$. This implies that ψ is a comultiplication on $X_{\mathcal{P}}$. □

Using the results above, we can define a comultiplication on a wedge of localized spheres as follows.

Definition 1. *Let $Q : \mathbb{S}^m_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ be any element of $[\mathbb{S}^m_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ satisfying*

$$\pi^1_{\mathcal{P}} Q = 0 = \pi^2_{\mathcal{P}} Q.$$

We define a comultiplication $\psi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ *by*

$$\begin{cases} \psi_Q k^1_{\mathcal{P}} &= \iota^1_{\mathcal{P}} k^1_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^1_{\mathcal{P}} \\ \psi_Q k^2_{\mathcal{P}} &= \iota^1_{\mathcal{P}} k^2_{\mathcal{P}} + \iota^2_{\mathcal{P}} k^2_{\mathcal{P}} + Q. \end{cases}$$

The homotopy class Q *is called a perturbation of the comultiplication* ψ_Q*.*

We note that if $e : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}_{\mathcal{P}}$ is the $\mathcal{P}$-localizing map, then the homomorphism

$$e^* : [\mathbb{S}^{2n-1}_{\mathcal{P}}; \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}] \xrightarrow{\cong} [\mathbb{S}^{2n-1}; \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}] \tag{6}$$

induced by e is an isomorphism of homotopy groups.

Convention. For our convenience in notation, using the isomorphism of homotopy groups in (6), we make use of the same notations of the homotopy classes on both sides of the isomorphism; that is,

$$\gamma \mapsto e^*(\gamma) = \gamma \circ e \longleftrightarrow \gamma \in [\mathbb{S}^{2n-1}_{\mathcal{P}}; \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}]$$

for any homotopy class γ in the homotopy group $[\mathbb{S}^{2n-1}_{\mathcal{P}}; \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}]$. In particular,

$$\gamma = [i^1_{\mathcal{P}}, i^2_{\mathcal{P}}] \in [\mathbb{S}^{2n-1}_{\mathcal{P}}; \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}]$$

as the generalized Whitehead product in the sense of Arkowitz [26]; see below.

Lemma 5. *Let* $[i^1, i^2] : \mathbb{S}^{2n-1} \to \mathbb{S}^n \vee \mathbb{S}^n$ *be the Whitehead product. Then we have*

$$[i^1, i^2]_{\mathcal{P}} = [i^1_{\mathcal{P}}, i^2_{\mathcal{P}}].$$

Proof. The functorial property of the localization shows that the following diagram

$$\begin{tikzcd} \mathbb{S}^{2n-1} \arrow[r, "a"] \arrow[d, "e"] & \mathbb{S}^n \vee \mathbb{S}^n \arrow[r, "i^1 \vee i^2"] \arrow[d, "e"] & \mathbb{S}^n \vee \mathbb{S}^n \vee \mathbb{S}^n \vee \mathbb{S}^n \arrow[r, "\nabla"] \arrow[d, "e"] & \mathbb{S}^n \vee \mathbb{S}^n \arrow[d, "e"] \\ \mathbb{S}^{2n-1}_{\mathcal{P}} \arrow[r, "a_{\mathcal{P}}"] & \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \arrow[r, "i^1_{\mathcal{P}} \vee i^2_{\mathcal{P}}"] & \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \arrow[r, "\nabla_{\mathcal{P}}"] & \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \end{tikzcd} \tag{7}$$

is commutative up to homotopy, where $a : \mathbb{S}^{2n-1} \to \mathbb{S}^n \vee \mathbb{S}^n$ is the attaching map and $e : Y \to Y_{\mathcal{P}}$ is a $\mathcal{P}$-localizing map. Since

$$[\mathbb{S}^{2n-1}; \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}] \cong [\mathbb{S}^{2n-1}_{\mathcal{P}}; \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}],$$

the commutative diagram in (7) shows the proof. □

Example 1. Let $\alpha : \mathbb{S}^m_{\mathcal{P}} \to \mathbb{S}^{2n-1}_{\mathcal{P}}$ be any homotopy class. We define a homotopy class Q in the homotopy group

$$\pi_m(X_{\mathcal{P}} \vee X_{\mathcal{P}}) = [\mathbb{S}^m; X_{\mathcal{P}} \vee X_{\mathcal{P}}] \cong [\mathbb{S}^m_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$$

as the composition

$$\mathbb{S}^m_{\mathcal{P}} \xrightarrow{\alpha} \mathbb{S}^{2n-1}_{\mathcal{P}} \xrightarrow{[i^1, i^2]_{\mathcal{P}}} \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \xrightarrow{k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}}} X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

of maps; that is,

$$Q = (k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})[i^1, i^2]_{\mathcal{P}}(\alpha) = [\iota^1_{\mathcal{P}} k^1_{\mathcal{P}}, \iota^2_{\mathcal{P}} k^1_{\mathcal{P}}]\alpha,$$

where $[i^1, i^2]_{\mathcal{P}}$ is the localization of the Whitehead product $[i^1, i^2]$. Then the map $\psi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ defined by Definition 1 is a comultiplication on $X_{\mathcal{P}} \vee X_{\mathcal{P}}$ whose perturbation is Q.

Let $q^2_{\mathcal{P}} : X_{\mathcal{P}} \to \mathbb{S}^m_{\mathcal{P}}$ be the projection. Then the cofibration

$$\mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \xrightarrow{k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}}} X_{\mathcal{P}} \vee X_{\mathcal{P}} \xrightarrow{q^2_{\mathcal{P}} \vee q^2_{\mathcal{P}}} \mathbb{S}^m_{\mathcal{P}} \vee \mathbb{S}^m_{\mathcal{P}},$$

asserts that if $Q \in \pi_m(X_{\mathcal{P}} \vee X_{\mathcal{P}})$ is any homotopy class satisfying

$$\pi^1_{\mathcal{P}} Q = 0 = \pi^2_{\mathcal{P}} Q,$$

then there exists a unique homotopy class $W \in \pi_m(\mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}})$ such that $Q = (k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})W$ and

$$p^1_{\mathcal{P}} W = 0 = p^2_{\mathcal{P}} W,$$

where $p^1_{\mathcal{P}}, p^2_{\mathcal{P}} : \mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}} \to \mathbb{S}^n_{\mathcal{P}}$ are the first and second projections induced by p^1 and p^2, respectively.

To investigate the fundamental property of homotopy comultiplications on a wedge of localized spheres, we need to study the perturbations of all homotopy comultiplications. This naturally raises the following question: How can we construct the perturbation of a comultiplication on the wedge of localized spheres? The following lemma gives an answer to this query.

We denote the (generalized) basic Whitehead products of $i^1_{\mathcal{P}}$ and $i^2_{\mathcal{P}}$ by $w_1, w_2, \ldots, w_s, \ldots$ and the height of w_s is denoted by h_s for each $s = 1, 2, 3, \ldots$. Then we have the following.

Lemma 6. *A perturbation $Q : \mathbb{S}^m_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ of any comultiplication $\psi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ can be expressed uniquely as in Definition 1 as follows:*

$$Q = (k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})_* \left(\sum_{s=3}^{\infty} w_s a_s\right),$$

where w_s is the sth (generalized) basic Whitehead product localized at $\mathcal{P}$ and a_s is any homotopy class in the homotopy group

$$[\mathbb{S}^m_{\mathcal{P}}; \mathbb{S}^{h_s+1}_{\mathcal{P}}] \cong [\mathbb{S}^m; \mathbb{S}^{h_s+1}_{\mathcal{P}}]$$

for $s = 3, 4, 5, \ldots$.

Proof. By using Lemmas 3–5, we see that every comultiplication $\psi_Q : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ has the type of Definition 1 satisfying

$$\pi^1_{\mathcal{P}} Q = 0 = \pi^2_{\mathcal{P}} Q.$$

Using Theorem 2, we obtain

$$\begin{aligned} \pi_m(\mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^n_{\mathcal{P}}) &\cong \bigoplus_{s=1}^{\infty} \pi_m(\mathbb{S}^{h_s+1}_{\mathcal{P}}) \\ &= \pi_m(\mathbb{S}^n_{\mathcal{P}}) \oplus \pi_m(\mathbb{S}^n_{\mathcal{P}}) \oplus \bigoplus_{s=3}^{\infty} \pi_m(\mathbb{S}^{h_s+1}_{\mathcal{P}}). \end{aligned}$$

Moreover, every homotopy class Q in the homotopy group $[\mathbb{S}^m_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ can be uniquely expressed as the following type:

$$(k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})_* \left(i^1_{\mathcal{P}} a_1 + i^2_{\mathcal{P}} a_2 + \sum_{s=3}^{\infty} w_s a_s\right) \in [\mathbb{S}^m_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}] \cong \pi_m(X_{\mathcal{P}} \vee X_{\mathcal{P}}).$$

Here,

- a_1 and a_2 are any homotopy classes in $[\mathbb{S}^m_{\mathcal{P}};\mathbb{S}^n_{\mathcal{P}}] \cong \pi_m(\mathbb{S}^n_{\mathcal{P}})$;
- a_s is any homotopy class in $[\mathbb{S}^m_{\mathcal{P}};\mathbb{S}^{h_s+1}_{\mathcal{P}}] \cong \pi_m(\mathbb{S}^{h_s+1}_{\mathcal{P}})$ for $s = 3,4,5,\ldots$; and
- w_s is the (generalized) basic Whitehead product for $s = 3,4,5\ldots$.

We recall that all of the (generalized) basic Whitehead products ω_s consists of at least one $i^1_{\mathcal{P}}$ and at least one $i^2_{\mathcal{P}}$, for example, $\omega_3 = [i^1_{\mathcal{P}}, i^2_{\mathcal{P}}]$, $\omega_4 = [i^1_{\mathcal{P}}, [i^1_{\mathcal{P}}, i^2_{\mathcal{P}}]]$, $\omega_5 = [i^2_{\mathcal{P}}, [i^1_{\mathcal{P}}, i^2_{\mathcal{P}}]]$ and so on. We now have

$$\begin{aligned}
\pi^1_{\mathcal{P}} \circ Q &= \pi^1_{\mathcal{P}}(k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})_*(\sum_{s=1}^{\infty} w_s a_s) \\
&= \pi^1_{\mathcal{P}}(k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})_*(i^1_{\mathcal{P}} a_1 + i^2_{\mathcal{P}} a_2 + \sum_{s=3}^{\infty} w_s a_s) \\
&= a_1 + 0_*(a_2) + 0 \\
&= a_1
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\pi^2_{\mathcal{P}} \circ Q &= \pi^2_{\mathcal{P}}(k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})_*(\sum_{s=3}^{\infty} w_s a_s) \\
&= \pi^2_{\mathcal{P}}(k^1_{\mathcal{P}} \vee k^1_{\mathcal{P}})_*(i^1_{\mathcal{P}} a_1 + i^2_{\mathcal{P}} a_2 + \sum_{s=3}^{\infty} w_s a_s) \\
&= 0_*(a_1) + a_2 + 0 \\
&= a_2,
\end{aligned} \tag{9}$$

where 0_* is the trivial homomorphism between homotopy groups and 0 is the trivial homotopy class. From the fact that $\pi^1_{\mathcal{P}} Q = 0 = \pi^2_{\mathcal{P}} Q$, the homotopy classes a_1 and a_2 in (8) and (9) should be zero, as required. □

For the 2-fold wedge sum of localized spheres, we have

Theorem 3. *The cardinality of the set of comultiplications on $\mathbb{S}^n_{\mathcal{P}} \vee \mathbb{S}^m_{\mathcal{P}}$ is*

$$\prod_{s=3}^{\infty} |\pi_m(\mathbb{S}^{h_s+1}_{\mathcal{P}})|,$$

where $2 \leq n < m$ and $\mathcal{P}$ is a collection of prime numbers.

Proof. By Lemma 6, we have the proof. □

Let $|C(X_{\mathcal{P}})|$ be the cardinality of the set $C(X_{\mathcal{P}})$ of comultiplications on $X_{\mathcal{P}}$.

Example 2. If $\mathcal{P} = \{2,3\}$, then we have

(1) $|C(\mathbb{S}^4_{\mathcal{P}} \vee \mathbb{S}^7_{\mathcal{P}})| = \infty$;
(2) $|C(\mathbb{S}^4_{\mathcal{P}} \vee \mathbb{S}^8_{\mathcal{P}})| = 2$; and
(3) $|C(\mathbb{S}^4_{\mathcal{P}} \vee \mathbb{S}^9_{\mathcal{P}})| = 2$.

Indeed, by Theorem 3, we have

$$\begin{array}{rcl}
|C(\mathbb{S}^4_{\mathcal{P}} \vee \mathbb{S}^8_{\mathcal{P}})| &=& |\pi_8(S^7_{\mathcal{P}})| \times |\pi_8(\mathbb{S}^{10}_{\mathcal{P}})| \times \cdots \\
&=& |\mathbb{Z}_2 \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times \cdots \\
&=& 2 \times 1 \times 1 \times \cdots \times 1 \times \cdots \\
&=& 2
\end{array},$$

where $\{e\}$ is the trivial group, and similarly for the other cases.

3.3. The k-Fold Wedge Sum of (Localized) Spheres

We now consider the k-fold wedge sum of 1-connected spheres $\{\mathbb{S}^{n_i} | i = 1, 2, \ldots, k\}$ and their localized spheres $\{\mathbb{S}^{n_i}_{\mathcal{P}} | i = 1, 2, \ldots, k\}$ as a generalization of the statements above; that is,

$$X = \mathbb{S}^{n_1} \vee \mathbb{S}^{n_2} \vee \cdots \vee \mathbb{S}^{n_k}$$

and

$$X_{\mathcal{P}} = \mathbb{S}^{n_1}_{\mathcal{P}} \vee \mathbb{S}^{n_2}_{\mathcal{P}} \vee \cdots \vee \mathbb{S}^{n_k}_{\mathcal{P}}$$

for $2 \leq n_1 < n_2 < \cdots < n_k$.

Notation. We make use of the following notations in the rest of this paper.

- $X := \bigvee_{i=1}^{k} \mathbb{S}^{n_i}$ for $2 \leq n_1 < n_2 < \cdots < n_k$.
- $\alpha^1_i, \alpha^2_i : \mathbb{S}^{n_i} \to X \vee X$ are the ith and $(k+i)$th inclusions, respectively, for $i = 1, 2, \ldots, k-1$.
- $\zeta^i : \mathbb{S}^{n_i} \to X$ is the ith inclusion for $i = 1, 2, \ldots, k$.
- $\iota^1, \iota^2 : X \to X \vee X$ are the first and second inclusions, respectively.
- $\pi^1, \pi^2 : X \vee X \to X$ are the first and second projections, respectively.
- $W^{\dagger}_j$ is the set of all (generalized) basic Whitehead products localized at $\mathcal{P}$ containing the homotopy classes from α^1_i for $i = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, k$.
- $W^{\sharp}_j$ is the set of all (generalized) basic Whitehead products localized at $\mathcal{P}$ containing the homotopy classes from α^2_i for $i = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, k$.
- W^{*}_j is the set of all (generalized) basic Whitehead products localized at $\mathcal{P}$ containing as a factor at least one of the homotopy classes from α^1_i and at least one of the homotopy classes from α^2_i for $i = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, k$.

We note that if $k = 2$, $n_1 = n$ and $n_2 = m$; that is, $X = \mathbb{S}^n \vee \mathbb{S}^m$, then we have

1. $\zeta^1 = k^1 : \mathbb{S}^n \hookrightarrow \mathbb{S}^n \vee \mathbb{S}^m$;
2. $\zeta^2 = k^2 : \mathbb{S}^m \hookrightarrow \mathbb{S}^n \vee \mathbb{S}^m$;
3. $\alpha^1_1 = (k^1 \vee k^1) \circ i^1 : \ \mathbb{S}^n \xrightarrow{i^1} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{k^1 \vee k^1} X \vee X$; and
4. $\alpha^2_1 = (k^1 \vee k^1) \circ i^2 : \ \mathbb{S}^n \xrightarrow{i^2} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{k^1 \vee k^1} X \vee X$

in the notations above.

To develop the basic Whitehead products in the k-fold wedge sum of spheres localized at $\mathcal{P}$, we order the basic Whitehead products of weight 1 as follows:

- $\alpha^1_i < \alpha^2_j$ for $i, j = 1, 2, \ldots, k-1$; and
- $\iota^1 \zeta^i < \iota^2 \zeta^j$ for $i, j = 1, 2, \ldots, k$.

Definition 2. *Let $Q_j : \mathbb{S}^{n_j}_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ be any element of $[\mathbb{S}^{n_j}_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ satisfying*

$$\pi^1_{\mathcal{P}} Q_j = 0 = \pi^2_{\mathcal{P}} Q_j$$

for each $j = 2, 3, \ldots, k$, where $\pi^1_{\mathcal{P}}, \pi^2_{\mathcal{P}} : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}}$ are the first and second projections, respectively. We define a comultiplication

$$\psi = \psi_{(Q_2, Q_3, \ldots, Q_k)} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$$

by

$$\begin{cases} \psi_Q\zeta^1_{\mathcal{P}} & = \iota^1_{\mathcal{P}}\zeta^1_{\mathcal{P}} + \iota^2_{\mathcal{P}}\zeta^1_{\mathcal{P}} \\ \psi_Q\zeta^2_{\mathcal{P}} & = \iota^1_{\mathcal{P}}\zeta^2_{\mathcal{P}} + \iota^2_{\mathcal{P}}\zeta^2_{\mathcal{P}} + Q_2 \\ \vdots & \\ \psi_Q\zeta^j_{\mathcal{P}} & = \iota^1_{\mathcal{P}}\zeta^j_{\mathcal{P}} + \iota^2_{\mathcal{P}}\zeta^j_{\mathcal{P}} + Q_j \\ \vdots & \\ \psi_Q\zeta^k_{\mathcal{P}} & = \iota^1_{\mathcal{P}}\zeta^k_{\mathcal{P}} + \iota^2_{\mathcal{P}}\zeta^k_{\mathcal{P}} + Q_k. \end{cases} \tag{10}$$

Here, we call $Q_j \in [\mathbb{S}^{n_j}_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ *the jth perturbation of* $\psi = \psi_{(Q_2,Q_3,\dots,Q_k)}$ *for* $j = 2, 3, \dots, k$.

We need to investigate the jth perturbation $Q_j \in [\mathbb{S}^{n_j}_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ of $\psi_{(Q_2,Q_3,\dots,Q_k)} : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ as a generalization of Lemma 6 more concretely.

Lemma 7. *The jth perturbation* $Q_j \in [\mathbb{S}^{n_i}_{\mathcal{P}}; X_{\mathcal{P}} \vee X_{\mathcal{P}}]$ *of any comultiplication* $\psi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ *can be uniquely expressed as in Definition 2 by*

$$Q_j = \sum_{w_j \in W^*_j} w_j \circ c_j. \tag{11}$$

for $j = 2, 3, \dots, k$. *Here,*

- $w_j : \mathbb{S}^{h_{w_j}+1}_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ *is the jth (generalized) basic Whitehead product localized at* $\mathcal{P}$*; and*
- $c_j : \mathbb{S}^{n_j}_{\mathcal{P}} \to \mathbb{S}^{h_{w_j}+1}_{\mathcal{P}}$ *is any homotopy class in the homotopy group* $[\mathbb{S}^{n_j}_{\mathcal{P}}; \mathbb{S}^{h_{w_j}+1}_{\mathcal{P}}] \cong [\mathbb{S}^{n_j}; \mathbb{S}^{h_{w_j}+1}_{\mathcal{P}}]$*, where* h_{w_j} *is the height of the (generalized) basic Whitehead product* $w_j \in W^*_j$ *for each* $j = 2, 3, \dots, k$.

Proof. The localized version of the Hilton's formula says that the jth perturbation Q_j of $\psi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$ can be uniquely written by

$$Q_j = \sum_{i=1}^{j-1} (\alpha^1_i)_{\mathcal{P}} \circ \beta^i_j + \sum_{i=1}^{j-1} (\alpha^2_i)_{\mathcal{P}} \circ \gamma^i_j + \sum_{v_j \in W^\dagger_j} u_j \circ a_j + \sum_{v_j \in W^\sharp_j} v_j \circ b_j + \sum_{w_j \in W^*_j} w_j \circ c_j \tag{12}$$

Here,

- β^i_j and γ^i_j are any homotopy classes in $[\mathbb{S}^{n_j}_{\mathcal{P}}; \mathbb{S}^{n_i}_{\mathcal{P}}] \cong \pi_{n_j}(\mathbb{S}^{n_i}_{\mathcal{P}})$ for $i = 1, 2, \dots, j-1$;
- a_j is any homotopy class in $[\mathbb{S}^{n_j}_{\mathcal{P}}; \mathbb{S}^{h_{u_j}+1}_{\mathcal{P}}] \cong \pi_{n_j}(\mathbb{S}^{h_{u_j}+1}_{\mathcal{P}})$ for $j = 2, 3, \dots k$;
- b_j is any homotopy class in $[\mathbb{S}^{n_j}_{\mathcal{P}}; \mathbb{S}^{h_{v_j}+1}_{\mathcal{P}}] \cong \pi_{n_j}(\mathbb{S}^{h_{v_j}+1}_{\mathcal{P}})$ for $j = 2, 3, \dots k$;
- c_j is any homotopy class in $[\mathbb{S}^{n_j}_{\mathcal{P}}; \mathbb{S}^{h_{w_j}+1}_{\mathcal{P}}] \cong \pi_{n_j}(\mathbb{S}^{h_{w_j}+1}_{\mathcal{P}})$ for $j = 2, 3, \dots k$; and
- u_j, v_j and w_j are the (generalized) basic Whitehead product localized at $\mathcal{P}$ for $j = 2, 3, \dots k$.

We show that the homotopy classes β^i_j, γ^i_j, a_j and b_j are all zero. By taking the first and second projections $\pi^1_{\mathcal{P}}, \pi^2_{\mathcal{P}} : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}}$ to the perturbation Q_j of any comultiplication $\psi : X_{\mathcal{P}} \to X_{\mathcal{P}} \vee X_{\mathcal{P}}$, we have

$$\pi^1_{\mathcal{P}} \circ Q_j = 0 = \pi^2_{\mathcal{P}} \circ Q_j \tag{13}$$

for each $j = 2, 3, \ldots, k$; that is,

$$\begin{aligned} 0 &= \pi^1_{\mathcal{P}} \circ Q_j \\ &= \sum_{i=1}^{j-1} \pi^1_{\mathcal{P}} \circ (\alpha^1_i)_{\mathcal{P}} \circ \beta^i_j + \sum_{i=1}^{j-1} \pi^1_{\mathcal{P}} \circ (\alpha^2_i)_{\mathcal{P}} \circ \gamma^i_j + \sum_{u_j \in W^\dagger_j} \pi^1_{\mathcal{P}} \circ u_j \circ a_j + \sum_{v_j \in W^\sharp_j} \pi^1_{\mathcal{P}} \circ v_j \circ b_j \\ &\quad + \sum_{w_j \in W^*_j} \pi^1_{\mathcal{P}} \circ w_j \circ c_j \end{aligned} \tag{14}$$

and similarly, for the second projection $\pi^2_{\mathcal{P}}$; see below. Since the map

$$\pi^1_{\mathcal{P}} \circ (\alpha^1_i)_{\mathcal{P}} : \ \mathbb{S}^{n_j}_{\mathcal{P}} \xrightarrow{(\alpha^1_i)_{\mathcal{P}}} X_{\mathcal{P}} \vee X_{\mathcal{P}} \xrightarrow{\pi^1_{\mathcal{P}}} X_{\mathcal{P}} \tag{15}$$

in (14) is the inclusion map, the localized version of the Hilton's formula asserts that the homotopy class β^i_j should be zero for each $i = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, k$. The choice of the (generalized) basic Whitehead products localized at $\mathcal{P}$ in $W^\dagger_j$ says that all the homotopy classes a_j should be zero. We note that the maps

$$\pi^1_{\mathcal{P}} \circ (\alpha^2_i)_{\mathcal{P}} : \ \mathbb{S}^{n_j}_{\mathcal{P}} \xrightarrow{(\alpha^2_i)_{\mathcal{P}}} X_{\mathcal{P}} \vee X_{\mathcal{P}} \xrightarrow{\pi^1_{\mathcal{P}}} X_{\mathcal{P}}, \tag{16}$$

$$\pi^1_{\mathcal{P}} \circ v_j : \ \mathbb{S}^{h_{v_j}+1}_{\mathcal{P}} \xrightarrow{v_j} X_{\mathcal{P}} \vee X_{\mathcal{P}} \xrightarrow{\pi^1_{\mathcal{P}}} X_{\mathcal{P}} \tag{17}$$

and

$$\pi^1_{\mathcal{P}} \circ w_j : \ \mathbb{S}^{h_{w_j}+1}_{\mathcal{P}} \xrightarrow{w_j} X_{\mathcal{P}} \vee X_{\mathcal{P}} \xrightarrow{\pi^1_{\mathcal{P}}} X_{\mathcal{P}} \tag{18}$$

are all zero by the choice of $(\alpha^2_i)_{\mathcal{P}}$, v_j and w_j for each $i = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, k$.

Similarly, by taking the second projection $\pi^2_{\mathcal{P}} : X_{\mathcal{P}} \vee X_{\mathcal{P}} \to X_{\mathcal{P}}$ to the perturbation Q_j again, we see that the map

$$\pi^2_{\mathcal{P}} \circ (\alpha^2_i)_{\mathcal{P}} : \ \mathbb{S}^{n_j}_{\mathcal{P}} \xrightarrow{(\alpha^2_i)_{\mathcal{P}}} X_{\mathcal{P}} \vee X_{\mathcal{P}} \xrightarrow{\pi^2_{\mathcal{P}}} X_{\mathcal{P}} \tag{19}$$

from (12) is the inclusion map so that $\gamma^i_j = 0$ for each $i = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, k$. The choice of the (generalized) basic Whitehead products localized at $\mathcal{P}$ in $W^\sharp_j$ says that all the homotopy classes b_j should be zero. Moreover, the localized version of the Hilton's formula asserts that the homotopy class $\pi^2_{\mathcal{P}} \circ (\alpha^1_i)_{\mathcal{P}}$, $\pi^2_{\mathcal{P}} \circ u_j$, and $\pi^2_{\mathcal{P}} \circ w_j$ are all zero by the choice of $(\alpha^1_i)_{\mathcal{P}}$, u_j and w_j for each $i = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, k$, as required. □

For the k-fold wedge sum of localized spheres, we have the following.

Theorem 4. *The cardinality of the set of comultiplications on* $X_{\mathcal{P}} := \mathbb{S}^{n_1}_{\mathcal{P}} \vee \mathbb{S}^{n_2}_{\mathcal{P}} \vee \cdots \vee \mathbb{S}^{n_k}_{\mathcal{P}}$ *is*

$$\prod_{w_2 \in W^*_2} |\pi_{n_2}(\mathbb{S}^{h_{w_2}+1}_{\mathcal{P}})| \times \prod_{w_3 \in W^*_3} |\pi_{n_3}(\mathbb{S}^{h_{w_3}+1}_{\mathcal{P}})| \times \cdots \times \prod_{w_k \in W^*_k} |\pi_{n_k}(\mathbb{S}^{h_{w_k}+1}_{\mathcal{P}})|,$$

where $2 \leq n_1 < n_2 < \cdots < n_k$ *and* $\mathcal{P}$ *is a collection of prime numbers.*

Proof. By Lemma 7, we complete the proof. □

Example 3. If $\mathcal{P} = \{2, 3, 5\}$, then we have

(1) $|C(\mathbb{S}^8_{\mathcal{P}} \vee \mathbb{S}^{12}_{\mathcal{P}} \vee \mathbb{S}^{26}_{\mathcal{P}})| = \infty$; and
(2) $|C(\mathbb{S}^8_{\mathcal{P}} \vee \mathbb{S}^{12}_{\mathcal{P}} \vee \mathbb{S}^{27}_{\mathcal{P}})| = 512$.

3.4. The (Localized) Moore Spaces

A comultiplication $\varphi : Y \to Y \vee Y$ is said to be *associative* if

$$(\varphi \vee 1) \circ \varphi \simeq (1 \vee \varphi) \circ \varphi : Y \to Y \vee Y \vee Y,$$

where $1 : Y \to Y$ is the identity map. If φ is associative, we call (Y, φ) an *associative co-H-space* or *co-H-group*. It is well known that all Moore spaces $M(G, n)$, $n \geq 2$, are co-H-groups, where G is an abelian group.

In general, the cardinality of the set $C(Y) \subseteq [Y; Y \vee Y]$ of all homotopy classes of comultiplications on Y is extremely difficult to detect. As a particular case, it was shown in [27] that the set of comultiplications $C(M(G, 2))$ is in one-one correspondence with the group $\mathrm{Ext}(G, G \otimes G)$, and that if $n \geq 3$, then $C(M(G, n))$ has one element as the standard comultiplication, where G is an abelian group. We note that this result is also valid for localizations.

4. Main Result and Its Illustration

Let $\mathcal{P}$ be a collection of prime numbers which may be empty and let $X_{\mathcal{P}}$ be the localization of the k-fold wedge sum of spheres

$$X := \bigvee_{i=1}^{k} \mathbb{S}^{n_i} = \mathbb{S}^{n_1} \vee \mathbb{S}^{n_2} \vee \cdots \vee \mathbb{S}^{n_k}$$

for $2 \leq n_1 < n_2 < \cdots < n_k$ or a Moore space $M(G, n)$, where G is an abelian group and $n \geq 2$. Then, by using the results in Sections 2 and 3, we have the following theorem.

Theorem 5. *Let $|C(X)|$ and $|C(X_{\mathcal{P}})|$ be the cardinalities of the sets of all the homotopy comultiplications on X and $X_{\mathcal{P}}$, respectively.*

(1) *If $|C(X)|$ is finite, then $|C(X)| \geq |C(X_{\mathcal{P}})|$.*
(2) *If $|C(X)|$ is infinite, then $|C(X)| = |C(X_{\mathcal{P}})|$.*

Proof. We prove the theorem in the case of the k-fold wedge sum of spheres and its localization. We first note that if the homotopy group $\pi_{n_j}(\mathbb{S}^{h_{w_j}+1})$ contains a q-torsion subgroup T_q with $(p, q) = 1$ for $w_j \in W_j^*$, $j = 2, 3, \ldots, k$, then all of the q-torsion subgroups of $\pi_{n_j}(\mathbb{S}^{h_{w_j}+1})$ could entirely vanish in $\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})$ for $w_j \in W_j^*$, $j = 2, 3, \ldots, k$, where q is a prime number with $(p, q) = 1$ for all $p \in \mathcal{P}$.

(1) If $|C(X)|$ is finite, then all of the homotopy groups $\pi_{n_j}(\mathbb{S}^{h_{w_j}+1})$, $j = 2, 3, \ldots, k$ must be the torsion abelian groups $T_{w_j}, j = 2, 3, \ldots, k$, where T_{w_j} is the torsion subgroup of

$$\prod_{j=2}^{k} \prod_{w_j \in W_j^*} \pi_{n_j}(\mathbb{S}^{h_{w_j}+1}) = \prod_{j=2}^{k} \prod_{w_j \in W_j^*} T_{w_j}. \tag{20}$$

We also note that

$$T_{w_j} \cong \mathbb{Z}_{{p_1}^{j_1}} \oplus \mathbb{Z}_{{p_2}^{j_2}} \oplus \cdots \oplus \mathbb{Z}_{{p_l}^{j_l}}, \tag{21}$$

where $p_1, p_2, \ldots, p_l$ are (not necessarily distinct) primes and $j_1, j_2, \ldots, j_l$ are (not necessarily distinct) positive integers. We thus have

$$
\begin{aligned}
|C(X)| &= \prod_{j=2}^{k} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}^{h_{w_j}+1})| \quad \text{(by Lemma 1)} \\
&= \prod_{j=2}^{k} \prod_{w_j \in W_j^*} |T_{w_j}| \\
&\geq \prod_{j=2}^{k} \prod_{w_j \in W_j^*} |T_{w_j} \otimes \mathbb{Z}_{\mathcal{P}}| \\
&= \prod_{j=2}^{k} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})| \quad \text{(by Theorem 4)} \\
&= |C(X_{\mathcal{P}})|.
\end{aligned}
\tag{22}
$$

We note that equality of the inequality in (22) holds only in the case that the collection $\mathcal{P}$ of primes contains all of the prime numbers expressed in the torsion subgroups T_{w_j}, $j = 2, 3, \ldots, k$.

(2) If $|C(X)|$ is infinite, then at least one of the homotopy groups $\pi_{n_j}(\mathbb{S}^{h_{w_j}+1})$, $j = 2, 3, \ldots, k$ contains an infinite cyclic group, say in dimension $n_s = h_{w_s} + 1$ or $n_s = 2(h_{w_s} + 1) - 1$, where $h_{w_s} + 1$ is an even integer for some $s = 2, 3, \ldots, k$. Moreover, it can be seen that the numbers of nonzero homotopy groups $\pi_{n_j}(\mathbb{S}^{h_{w_j}+1})$ or $\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})$, $j = 2, 3, \ldots, k$ are always finite because the height h_{w_j} goes to infinity.

Let F_{w_s} be a free abelian group of rank 1 and let T_{w_s} be a torsion subgroup of $\pi_{n_s}(\mathbb{S}^{h_{w_s}+1})$ for some $s = 2, 3, \ldots, k$. Then we have

$$
\begin{aligned}
|C(X)| &= \prod_{j=2}^{k} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}^{h_{w_j}+1})| \\
&= |F_{w_s} \oplus T_{w_s}| \times \prod_{j \neq s} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}^{h_{w_j}+1})| \\
&\approx |\mathbb{Z}|.
\end{aligned}
\tag{23}
$$

Here, $\approx$ means the same cardinality as sets and

$$
F_{w_s} \oplus T_{w_s} \cong \begin{cases} \mathbb{Z} & \text{for } n_s = h_{w_s} + 1 \\ \mathbb{Z} \oplus T_{w_s} & \text{for } n_s = 2(h_{w_s} + 1) - 1 \text{ and } h_{w_s} + 1 \text{ even.} \end{cases}
$$

By Theorem 4 or by tensoring $\mathbb{Z}_{\mathcal{P}}$ on the homotopy groups described in (23), we see that if $n_s = h_{w_s} + 1$ for some s, then

$$
\begin{aligned}
|C(X_{\mathcal{P}})| &= \prod_{j=2}^{k} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})| \\
&= |(F_{w_s} \oplus T_{w_s}) \otimes \mathbb{Z}_{\mathcal{P}}| \times \prod_{j \neq s} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})| \\
&= |\mathbb{Z} \otimes \mathbb{Z}_{\mathcal{P}}| \times \prod_{j \neq s} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})| \\
&\approx |\mathbb{Z}_{\mathcal{P}}|
\end{aligned}
\tag{24}
$$

and if $h_{w_s} + 1$ is even and $n_s = 2(h_{w_s} + 1) - 1$ for some s, then

$$\begin{aligned} |C(X_{\mathcal{P}})| &= \prod_{j=2}^{k} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})| \\ &= |(F_{w_s} \oplus T_{w_s}) \otimes \mathbb{Z}_{\mathcal{P}}| \times \prod_{j \neq s} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})| \\ &= |(\mathbb{Z} \oplus F_{w_s}) \otimes \mathbb{Z}_{\mathcal{P}}| \times \prod_{j \neq s} \prod_{w_j \in W_j^*} |\pi_{n_j}(\mathbb{S}_{\mathcal{P}}^{h_{w_j}+1})| \\ &\approx |\mathbb{Z}_{\mathcal{P}}| \end{aligned} \tag{25}$$

We note that some finite groups in (24) and (25) will vanish for some torsion groups, each of whose orders do not belong to the collection $\mathcal{P}$ of prime numbers. Moreover, because $\mathcal{P}$ is a collection of primes and the free abelian group $\mathbb{Z}$ is a subgroup of $\mathbb{Z}_{\mathcal{P}}$ as a subring of the ring of rational numbers $\mathbb{Q}$; that is,

$$\mathbb{Z} \subseteq \mathbb{Z}_{\mathcal{P}} \subseteq \mathbb{Q},$$

their cardinalities are the same as countably infinite sets.

The proof in the case of a Moore space $M(G, n)$ and its localization runs through the similar way to the case of the k-fold wedge sum of spheres and its localization. □

Remark 1. *We note that if the collection $\mathcal{P}$ of prime numbers is the empty collection, then the above statement is also true as a rationalization $X_{\mathbb{Q}}$ of a 1-connected CW-complex $X := \bigvee_{i=1}^{k} \mathbb{S}^{n_i}$ for $2 \leq n_1 < n_2 < \cdots < n_k$.*

We finally give an example to illustrate our results.

Example 4. The finite (resp. infinite) cases are described in Tables 1–3 (resp. Tables 4 and 5). In the tables, ϕ denotes the empty collection of primes and $\aleph_0$ is aleph-zero.

Table 1. The finite case.

$X = \mathbb{S}^n \vee \mathbb{S}^m$	$\mathcal{P}$	$\lvert C(X)\rvert$	$\lvert C(X_{\mathcal{P}})\rvert$
$\mathbb{S}^3 \vee \mathbb{S}^6$	$\{2,5\}$	2	2
$\mathbb{S}^3 \vee \mathbb{S}^6$	$\{3,5\}$	2	1
$\mathbb{S}^8 \vee \mathbb{S}^{26}$	$\{11,13\}$	504	1
$\mathbb{S}^3 \vee \mathbb{S}^6$	ϕ	2	1

Table 2. The finite case for $k \geq 3$.

$X = \bigvee_{i=1}^{k} \mathbb{S}^{n_i}$	$\mathcal{P}$	$\lvert C(X)\rvert$	$\lvert C(X_{\mathcal{P}})\rvert$
$\mathbb{S}^4 \vee \mathbb{S}^6 \vee \mathbb{S}^8$	$\{3,5\}$	2	1
$\mathbb{S}^5 \vee \mathbb{S}^6 \vee \mathbb{S}^7$	$\{3,5\}$	1	1
$\mathbb{S}^5 \vee \mathbb{S}^7 \vee \mathbb{S}^{10}$	$\{3,5\}$	2	1
$\mathbb{S}^7 \vee \mathbb{S}^9 \vee \mathbb{S}^{11} \vee \mathbb{S}^{22}$	$\{3,5\}$	19,568,944,742,400	1

Table 3. The finite case: Moore spaces.

$Y = M(G,n)$	$\mathcal{P}$	$\lvert C(Y)\rvert$	$\lvert C(Y_{\mathcal{P}})\rvert$
$M(\mathbb{Z},2)$	$\{2,5\}$	1	1
$M(\mathbb{Z}_3,2)$	$\{2,5\}$	3	1
$M(\mathbb{Q},2)$	$\{3,5\}$	1	1
$M(G,n), n \geq 3$	ϕ	1	1

Table 4. The infinite case.

$X = \mathbb{S}^n \vee \mathbb{S}^m$	$\mathcal{P}$	$\lvert C(X)\rvert$	$\lvert C(X_{\mathcal{P}})\rvert$
$\mathbb{S}^4 \vee \mathbb{S}^7$	$\{2,5\}$	$\lvert\mathbb{Z}\rvert = \aleph_0$	$\lvert\mathbb{Z}_{\mathcal{P}}\rvert = \aleph_0$
$\mathbb{S}^5 \vee \mathbb{S}^9$	$\{2,5\}$	$\lvert\mathbb{Z}\rvert = \aleph_0$	$\lvert\mathbb{Z}_{\mathcal{P}}\rvert = \aleph_0$
$\mathbb{S}^{10} \vee \mathbb{S}^{19}$	$\{2,5\}$	$\lvert\mathbb{Z}\rvert = \aleph_0$	$\lvert\mathbb{Z}_{\mathcal{P}}\rvert = \aleph_0$
$\mathbb{S}^4 \vee \mathbb{S}^7$	ϕ	$\lvert\mathbb{Z}\rvert = \aleph_0$	$\lvert\mathbb{Q}\rvert = \aleph_0$

Table 5. The infinite case for $k \geq 3$.

$X = \bigvee_{i=1}^{k} \mathbb{S}^{n_i}$	$\mathcal{P}$	$\lvert C(X)\rvert$	$\lvert C(X_{\mathcal{P}})\rvert$
$\mathbb{S}^3 \vee \mathbb{S}^4 \vee \mathbb{S}^5$	$\{2,5\}$	$\lvert\mathbb{Z}\rvert = \aleph_0$	$\lvert\mathbb{Z}_{\mathcal{P}}\rvert = \aleph_0$
$\mathbb{S}^4 \vee \mathbb{S}^6 \vee \mathbb{S}^{12}$	$\{2,5\}$	$\lvert\mathbb{Z}\rvert = \aleph_0$	$\lvert\mathbb{Z}_{\mathcal{P}}\rvert = \aleph_0$
$\mathbb{S}^7 \vee \mathbb{S}^{12} \vee \mathbb{S}^{24}$	$\{2,5\}$	$\lvert\mathbb{Z}\rvert = \aleph_0$	$\lvert\mathbb{Z}_{\mathcal{P}}\rvert = \aleph_0$
$\mathbb{S}^6 \vee \mathbb{S}^8 \vee \mathbb{S}^{20}$	ϕ	$\lvert\mathbb{Z}\rvert = \aleph_0$	$\lvert\mathbb{Q}\rvert = \aleph_0$

Proof. To prove the example, we use the previous results in Sections 2 and 3, and prove the first two cases to illustrate the method. We note that $\pi_6(\mathbb{S}^5)$ is isomorphic to the cyclic group of order 2; that is, $\pi_6(\mathbb{S}^5) \cong \mathbb{Z}_2$; see [28]. If $X = \mathbb{S}^3 \vee \mathbb{S}^6$ and $\mathcal{P} = \{2,5\}$, then we have

$$\begin{aligned} |C(X)| &= |\pi_6(\mathbb{S}^5)| \times |\pi_6(\mathbb{S}^7)| \times |\pi_6(\mathbb{S}^7)| \\ &\quad \times |\pi_6(\mathbb{S}^9)| \times |\pi_6(\mathbb{S}^9)| \times |\pi_6(\mathbb{S}^9)| \times |\pi_6(\mathbb{S}^{11})| \times \cdots \\ &= 2 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times \cdots \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} |C(X_{\mathcal{P}})| &= |\pi_6(\mathbb{S}^5) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^7) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^7) \otimes \mathbb{Z}_{\mathcal{P}}| \\ &\quad \times |\pi_6(\mathbb{S}^9) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^9) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^9) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^{11}) \otimes \mathbb{Z}_{\mathcal{P}}| \times \cdots \\ &= |\mathbb{Z}_2 \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \\ &\quad \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times \cdots \\ &= 2 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times \cdots \\ &= 2 \end{aligned},$$

where $\{e\}$ is the trivial group. If $X = \mathbb{S}^3 \vee \mathbb{S}^6$ and $\mathcal{P} = \{3,5\}$, then we get

$$\begin{aligned} |C(X_{\mathcal{P}})| &= |\pi_6(\mathbb{S}^5) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^7) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^7) \otimes \mathbb{Z}_{\mathcal{P}}| \\ &\quad \times |\pi_6(\mathbb{S}^9) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^9) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^9) \otimes \mathbb{Z}_{\mathcal{P}}| \times |\pi_6(\mathbb{S}^{11}) \otimes \mathbb{Z}_{\mathcal{P}}| \times \cdots \\ &= |\mathbb{Z}_2 \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \\ &\quad \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times |\{e\} \otimes \mathbb{Z}_{\mathcal{P}}| \times \cdots \\ &= 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times \cdots \\ &= 1 \end{aligned}.$$

Therefore, if $X = \mathbb{S}^3 \vee \mathbb{S}^6$, then we have

$$\begin{cases} |C(X)| = 2 = |C(X_{\mathcal{P}})| & \text{for } \mathcal{P} = \{2,5\} \\ |C(X)| = 2 > 1 = |C(X_{\mathcal{P}})| & \text{for } \mathcal{P} = \{3,5\} \end{cases},$$

as required. □

5. Conclusions and Further Prospects

Co-H-spaces, also called spaces with a comultiplication, and localization theory play a pivotal role in (equivariant) homotopy theory. In general, it turns out that the computation of the cardinality of the set of homotopy comultiplications is very difficult. In this paper, we have investigated the inequalities and relationships between the cardinalities of the sets of homotopy comultiplications on the wedge of localized spheres and localized Moore spaces.

We do hope that our methods can be used to study homotopy comultiplications on the wedge sum of any number localized spheres and localized Moore spaces. We also hope that the results in this paper can be applied to the notions of algebra comultiplications on the (localized) algebraic objects such as (free) Lie algebras by considering the coproduct of Lie algebras or cohomology modules over $\mathbb{Z}_{\mathcal{P}}$.

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