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The 3-Rainbow Domination Number of the Cartesian Product of Cycles

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Abstract: We have studied the k-rainbow domination number of $C_n \square C_m$ for $k \geq 4$ (Gao et al. 2019), in which we present the 3-rainbow domination number of $C_n \square C_m$, which should be bounded above by the four-rainbow domination number of $C_n \square C_m$. Therefore, we give a rough bound on the 3-rainbow domination number of $C_n \square C_m$. In this paper, we focus on the 3-rainbow domination number of the Cartesian product of cycles, $C_n \square C_m$. A 3-rainbow dominating function (3RDF) f on a given graph G is a mapping from the vertex set to the power set of three colors $\{1, 2, 3\}$ in such a way that every vertex that is assigned to the empty set has all three colors in its neighborhood. The weight of a 3RDF on G is the value $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The 3-rainbow domination number, $\gamma_{r3}(G)$, is the minimum weight among all weights of 3RDFs on G. In this paper, we determine exact values of the 3-rainbow domination number of $C_3 \square C_m$ and $C_4 \square C_m$ and present a tighter bound on the 3-rainbow domination number of $C_n \square C_m$ for $n \geq 5$.

Keywords: rainbow domination; graph domination; Cartesian product graph; cycle

1. Introduction

In a graph G with vertex set $V(G)$ and edge set $E(G)$, the open neighborhood of a vertex $v \in V(G)$ is a set $\{u | (u, v) \in E\}$, denoted by $N(v)$. The degree of a vertex $v \in V$ is $\deg(v) = |N(v)|$. The minimum degree of G is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. If for each vertex $u \in V \backslash S$, there is a $v \in S$ such that $(u, v) \in E$, then S is a dominating set. The domination number is the minimum cardinality among all dominating sets in G and it is denoted by $\gamma(G)$.

Domination in graphs originates from location problems in operations research. As a variation of domination in graphs, rainbow domination was introduced by Brešar et al. [1]. The essence of the rainbow domination is to study how to dispatch many types of "guards" to dominate a graph. It is required that each vertex in a graph that is not settled by a "guard" has all types of "guards" in its adjacent vertices.

A function f is called a k-rainbow dominating function (kRDF) on a graph G satisfying the condition that for each vertex v such that $f(v) = \emptyset$, $\cup_{u \in N(v)} f(u) = \{1, 2, \cdots, k\}$. The weight of kRDF on G is the value $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight among all weights of kRDFs on G is called the k-rainbow domination number, denoted by $\gamma_{rk}(G)$. Let f be a kRDF function on G; if $w(f) = \gamma_{rk}(G)$, then f is called the $\gamma_{rk}(G)$-function.

The k-rainbow domination has many practical applications, such as storage hierarchy optimization, in information transfer or people allocation between company departments, channel assignment, network security, logistics scheduling, and so on. Therefore, it has been extensively studied by scholars. There are numerous studies on two-rainbow domination [2–10].

For $k \geq 3$, it is more difficult to determine the k-rainbow domination number of a graph. Chang et al. [11] proved that the k-rainbow domination is NP-complete, and they studied the k-rainbow domination problem on trees. Shao et al. [12] gave bounds for the k-rainbow domination number on an arbitrary graph, and they investigated the 3-rainbow domination numbers of cycles, paths, and generalized Petersen graphs. They determined the 3-rainbow domination number of $P(n, 1)$ and the upper bounds for $P(n, 2)$ and $P(n, 3)$. Fujita et al. [13] proved sharp upper bounds on the k-rainbow domination number for all values of k. Hao et al. [14] studied the k-rainbow domination number of directed graphs and presented the exact values of the k-rainbow domination number on the Cartesian product graph of two directed cycles. Wang et al. [15] determined the k-rainbow domination number of $P_3 \square P_n$ for $k \in \{2, 3, 4\}$. Brezovnik et al. [16] studied the complexity of k-rainbow independent domination and presented sharp bounds for the k-rainbow independent domination number of the lexicographic product and the exact formula for $k = 2$. Kang et al. [17] initiated the study of outer-independent k-rainbow domination and presented sharp lower and upper bounds on the outer-independent two-rainbow domination number. There are also some references related to k-coloring of a graph [18,19].

$G_1 \square G_2$, the Cartesian product of G_1 and G_2, is the graph with the vertex set $V(G_1) \times V(G_2)$, and $(u, v)(u', v') \in E(G_1 \square G_2)$ if either $uu' \in E(G_1)$ and $v = v'$ or $vv' \in E(G_2)$ and $u = u'$. Figure 1 shows the graph of $C_n \square C_m$.

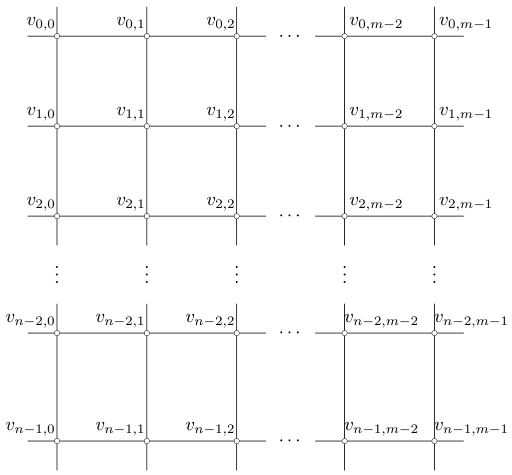


Figure 1. Graph $C_n \square C_m$.

Vizing initiated the problem of domination on Cartesian product graphs [20]. Since then, various domination numbers of $G \square H$ were extensively studied [21–24].

In this paper, we focus on the study of the 3-rainbow domination number of Cartesian products of two undirected cycles, $C_n \square C_m$. Here, we recall some important results.

Theorem 1. ([12]) *Let G be a connected graph. Then, $\gamma_{rt}(G) \geq \left\lceil \frac{|V(G)|t}{\Delta(G)+t} \right\rceil$.*

Theorem 2. ([25]) *Let G be a connected graph of order $n \geq 8$ with $\delta(G) \geq 2$. Then, $\gamma_{r3}(G) \leq \frac{5n}{6}$.*

In $G = C_n \square C_m$, $|V(G)| = mn$, $\Delta(G) = \delta(G) = 4$, by Theorems 1 and 2, we can get:

$$\lceil \frac{3mn}{7} \rceil \leq \gamma_{r3}(G) \leq \frac{5mn}{6}. \tag{1}$$

We studied the k-rainbow domination number of $C_n \square C_m$ for $k \geq 4$ [26], in which we presented that the 3-rainbow domination number of $C_n \square C_m$ is bounded above by the four-rainbow domination number of $C_n \square C_m$, i.e.:

$$\gamma_{r3}(C_n \square C_m) \leq \gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + m + \frac{n}{2} - 1. \tag{2}$$

Since $\frac{mn}{2} + m + \frac{n}{2} - 1 \leq \frac{5mn}{6}$ for $m, n \geq 4$, by Equations (1) and (2), we can get a rough bound of $\gamma_{r3}(C_n \square C_m)$ as described in the following Lemma 1.

Lemma 1. *For $G = C_n \square C_m$, $\lceil \frac{3mn}{7} \rceil \leq \gamma_{r3}(G) \leq \frac{mn}{2} + m + \frac{n}{2} - 1$.*

It is very difficult to determine the k-rainbow domination number of a graph for $k \geq 3$, since the problem is NP-complete. Only with some effective methods, one can present a sharp bound on the k-rainbow domination number, or make the known bound tighter, or determine the exact k-rainbow domination number for a given family of graphs. In this paper, we lower the upper bound in Lemma 1 by constructing some good enough 3RDFs; upon these functions, we can get a sharp upper bound of $\gamma_{r3}(C_n \square C_m)$. Furthermore, we promote the lower bound in Lemma 1 for $C_3 \square C_m$ and $C_4 \square C_m$ by providing proofs for the new lower bounds. Thus, we determine the exact values of $\gamma_{r3}(C_3 \square C_m)$ and $\gamma_{r3}(C_4 \square C_m)$. For $n \geq 5$ and any integer m, we present a tighter bound of $\gamma_{r3}(C_n \square C_m)$ than in Lemma 1.

2. Upper Bounds on the 3-rainbow Domination Number of $C_n \square C_m$

In this section, we construct some 3-rainbow dominating functions according to the characteristics of $C_n \square C_m$; upon these functions, we can get upper bounds of $\gamma_{r3}(C_n \square C_m)$. Figure 2a shows a 3RDF on $C_8 \square C_8$. We use 0, 1, 2, 3 to represent the color sets $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, respectively, and use 2, 3 to encode the color set $\{2, 3\}$. In this way, we could use Figure 2b to show a function in the following sections.

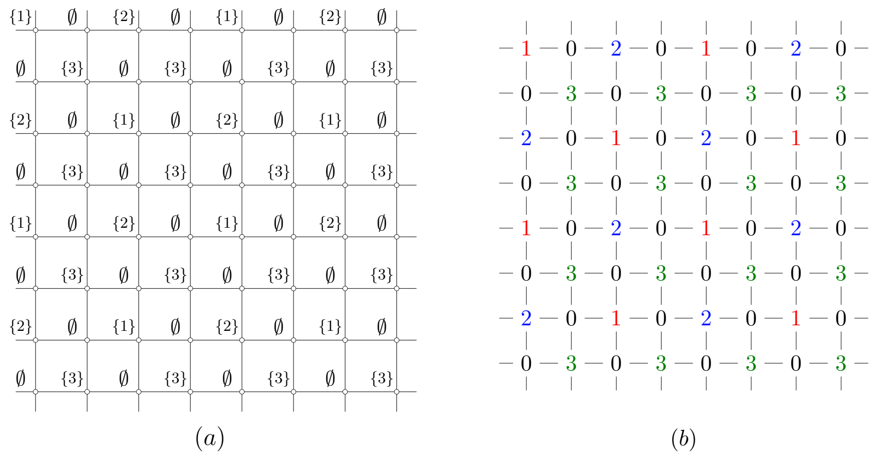


Figure 2. A 3RDF on $C_8 \square C_8$. (**a**) Vertex labeled with color sets. (**b**) Vertex labeled with codes.

By symmetry of $C_n \square C_m$, we can only discuss the cases of $n(\text{mod } 4) \geq m(\text{mod } 4)$.

Lemma 2. *For $n \not\equiv 2(\text{mod } 4)$ and $m \not\equiv 2(\text{mod } 4)$, $\gamma_{r3}(C_n \square C_m) \leq \lceil \frac{mn}{2} \rceil$.*

Proof. We define a 3RDF f as follows.

$$f(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i(\text{mod } 2) = j(\text{mod } 2) = 0 \wedge i(\text{mod } 4) = j(\text{mod } 4), \\ \{2\}, & i(\text{mod } 4) + j(\text{mod } 4) = 2 \wedge i(\text{mod } 4) \neq 1, \\ \{3\}, & \textit{otherwise}. \end{cases}$$

Figure 3 shows f on $C_8 \square C_8$, $C_9 \square C_8$, $C_9 \square C_9$, $C_{11} \square C_8$, $C_{11} \square C_9$, and $C_{11} \square C_{11}$, where R_m means we repeat the four columns with m becoming bigger and R_n means we repeat the four rows with n becoming bigger.

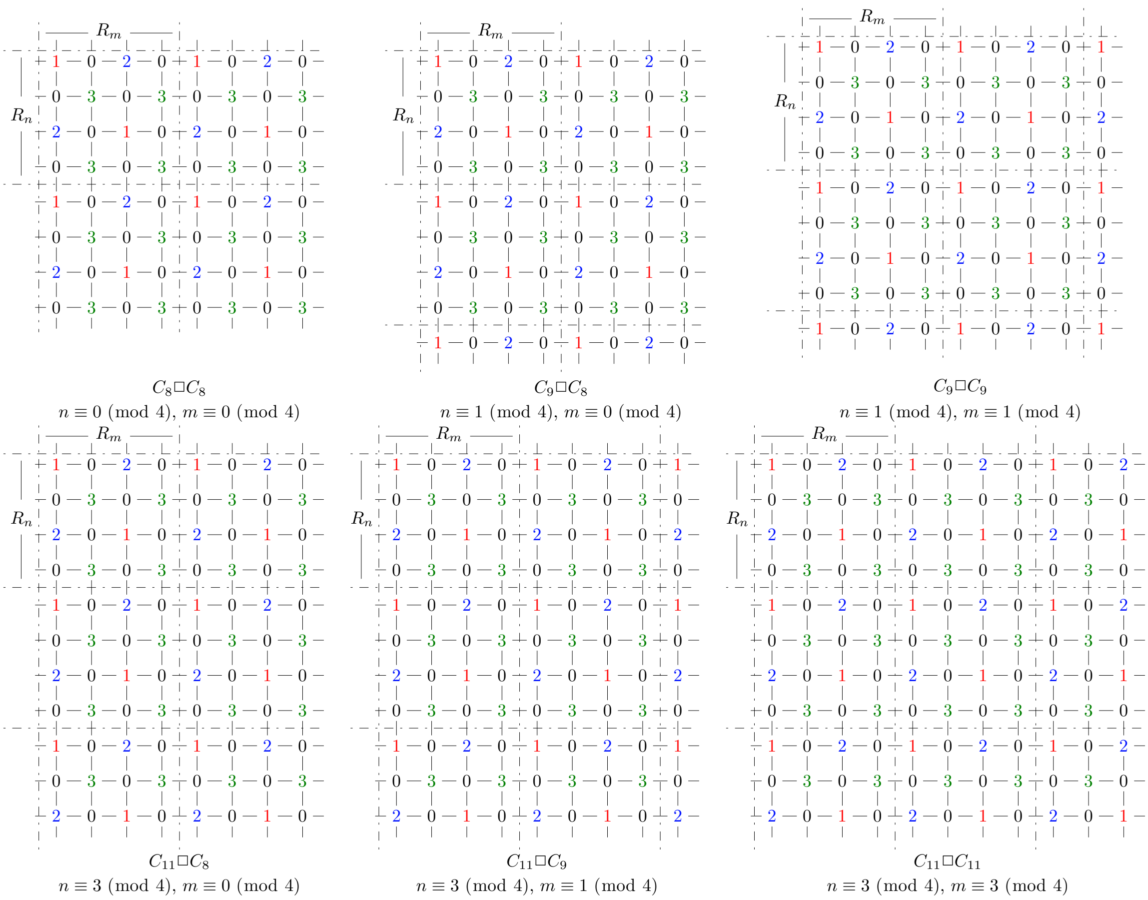


Figure 3. f on some $C_n \square C_m$ for $n \not\equiv 2 (\bmod\ 4) \wedge m \not\equiv 2 (\bmod\ 4)$.

One can check that f is a 3RDF, and its weight is shown in Table 1.

Table 1. The weight of f on $C_n \square C_m$ ($n \not\equiv 2 (\bmod\ 4) \wedge m \not\equiv 2 (\bmod\ 4)$).

$n \not\equiv 2 (\bmod\ 4) \wedge m \not\equiv 2 (\bmod\ 4)$	The Weight of f
$n \equiv 0 \pmod 4, m \equiv 0 \pmod 4$	$\omega(f) = \frac{m}{4} \times \frac{n}{4} \times 8 = \frac{mn}{2}$
$n \equiv 1 \pmod 4, m \equiv 0 \pmod 4$	$\omega(f) = \frac{m}{4} \times \frac{n-1}{4} \times 8 + \frac{m}{4} \times 2 = \frac{mn}{2}$
$n \equiv 1 \pmod 4, m \equiv 1 \pmod 4$	$\omega(f) = \frac{m-1}{4} \times \frac{n-1}{4} \times 8 + \frac{n-1}{4} \times 2 + \frac{m-1}{4} \times 2 + 1 = \lceil \frac{mn}{2} \rceil$
$n \equiv 3 \pmod 4, m \equiv 0 \pmod 4$	$\omega(f) = \frac{m}{4} \times \frac{n-3}{4} \times 8 + \frac{m}{4} \times 6 = \frac{mn}{2}$
$n \equiv 3 \pmod 4, m \equiv 1 \pmod 4$	$\omega(f) = \frac{m-1}{4} \times \frac{n-3}{4} \times 8 + \frac{n-3}{4} \times 2 + \frac{m-1}{4} \times 6 + 2 = \lceil \frac{mn}{2} \rceil$
$n \equiv 3 \pmod 4, m \equiv 3 \pmod 4$	$\omega(f) = \frac{m-3}{4} \times \frac{n-3}{4} \times 8 + \frac{n-3}{4} \times 6 + \frac{m-3}{4} \times 6 + 5 = \lceil \frac{mn}{2} \rceil$

Hence, $\gamma_{r3}(C_n \square C_m) \leq \lceil \frac{mn}{2} \rceil$ for $n \not\equiv 2 (\bmod\ 4)$ and $m \not\equiv 2 (\bmod\ 4)$. $\square$

Lemma 3. *For $n \equiv 2(\bmod 4)$, $m \equiv 0(\bmod 4)$, $\gamma_{r3}(C_n \Box C_m) \leq \frac{mn}{2}$.*

Proof. We first define a 3RDF g on $C_4 \Box C_4$.

$$g(v_{i,j}) = \begin{cases} \emptyset, & i(\bmod 2) \neq j(\bmod 2), \\ \{1\}, & i = j \wedge i \neq 1, \\ \{2\}, & i + j = 2 \wedge i \neq 1, \\ \{3\}, & i = 1 \wedge j(\bmod 2) = 1 \vee i = 3 \wedge j = 1. \end{cases}$$

Then, for $C_n \Box C_m$, $n \equiv 2(\bmod 4)$, $m \equiv 0(\bmod 4)$, we define a 3RDF f as follows.

$$f(v_{i,j}) = \begin{cases} g(v_{i(\bmod 4),j(\bmod 4)}), & 0 \leq i \leq n-3, \\ h(v_{i,j(\bmod 4)}), & n-2 \leq i \leq n-1, \end{cases}$$

where h is a function defined on $\{v_{i,j} | n-2 \leq i \leq n-1, 0 \leq j \leq 3\}$,

$$h(v_{i,j}) = \begin{cases} \emptyset, & i(\bmod 2) \neq j(\bmod 2), \\ \{1\}, & i = n-1 \wedge j = 1, \\ \{2\}, & i = n-2 \wedge j = 0, 2, \\ \{3\}, & i = n-1 \wedge j = 3. \end{cases}$$

Figure 4 shows f on $C_{10} \Box C_8$, where R_m means we repeat the four columns with m becoming bigger and R_n means we repeat the four rows with n becoming bigger.

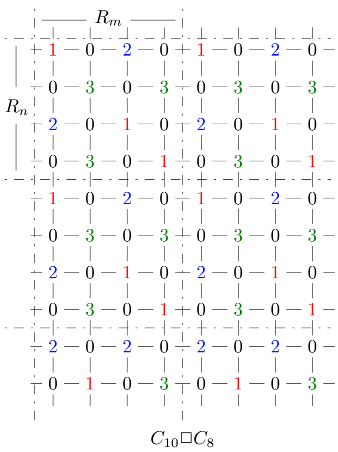


Figure 4. f on $C_{10} \Box C_8$.

One can check that f is a 3RDF, and the weight of f is $\omega(f) = \frac{m}{4} \times \frac{n-2}{4} \times 8 + \frac{m}{4} \times 4 = \frac{mn}{2}$. Hence, $\gamma_{r3}(C_n \Box C_m) \leq \frac{mn}{2}$ for $n \equiv 2(\bmod 4)$, $m \equiv 0(\bmod 4)$. □

Lemma 4. *For $n \equiv 2(\bmod 4)$, $m \equiv 1(\bmod 4)$, $\gamma_{r3}(C_n \Box C_m) \leq \frac{mn}{2}$.*

Proof. **Case 1.** $m = 5$.

Case 1.1. For $n \equiv 2(\bmod 12)$, we first define a 3RDF g_1 on $C_6 \Box C_5$,

$$g_1(v_{i,j}) = \begin{cases} \emptyset, & i(\bmod 2) \neq j(\bmod 2), \\ \{1\}, & i = 0 \wedge j(\bmod 2) = 0 \vee i = 3 \wedge j(\bmod 2) = 1, \\ \{2\}, & i = 1 \wedge j = 1 \vee i = 2 \wedge j = 2, 4 \vee i = 4 \wedge j = 0 \vee i = 5 \wedge j = 3, \\ \{3\}, & i = 1 \wedge j = 3 \vee i = 2 \wedge j = 0 \vee i = 4 \wedge j = 2, 4 \vee i = 5 \wedge j = 1. \end{cases}$$

Then, we construct a 3RDF f on $C_n \Box C_5$,

$$f(v_{i,j}) = \begin{cases} g_1(v_{i(\bmod 6),j}), & 0 \leq i \leq n-7, \\ h(v_{i,j}), & n-6 \leq i \leq n-1, \end{cases}$$

where h is a function defined on $\{v_{i,j} | n-6 \le i \le n-1, 0 \le j \le 4\}$,

$$h(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = n-2, n-6 \wedge j = 2 \vee i = n-4 \wedge j(\text{mod } 2) = 0, \\ \{2\}, & i = n-1, n-3 \wedge j = 3 \vee i = n-2 \wedge j = 0 \vee i = n-5 \\ & \wedge j = 1 \vee i = n-6 \wedge j = 4, \\ \{3\}, & otherwise. \end{cases}$$

Case 1.2. For $n \equiv 6(\text{mod } 12)$, define $f(v_{i,j}) = g_1(v_{i(\text{ mod } 6),j})$.
Case 1.3. For $n \equiv 10(\text{mod } 12)$, we first define g_2 on $C_6 \Box C_5$,

$$g_2(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = 0 \wedge j = 0, 2 \vee i = 2 \wedge j = 4 \vee i = 3 \wedge j = 1 \vee i = 5 \wedge j = 1, 3 \\ \{2\}, & i = 0 \wedge j = 4 \vee i = 1 \wedge j = 1 \vee i = 3 \wedge j = 3 \vee i = 4 \wedge j = 0, \\ \{3\}, & i = 1 \wedge j = 3 \vee i = 2 \wedge j = 0, 2 \vee i = 4 \wedge j = 2, 4. \end{cases}$$

Then, we define f as follows.

$$f(v_{i,j}) = \begin{cases} g_2(v_{i(\text{ mod } 6),j}), & 0 \le i \le n-7, \\ h(v_{i,j}), & n-6 \le i \le n-1, \end{cases}$$

where h is a function defined on $\{v_{i,j} | n-6 \le i \le n-1, 0 \le j \le 4\}$,

$$h(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = n-2 \wedge j = 4 \vee i = n-3 \wedge j = 1 \vee i = n-5 \wedge j = 1, 3, \\ \{2\}, & i = n-2 \wedge j = 0, 2 \vee i = n-4 \wedge j = 2, 4 \vee i = n-6 \wedge j = 0, \\ \{3\}, & i = n-1 \wedge j = 1, 3 \vee i = n-3 \wedge j = 3 \vee i = n-4 \wedge j = 0 \\ & \vee i = n-6 \wedge j = 0. \end{cases}$$

Figure 5 shows f on $C_{26} \Box C_5$, $C_{18} \Box C_5$, and $C_{22} \Box C_5$, where R_n means we repeat the six rows with n becoming bigger.

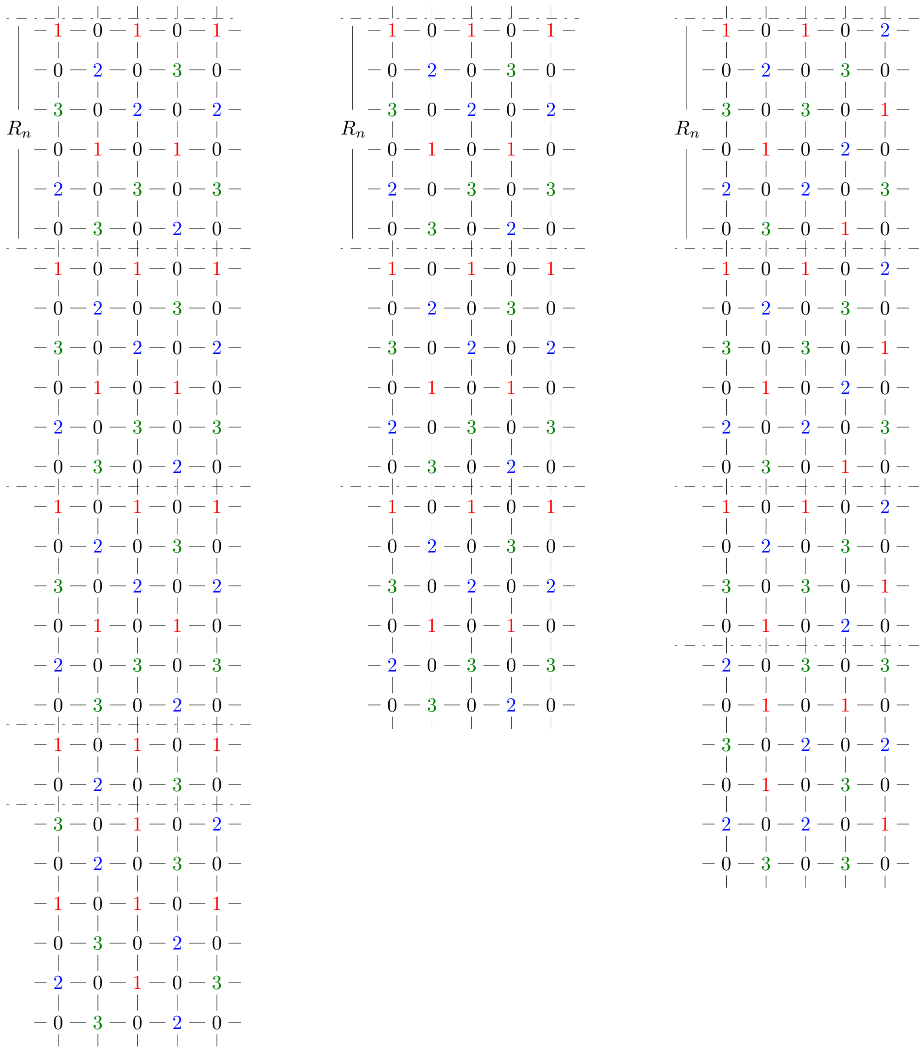
One can check that f is a 3RDF, and its weight is shown in Table 2.

Table 2. The weight of f on $C_n \Box C_5$ $(n \equiv 2(\text{mod } 4))$.

$m = 5 \wedge n \equiv 2 \ (\text{mod } 4)$	The Weight of f
$m = 5, n \equiv 2 \ (\text{mod } 12)$	$\omega(f) = \frac{15 \times (n-8)}{6} + \frac{5 \times 8}{2} = \frac{5n}{2}$
$m = 5, n \equiv 6 \ (\text{mod } 12)$	$\omega(f) = \frac{15 \times n}{6} = \frac{5n}{2}$
$m = 5, n \equiv 10 \ (\text{mod } 12)$	$\omega(f) = \frac{15 \times (n-10)}{6} + \frac{5 \times 10}{2} = \frac{5n}{2}$

Case 2. For $m \ge 9$, we first define a function g on $C_4 \Box C_4$.

$$g(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i(\text{mod } 2) = j(\text{mod } 2) = 0 \wedge i = j, \\ \{2\}, & i + j = 2 \wedge i \neq 1 \vee i = j = 3, \\ \{3\}, & i = 1 \wedge j(\text{mod } 2) = 1 \vee i = 3 \wedge j = 1. \end{cases}$$



$C_{26}\square C_5$ $n \equiv 2 \pmod{12}$ $C_{18}\square C_5$ $n \equiv 6 \pmod{12}$ $C_{22}\square C_5$ $n \equiv 10 \pmod{12}$

Figure 5. f on some $C_n \square C_5$ for $n \equiv 2(\text{mod } 4)$.

Then, we define f as follows.

$$f(v_{i,j}) = \begin{cases} g(v_{i(\text{ mod } 4),j(\text{ mod } 4)}), & 0 \le i \le n-4 \wedge 0 \le j \le m-6, \\ h_1(v_{i(\text{ mod } 4),j}), & 0 \le i \le n-4 \wedge m-5 \le j \le m-1, \\ h_2(v_{i,j(\text{ mod } 4)}), & n-3 \le i \le n-1 \wedge 0 \le j \le m-6, \\ h_3(v_{i,j}), & n-3 \le i \le n-1 \wedge m-5 \le j \le m-1, \end{cases}$$

where $h_1(v_{i,j})$ $(\{v_{i,j} | 0 \le i \le 4, m-5 \le j \le m-1\})$, $h_2(v_{i,j})$ $(\{v_{i,j} | n-3 \le i \le n-1, 0 \le j \le 3\})$, $h_3(v_{i,j})$ $(\{v_{i,j} | n-3 \le i \le n-1, m-5 \le j \le m-1\})$ are defined as follows.

$$h_1(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = 0,2 \wedge j = m-5 \vee i = 1,3 \wedge j = m-2, \\ \{2\}, & i = 0 \wedge j = m-1 \vee i = 1 \wedge j = m-4 \vee i = 2 \wedge j = m-3, \\ \{3\}, & \textit{otherwise}. \end{cases}$$

$$h_2(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = n-1 \wedge j = 3 \vee i = n-3 \wedge j = 1, \\ \{2\}, & i = n-1 \wedge j = 1 \vee i = n-3 \wedge j = 3, \\ \{3\}, & \textit{otherwise}. \end{cases}$$

$$h_3(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = n-2 \wedge j = m-1, m-3, \\ \{2\}, & i = n-1 \wedge j = m-4 \vee i = n-3 \wedge j = m-2, \\ \{3\}, & \textit{otherwise}. \end{cases}$$

Figure 6 shows f on $C_{18}\square C_{17}$, where R_m means we repeat the four columns with m becoming bigger and R_n means we repeat the four rows with n becoming bigger.

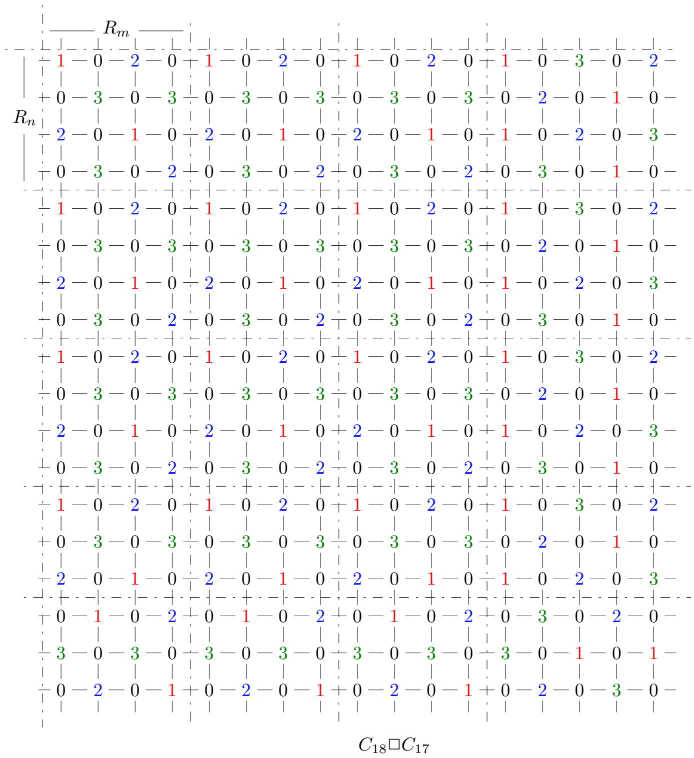


Figure 6. f on $C_{18}\square C_{17}$.

One can check that f is a 3RDF, and the weight of f is $\omega(f) = \frac{n-6}{4} \times \frac{m-5}{4} \times 8 + \frac{m-5}{4} \times 12 + \frac{n-6}{4} \times 10 + 15 = \frac{mn}{2}$.

Hence, $\gamma_{r3}(C_n\square C_m) \leq \frac{mn}{2}$ for $n \equiv 2(\text{mod } 4)$, $m \equiv 1(\text{mod } 4)$. $\square$

Lemma 5. *For* $n \equiv 2(\text{mod } 4)$, $m \equiv 2(\text{mod } 4)$, $\gamma_{r3}(C_n\square C_m) \leq \frac{mn}{2}$.

Proof. We first define a function g on $C_4\square C_4$.

$$g(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i(\text{mod } 2) = j(\text{mod } 2) = 0 \wedge i = j, \\ \{2\}, & i + j = 2 \wedge i \neq 1 \vee i = j = 3, \\ \{3\}, & i = 1 \wedge j(\text{mod } 2) = 1 \vee i = 3 \wedge j = 1. \end{cases}$$

Then, for $n \equiv 2(\text{mod } 4)$, $m \equiv 2(\text{mod } 4)$, we define a 3RDF f on $C_n\square C_m$ as follows.

$$f(v_{i,j}) = \begin{cases} g(v_{i(\text{ mod } 4),j(\text{ mod } 4)}), & 0 \leq i \leq n-3 \wedge 0 \leq j \leq m-3, \\ h_1(v_{i(\text{ mod } 4),j}), & 0 \leq i \leq n-3 \wedge m-2 \leq j \leq m-1, \\ h_2(v_{i,j(\text{ mod } 4)}), & n-2 \leq i \leq n-1 \wedge 0 \leq j \leq m-3, \\ h_3(v_{i,j}), & n-2 \leq i \leq n-1 \wedge m-2 \leq j \leq m-1, \end{cases}$$

where $h_1(v_{i,j})$ $(\{v_{i,j}|0 \le i \le 4, m-2 \le j \le m-1\})$, $h_2(v_{i,j})$ $(\{v_{i,j}|n-2 \le i \le n-1, 0 \le j \le 3\})$, $h_3(v_{i,j})$ $(\{v_{i,j}|n-2 \le i \le n-1, m-2 \le j \le m-1\})$ are defined as follows.

$$h_1(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = 0,2 \wedge j = m-2, \\ \{2\}, & i = 1 \wedge j = m-1, \\ \{3\}, & i = 3 \wedge j = m-1. \end{cases}$$

$$h_2(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = n-2 \wedge j = 0,2, \\ \{2\}, & i = n-1 \wedge j = 1, \\ \{3\}, & i = n-1 \wedge j = 3. \end{cases}$$

$$h_3(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{2\}, & i = n-2 \wedge j = m-2, \\ \{3\}, & i = n-1 \wedge j = m-1. \end{cases}$$

Figure 7 shows f on $C_{10}\square C_{10}$, where R_m means we repeat the four columns with m becoming bigger and R_n means we repeat the four rows with n becoming bigger.

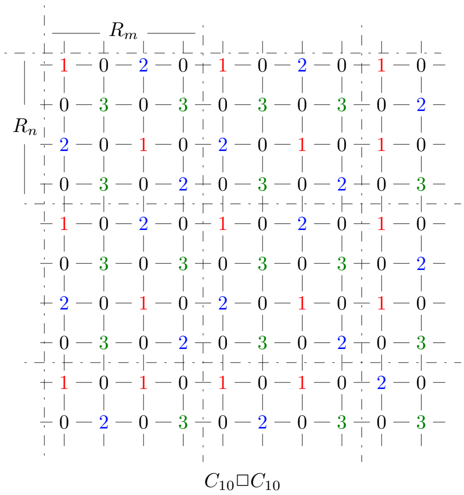


Figure 7. f on $C_{10}\square C_{10}$.

One can check that f is a 3RDF, and the weight of f is $\omega(f) = \frac{m-2}{4} \times \frac{n-2}{4} \times 8 + \frac{m-2}{4} \times 4 + \frac{n-2}{4} \times 4 + 2 = \frac{mn}{2}$.

Hence, $\gamma_{r3}(C_n\square C_m) \le \frac{mn}{2}$ for $n \equiv 2(\text{mod } 4)$, $m \equiv 2(\text{mod} 4)$. $\square$

Lemma 6. *For $n \equiv 3(\text{mod } 4)$, $m = 2(\text{mod } 4)$,*

$$\gamma_{r3}(C_n\square C_m) \le \begin{cases} \frac{3m+2}{2}, & n = 3, m \equiv 2, 10(\text{mod } 12), \\ \frac{mn}{2}, & \textit{otherwise}. \end{cases}$$

Proof. **Case 1.** For $n = 3$, we define f in two subcases.

Case 1.1. $m \equiv 6(\text{mod } 12)$.

$$f(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i(\text{mod } 2) = 0 \wedge j(\text{mod } 6) = 0 \vee i = 1 \wedge j(\text{mod } 6) = 3, \\ \{2\}, & i(\text{mod } 2) = 0 \wedge j(\text{mod } 6) = 2 \vee i = 1 \wedge j(\text{mod } 6) = 5, \\ \{3\}, & \textit{otherwise}. \end{cases}$$

Figure 8 (above) shows f on $C_3\square C_{18}$, where R_m means we repeat the six columns with m becoming bigger. One can check that f is a 3RDF. The weight of f is $\omega(f)=\frac{m}{6}\times 9=\frac{3m}{2}$. Hence, $\gamma_{r3}(C_3\square C_m)\geq \frac{3m}{2}$ for $m\equiv 6(\bmod 12)$.

Case 1.2. $m\equiv 2,10(\bmod\ 12)$. We first define a function g on $C_3\square C_4$.

$$g(v_{i,j})=\begin{cases}\emptyset, & i(\bmod\ 2)\neq j(\bmod\ 2),\\ \{1\}, & i=j=0\vee i=j=2,\\ \{2\}, & i+j=2\wedge i\neq 1,\\ \{3\}, & i=1\wedge j=1,3.\end{cases}$$

Then, we define f as follows.

$$f(v_{i,j})=\begin{cases} g(v_{i,j(\bmod 4)}), & 0\leq j\leq m-4,\\ h(v_{i,j}), & m-3\leq j\leq m-1,\end{cases}$$

where h is a function defined on $\{v_{i,j}|0\leq i\leq 2\wedge m-3\leq j\leq m-1\}$,

$$h(v_{i,j})=\begin{cases}\emptyset, & i=0\wedge j=m-3,m-1\vee i=1\wedge j=m-2\vee i=2\wedge j=m-1,\\ \{1\}, & i=2\wedge j\neq m-1,\\ \{2\}, & i=0\wedge j=m-2,\\ \{3\}, & i=1\wedge j\neq m-2.\end{cases}$$

Figure 8 (below) shows f on $C_3\square C_{14}$, where R_m means we repeat the four columns with m becoming bigger. One can check that f is a 3RDF. The weight of f is f is $\omega(f)=\frac{m-6}{4}\times 6+10=\frac{3m+2}{2}$. Hence, $\gamma_{r3}(C_3\square C_m)\leq\frac{3m+2}{2}$ for $m\not\equiv 6(\bmod 12)$.

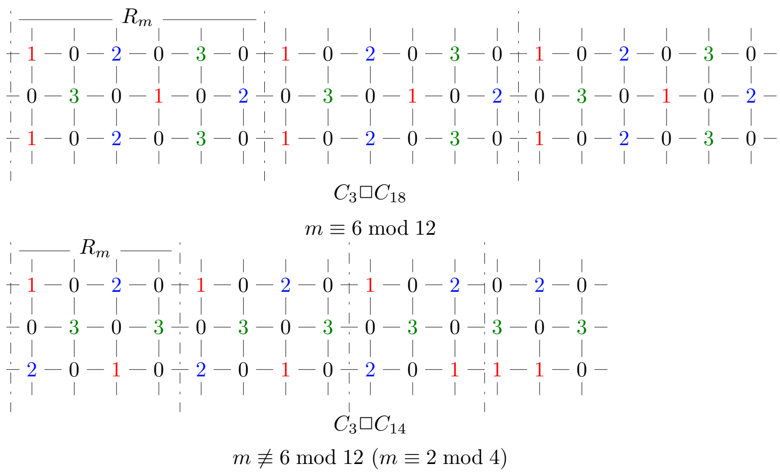


Figure 8. 3RDFs on $C_3\square C_{18}$ and $C_3\square C_{14}$.

Hence, for $m\equiv 2(\bmod\ 4)$,

$$\gamma_{r3}(C_3\square C_m)\leq\begin{cases}\frac{3m}{2}, & m\equiv 6(\bmod\ 12),\\ \frac{3m+2}{2}, & m\not\equiv 6(\bmod\ 12).\end{cases}$$

Case 2. For $n=7$, we define f in two subcases.

Case 2.1. $m\equiv 6(\bmod\ 12)$.

$$f(v_{i,j})=\begin{cases}\emptyset, & i(\bmod\ 2)\neq j(\bmod\ 2),\\ \{1\}, & i(\bmod\ 2)=0\wedge j(\bmod\ 6)=0\vee i(\bmod\ 2)=1\wedge j(\bmod\ 6)=3,\\ \{2\}, & i(\bmod\ 2)=0\wedge j(\bmod\ 6)=2\vee i(\bmod\ 2)=1\wedge j(\bmod\ 6)=5,\\ \{3\}, & \textit{otherwise}.\end{cases}$$

Figure 9 (above) shows f on $C_7 \square C_{18}$, where R_m means that we repeat the six columns with m becoming bigger and R_n means we repeat the two rows with n becoming bigger. One can check that f is a 3RDF, and the weight of f is $\omega(f) = \frac{7-1}{2} \times \frac{m}{6} \times 6 + \frac{m}{6} \times 3 = \frac{7m}{2}$.

Case 2.2. $m \equiv 2, 10 (\text{mod } 12)$. We first define a function g on $C_7 \square C_4$.

$$g(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{1\}, & i = 0 \wedge j = 0 \vee i = 2 \wedge j = 2 \vee i = 3 \wedge j = 1 \vee i = 5 \wedge j = 1, 3, \\ \{2\}, & i = 0 \wedge j = 2 \vee i = 2 \wedge j = 0 \vee i = 4 \wedge j = 2 \vee i = 6 \wedge j = 0, \\ \{3\}, & \textit{otherwise}. \end{cases}$$

Then, we define f as follows.

$$f(v_{i,j}) = \begin{cases} g(v_{i,j(\text{ mod } 4)}), & 0 \leq j \leq m - 7, \\ h(v_{i,j}), & m - 6 \leq j \leq m - 1, \end{cases}$$

where h is a function defined on $\{v_{i,j} | 0 \leq i \leq 6 \wedge m - 6 \leq j \leq m - 1\}$,

$$h(v_{i,j}) = \begin{cases} \emptyset, & i(\text{mod } 2) \neq j(\text{mod } 2), \\ \{2\}, & i(\text{mod } 2) = 0 \wedge i \neq 0 \wedge j = m - 6 \vee i = 0, 2 \wedge j = m - 4 \\ & \vee i = 5 \wedge j = m - 3 \vee i = 4 \wedge j = m - 2 \vee i = 1 \wedge j = m - 1, \\ \{3\}, & i(\text{mod } 2) = 1 \wedge i \neq 3 \wedge j = m - 5 \vee i = 3 \wedge j = m - 1, m - 3 \\ & \vee i = 0, 6 \wedge j = m - 2, \\ \{1\}, & \textit{otherwise}. \end{cases}$$

Figure 9 (below) shows f on $C_7 \square C_{22}$, where R_m means we repeat the four columns with m becoming bigger. One can check that f is a 3RDF, and the weight of f is $\omega(f) = \frac{m-6}{4} \times 14 + 21 = \frac{7m}{2}$.

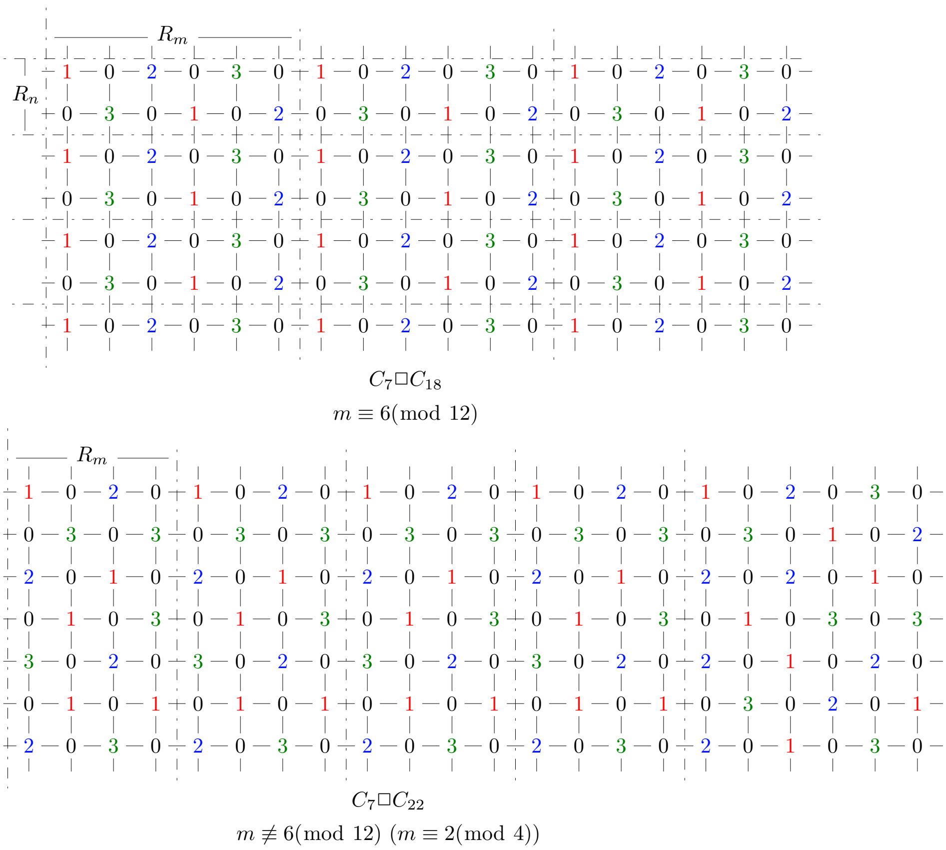


Figure 9. 3RDFs on $C_7 \square C_{18}$ and $C_7 \square C_{22}$.

Hence, $\gamma_{r3}(C_7 \square C_m) \leq \frac{7m}{2}$, $m \equiv 2(\text{mod } 4)$.

Case 3. $n \geq 8$.

We first define a function g on $C_4 \square C_4$.

$$g(v_{i,j}) = \begin{cases} \emptyset, & i(\bmod 2) \neq j(\bmod 2), \\ \{1\}, & i = j \wedge i \neq 1, \\ \{2\}, & i + j = 2 \wedge i \neq 1, \\ \{3\}, & otherwise. \end{cases}$$

Then, we define f as follows.

$$f(v_{i,j}) = \begin{cases} g(v_{i(\bmod 4), j(\bmod 4)}), & 0 \leq i \leq n-7 \wedge 0 \leq j \leq m-4, \\ h_1(v_{i(\bmod 4),j}), & 0 \leq i \leq n-7 \wedge m-3 \leq j \leq m-1, \\ h_2(v_{i,j(\bmod 4)}), & n-6 \leq i \leq n-1 \wedge 0 \leq j \leq m-4, \\ h_3(v_{i,j}), & n-6 \leq i \leq n-1 \wedge m-3 \leq j \leq m-1, \end{cases}$$

where $h_1(v_{i,j})$ $(\{v_{i,j} | 0 \leq i \leq 3, m-3 \leq j \leq m-1\})$, $h_2(v_{i,j})$ $(\{v_{i,j} | n-6 \leq i \leq n-1, 0 \leq j \leq 3\})$, $h_3(v_{i,j})$ $(\{v_{i,j} | n-6 \leq i \leq n-1, m-3 \leq j \leq m-1\})$ are defined as follows.

$$h_1(v_{i,j}) = \begin{cases} \emptyset, & i(\bmod 2) \neq j(\bmod 2), \\ \{1\}, & i = 1 \wedge j = m-3 \vee i = 3 \wedge j = m-1, \\ \{2\}, & i = 1 \wedge j = m-1 \vee i = 3 \wedge j = m-3, \\ \{3\}, & otherwise. \end{cases}$$

$$h_2(v_{i,j}) = \begin{cases} \emptyset, & i(\bmod 2) \neq j(\bmod 2), \\ \{1\}, & i = n-6 \wedge j = 1 \vee i = n-4 \wedge j = 3 \\ & \vee i = n-3 \wedge j = 2 \vee i = n-1 \wedge j = 0, \\ \{2\}, & i = n-6 \wedge j = 3 \vee i = n-4 \wedge j = 1 \\ & \vee i = n-3 \wedge j = 0 \vee i = n-1 \wedge j = 2, \\ \{3\}, & i = n-2 \wedge j(\bmod 2) = 1 \vee i = n-5 \wedge j(\bmod 2) = 0. \end{cases}$$

$$h_3(v_{i,j}) = \begin{cases} \emptyset, & i(\bmod 2) \neq j(\bmod 2), \\ \{1\}, & i = n-2, n-6 \wedge j = m-3 \\ & \vee i = n-5 \wedge j = m-2 \vee i = n-4 \wedge j = m-1, \\ \{2\}, & i = n-2, n-6 \wedge j = m-1 \\ & \vee i = n-4 \wedge j = m-3, \\ \{3\}, & i = n-1, n-3 \wedge j = m-2. \end{cases}$$

Figure 10 shows f on $C_{15} \square C_{14}$, where R_m means we repeat the four columns with m becoming bigger and R_n means we repeat the four rows with n becoming bigger.

One can check that f is a 3RDF, and the weight of f is $\omega(f) = \frac{m-6}{4} \times \frac{n-7}{4} \times 8 + \frac{m-6}{4} \times 14 + \frac{n-7}{4} \times 12 + 21 = \frac{mn}{2}$.

Hence, $\gamma_{r3}(C_n \square C_m) \leq \frac{mn}{2}$ for $n \equiv 3(\bmod 4)$ $(n > 7)$, $m \equiv 2(\bmod 4)$. $\square$

By Lemmas 2–6, we have:

Theorem 3.

$$\gamma_{r3}(C_n \square C_m) \leq \begin{cases} \frac{3m+2}{2}, & n = 3 \wedge m \equiv 2, 10(\bmod 12), \\ \lceil \frac{mn}{2} \rceil, & otherwise. \end{cases}$$

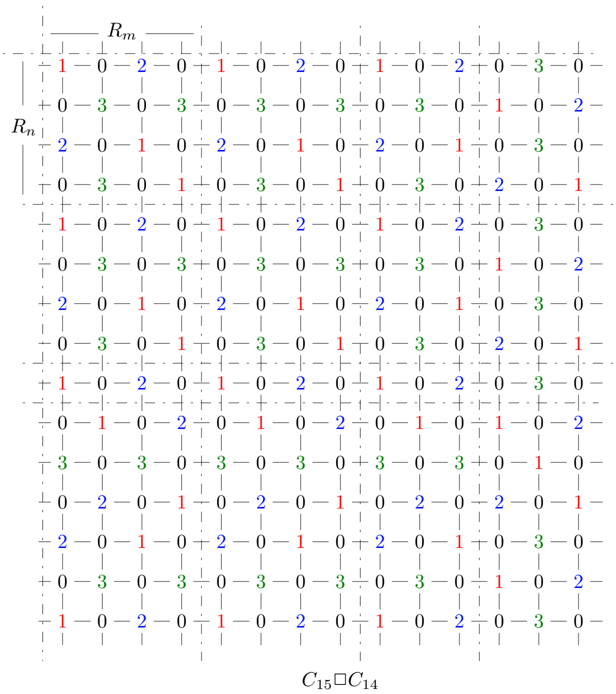


Figure 10. f on $C_{15}\Box C_{14}$.

3. The 3-rainbow Domination Number of $C_3\Box C_m$

Let f be an arbitrary 3RDF on $C_3\Box C_m$; we denote $\omega(f_j) = |f(v_{0,j})| + |f(v_{1,j})| + |f(v_{2,j})|$ $(0 \le j \le m-1)$.

Lemma 7. *Let f be a 3-rainbow dominating function on $C_3\Box C_m$. Then:*

(1) *if $\omega(f_j) = 0$, then $\omega(f_{j-1}) + \omega(f_{j+1}) \ge 9$;*
(2) *if $\omega(f_j) = 1$, then $\omega(f_{j-1}) + \omega(f_{j+1}) \ge 4$.*

Proof. (1) Since $\omega(f_j) = 0$, then $|f(v_{i,j-1})| + |f(v_{i,j+1})| \ge 3$ $(i = 0, 1, 2)$. It follows that $\omega(f_{j-1}) + \omega(f_{j+1}) \ge 9$.

(2) Since $\omega(f_j) = 1$, without loss of generality, let $|f(v_{0,j})| = 1$, then $|f(v_{i,j-1})| + |f(v_{i,j+1})| \ge 2$ $(i = 1, 2)$. It follows that $\omega(f_{j-1}) + \omega(f_{j+1}) \ge 4$.
□

Theorem 4. *For $m = 3, 4$, $\gamma_{r3}(C_3\Box C_m) = \lceil \frac{3m}{2} \rceil$.*

Proof. By Theorem 2, $\gamma_{r3}(C_3\Box C_m) \le \lceil \frac{3m}{2} \rceil$ $(m = 3, 4)$.

If there exists $\omega(f_j) = 0$, by Lemma 7 (1), it follows that $\omega(f) \ge \omega(f_{j-1}) + \omega(f_{j+1}) \ge 9 \ge \lceil \frac{3m}{2} \rceil$.

If $\omega(f_j) \ge 1$ for $0 \le j \le m-1$, and if there exists $\omega(f_j) = 1$, then by Lemma 7 (2), $\omega(f) = \omega(f_{j-1}) + \omega(f_{j+1}) + \omega(f_j) \ge 4 + 1 = 5 = \lceil \frac{3m}{2} \rceil$ for $m = 3$, and $\omega(f) = \omega(f_{j-1}) + \omega(f_{j+1}) + \omega(f_j) + \omega(f_{j+2}) \ge 4 + 1 + 1 = 6 = \lceil \frac{3m}{2} \rceil$ for $m = 4$.

If $\omega(f_j) \ge 2$ for $0 \le j \le m-1$, then $\omega(f) = \sum_{0 \le j \le m-1} \omega(f_j) \ge 2 \times m = 2m$. Thus, $\gamma_{r3}(C_3\Box C_m) \ge \lceil \frac{3m}{2} \rceil$ for $m = 3, 4$.
□

Lemma 8. *Let f be a $\gamma_{r3}(C_3\Box C_m)$-function $(m \ge 5)$, then $\omega(f_j) \ge 1$ for $0 \le j \le m-1$.*

Proof. By contrast, suppose f is an arbitrary γ_{r3}-function and there exists a j with $\omega(f_j) = 0$; by Lemma 7 (1), we have $\omega(f_{j-1}) + \omega(f_j) + \omega(f_{j+1}) \ge 9$.

We construct a function f' as follows, and Figure 11 shows the sketch of f'.

$$f'(v_{i,t}) = \begin{cases} \emptyset, & i = 0,2 \wedge t = j-1, j+1 \vee i = 1 \wedge t = j, \\ \{1\}, & i = 2 \wedge t = j, \\ \{2\}, & i = 0 \wedge t = j, \\ \{3\}, & i = 1 \wedge t = j-1, j+1, \\ \{1\} \cup f(v_{i,t}), & i = 0 \wedge t = j-2, j+2, \\ \{2\} \cup f(v_{i,t}), & i = 2 \wedge t = j-2, j+2, \\ f(v_{i,t}), & \textit{otherwise}. \end{cases}$$

$$\begin{array}{ccccccc} j-3 & j-2 & j-1 & j & j+1 & j+2 & j+3 \\ f(v_{0,j-3}) & \{1\} \cup f(v_{0,j-2}) & \emptyset & \{2\} & \emptyset & \{1\} \cup f(v_{0,j+2}) & f(v_{0,j+3}) \\ f(v_{1,j-3}) & f(v_{1,j-2}) & \{3\} & \emptyset & \{3\} & f(v_{1,j+2}) & f(v_{1,j+3}) \\ f(v_{2,j-3}) & \{2\} \cup f(v_{2,j-2}) & \emptyset & \{1\} & \emptyset & \{2\} \cup f(v_{2,j+2}) & f(v_{2,j+3}) \end{array}$$

Figure 11. The sketch of f' in Lemma 8.

Thus, $\omega(f') = \omega(f) - 9 + 8 < \omega(f)$, a contradiction with that f being a γ_{r3}-function. □

Theorem 5. *For $m \geq 5$,*

$$\gamma_{r3}(C_3 \square C_m) \geq \begin{cases} \frac{3m+2}{2}, & m \equiv 2, 10 (\bmod\ 12), \\ \lceil \frac{3m}{2} \rceil, & \textit{otherwise}. \end{cases}$$

Proof. First, we prove $\gamma_{r3}(C_3 \square C_m) \geq \lceil \frac{3m}{2} \rceil$ for $m \geq 5$.

Let f be a $\gamma_{r3}(C_3 \square C_m)$-function. By Lemma 8, $\omega(f_j) \geq 1$ for $0 \leq j \leq m-1$.

For $\omega(f_j) = 1$, by Lemma 7 (2), $\omega(f_{j-1}) + \omega(f_j) + \omega(f_j) + \omega(f_{j+1}) \geq 1 \times 2 + 4 = 6$.

For $\omega(f_j) = 2$, $\omega(f_{j-1}) + \omega(f_j) + \omega(f_j) + \omega(f_{j+1}) \geq 1 + 2 \times 2 + 1 = 6$.

For $\omega(f_j) \geq 3$, $\omega(f_{j-1}) + \omega(f_j) + \omega(f_j) + \omega(f_{j+1}) \geq 1 + 2 \times 3 + 1 = 8 > 6$.

Hence,

$$\begin{aligned} 4\omega(f) &= 4 \textstyle\sum_{0 \leq j \leq m-1} \omega(f_j) \\ &= \textstyle\sum_{0 \leq j \leq m-1} \omega(f_{j-1}) + \omega(f_j) + \omega(f_j) + \omega(f_{j+1}) \\ &\geq 6m. \end{aligned} \tag{3}$$

Thus, $\omega(f) \geq \lceil \frac{3m}{2} \rceil$.

Then, we prove for $m \equiv 2, 10 (\bmod\ 12)$ that the lower bounds of $\gamma_{r3}(C_3 \square C_m)$ can be improved to $\frac{3m+2}{2}$ instead of $\lceil \frac{3m}{2} \rceil$.

If there exists $\omega(f_j) \geq 3$, or $\omega(f_j) = 2 \wedge (\omega(f_{j-1}) + \omega(f_j) + \omega(f_j) + \omega(f_{j+1}) > 6)$, or $\omega(f_j) = 1 \wedge (\omega(f_{j-1}) + \omega(f_j) + \omega(f_j) + \omega(f_{j+1}) > 6)$, then the inequality in (1) is strictly true, that is $\omega(f) > \lceil \frac{3m}{2} \rceil$, i.e., $\omega(f) \geq \frac{3m+2}{2}$.

Excluding the above cases, we will prove that the remaining cases $\omega(f_j) = 1 \wedge \omega(f_{j-1}) = \omega(f_{j+1}) = 2$ and $\omega(f_j) = 2 \wedge \omega(f_{j-1}) = \omega(f_{j+1}) = 1$ cannot exist in $C_3 \square C_m$ for $m \equiv 2, 10 \pmod{12}$.

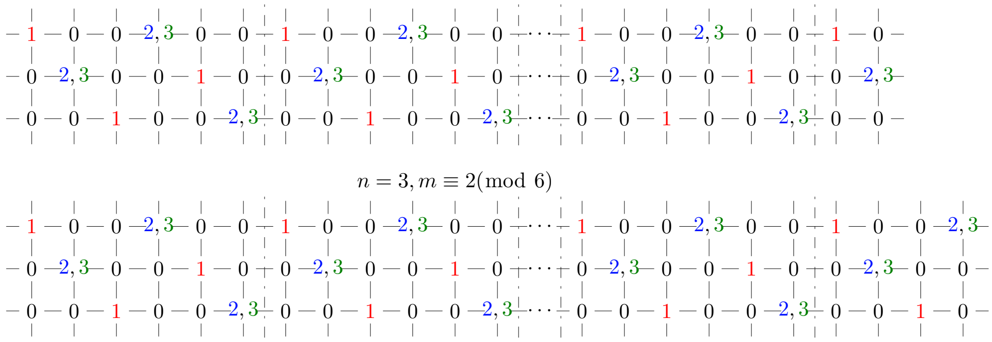
By contrast, without loss of generality, let $\omega(f_0) = 1$ and $|f(v_{0,0})| = 1$, then $\omega(f_1) = 2$. By the definition of 3RDF, $|f(v_{1,1})| = 2$ or $|f(v_{1,1})| = |f(v_{2,1})| = 1$.

Case 1. $|f(v_{1,1})| = 2$. In this case, by the definition of 3RDF, $|f(v_{2,2})| = 1$, $|f(v_{0,3})| = 2$, $|f(v_{1,4})| = 1$, $|f(v_{2,5})| = 2$, $|f(v_{0,6})| = 1$. Continuing in this way, we have:

$$|f(v_{i,t})| = \begin{cases} 1, & i = 0 \wedge t = 6k \vee i = 1 \wedge t = 6k+4 \vee i = 2 \wedge t = 6k+2, \\ 2, & i = 0 \wedge t = 6k+3 \vee i = 1 \wedge t = 6k+1 \vee i = 2 \wedge t = 6k+5, \\ 0, & otherwise, \end{cases}$$

where $k \geq 0$.

If we let $f(v_{0,0}) = \{1\}$, then $f(v_{1,1}) = \{2,3\}$. For $m \equiv 2, 10 (\bmod 12)$, i.e., $m \equiv 2, 4 (\bmod 6)$, f on $C_3 \square C_m$ is shown in Figure 12. Then, $f(v_{2,0}) = \emptyset$ and $\cup_{u \in N(v_{2,0})} f(u) \neq \{1,2,3\}$; this is a contradiction to the definition of 3RDF.



$n = 3, m \equiv 2(\bmod 6)$

$n = 3, m \equiv 4(\bmod 6)$

Figure 12. f on $C_3 \square C_m$ for $m \equiv 2, 4(\bmod 6)$.

Case 2. $|f(v_{1,1})| = |f(v_{2,1})| = 1$. In this case, we have $|f(v_{0,2})| = 1$, $|f(v_{1,3})| = |f(v_{2,3})| = |f(v_{0,4})| = 1$. Continuing in this way, we have:

$$|f(v_{i,t})| = \begin{cases} 1, & i = 0 \wedge t = 2k \vee i = 1 \wedge t = 2k+1 \vee i = 2 \wedge t = 2k+1, \\ 0, & otherwise, \end{cases}$$

where $k \geq 0$.

If we let $f(v_{0,0}) = \{1\}$, $f(v_{1,1}) = \{2\}$, then we have $f(v_{2,1}) \in \{\{2\}, \{3\}\}$.

Case 2.1. $f(v_{2,1}) = \{2\}$. It follows that $f(v_{1,3}) = f(v_{1,3}) = f(v_{2,3}) = \{1\}$, $f(v_{0,4}) = \{2\}$, $f(v_{0,2}) = f(v_{1,5}) = f(v_{2,5}) = \{3\}$. Continuing in this way, we have:

$$f(v_{i,t}) = \begin{cases} \{1\}, & i = 0 \wedge t = 6k \vee i = 1, 2 \wedge t = 6k+3, \\ \{2\}, & i = 0 \wedge t = 6k+4 \vee i = 1, 2 \wedge t = 6k+1, \\ \{3\}, & i = 0 \wedge t = 6k+2 \vee i = 1, 2 \wedge t = 6k+5, \\ \emptyset, & otherwise, \end{cases}$$

where $k \geq 0$.

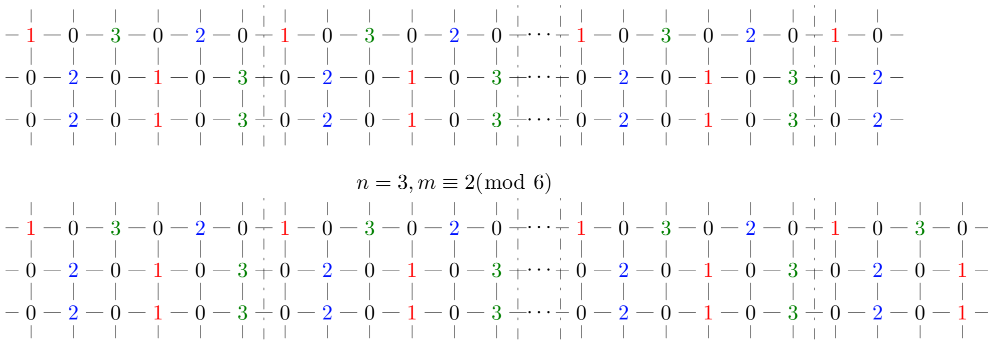
For $m \equiv 2, 10 (\bmod 12)$, i.e., $m \equiv 2, 4 (\bmod 6)$, f on $C_3 \square C_m$ is shown in Figure 13. Then, $f(v_{2,0}) = \emptyset$ and $\cup_{u \in N(v_{2,0})} f(u) \neq \{1,2,3\}$, a contradiction.

Case 2.2. $f(v_{2,1}) = \{3\}$. It follows the $f(v_{1,3}) = \{3\}$ and $f(v_{2,3}) = \{2\}$. Continuing in this way, we have:

$$f(v_{i,t}) = \begin{cases} \{1\}, & i = 0 \wedge t = 2k, \\ \{2\}, & i = 1 \wedge t = 4k+1 \vee i = 2 \wedge t = 4k+3, \\ \{3\}, & i = 1 \wedge t = 4k+3 \vee i = 2 \wedge t = 4k+1, \\ \emptyset, & otherwise, \end{cases}$$

where $k \geq 0$.

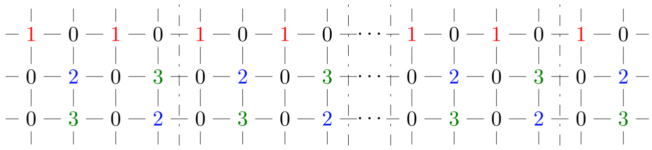
For $m \equiv 2, 10 (\bmod 12)$, i.e., $m \equiv 2 (\bmod 4)$, f on $C_3 \square C_m$ is shown in Figure 14. Then, $f(v_{2,0}) = \emptyset$ and $\cup_{u \in N(v_{2,0})} f(u) \neq \{1,2,3\}$, a contradiction.



$n = 3, m \equiv 2(\text{mod } 6)$

$n = 3, m \equiv 4(\text{mod } 6)$

Figure 13. Graph of f in Case 4.2.1.



$n = 3, m \equiv 2(\text{mod } 4)$

Figure 14. Graph of f in Case 4.2.2.

Thus, $\omega(f) \geq \frac{3m+2}{2}$ for $m \equiv 2, 10(\text{mod}\, 12)$. $\square$

By Theorems 3–5, we have:

Theorem 6. *For $m \geq 3$,*

$$\gamma_{r3}(C_3 \square C_m) = \begin{cases} \frac{3m+2}{2}, & m \equiv 2, 10(\text{mod } 12), \\ \lceil \frac{3m}{2} \rceil, & \textit{otherwise}. \end{cases}$$

4. The 3-rainbow Domination Number of $C_4 \square C_m$

Let f be an arbitrary 3RDF on $C_4 \square C_m$; we denote $\omega(f_j) = |f(v_{0,j})| + |f(v_{1,j})| + |f(v_{2,j})| + |f(v_{3,j})|$ $(0 \leq j \leq m-1)$.

Lemma 9. *Let f be a 3-rainbow dominating function on $C_4 \square C_m$. Then:*

(1) *if $\omega(f_j) = 0$, then $\omega(f_{j-1}) + \omega(f_{j+1}) \geq 12$;*
(2) *if $\omega(f_j) = 1$, then $\omega(f_{j-1}) + \omega(f_{j+1}) \geq 7$.*

Proof. (1) Since $\omega(f_j) = 0$, then $|f(v_{i,j-1})| + |f(v_{i,j+1})| \geq 3$ $(i = 0, 1, 2, 3)$. It follows that $\omega(f_{j-1}) + \omega(f_{j+1}) \geq 12$.

(2) Since $\omega(f_j) = 1$, we can let $|f(v_{0,j})| = 1$, then $|f(v_{1,j-1})| + |f(v_{1,j+1})| \geq 2$, $|f(v_{2,j-1})| + |f(v_{2,j+1})| \geq 3$ and $|f(v_{3,j-1})| + |f(v_{3,j+1})| \geq 2$. It follows that $\omega(f_{j-1}) + \omega(f_{j+1}) \geq 7$. $\square$

Theorem 7. $\gamma_{r3}(C_4 \square C_4) = 8$.

Proof. By Theorem 2, $\gamma_{r3}(C_4 \square C_4) \leq 8$.

If there exists $\omega(f_j) = 0$, by Lemma 9 (1), it follows that $\omega(f) \geq \omega(f_{j-1}) + \omega(f_{j+1}) \geq 12 > 8$.

If $\omega(f_j) \geq 1$ for $0 \leq j \leq m-1$, and if there exists $\omega(f_j) = 1$, then by Lemma 9 (2), it follows that $\omega(f) \geq \omega(f_{j-1}) + \omega(f_j) + \omega(f_{j+1}) \geq 8$.

If $\omega(f_j) \geq 2$ for $0 \leq j \leq m-1$, then $\omega(f) = \sum_{0 \leq j \leq 3} \omega(f_j) \geq 4 \times 2 = 8$.

Thus, $\gamma_{r3}(C_4 \Box C_4) \geq 8$, together with $\gamma_{r3}(C_4 \Box C_4) \leq 8$, and we have $\gamma_{r3}(C_4 \Box C_4) = 8$. $\square$

Lemma 10. *Let f be a $\gamma_{r3}(C_4 \Box C_m)$-function $(m \geq 5)$, then $\omega(f_j) \geq 1$ for $0 \leq j \leq m-1$.*

Proof. By contrast, suppose f is an arbitrary γ_{r3}-function and there exists a j with $\omega(f_j) = 0$. By Lemma 9 (1), we have $\omega(f_{j-1}) + \omega(f_j) + \omega(f_{j+1}) \geq 12$.

We construct a function f' as follows, and Figure 15 shows the sketch of f'.

$$f'(v_{i,t}) = \begin{cases} \emptyset, & i = 0,2 \wedge t = j-1, j+1 \vee i = 1,3 \wedge t = j, \\ \{1\}, & i = 0 \wedge t = j, \\ \{2\}, & i = 2 \wedge t = j, \\ \{3\}, & i = 1,3 \wedge t = j-1, j+1, \\ \{1\} \cup f(v_{i,t}), & i = 2 \wedge t = j-2, j+2, \\ \{2\} \cup f(v_{i,t}), & i = 0 \wedge t = j-2, j+2, \\ f(v_{i,t}), & \textit{otherwise}. \end{cases}$$

$$\begin{array}{ccccccc} j-3 & j-2 & j-1 & j & j+1 & j+2 & j+3 \\ f(v_{0,j-3}) & \{2\} \cup f(v_{0,j-2}) & \emptyset & \{1\} & \emptyset & \{2\} \cup f(v_{0,j+2}) & f(v_{0,j+3}) \\ f(v_{1,j-3}) & f(v_{1,j-2}) & \{3\} & \emptyset & \{3\} & f(v_{1,j+2}) & f(v_{1,j+3}) \\ f(v_{2,j-3}) & \{1\} \cup f(v_{2,j-2}) & \emptyset & \{2\} & \emptyset & \{1\} \cup f(v_{2,j+2}) & f(v_{2,j+3}) \\ f(v_{3,j-3}) & f(v_{3,j-2}) & \{3\} & \emptyset & \{3\} & f(v_{3,j+2}) & f(v_{3,j+3}) \end{array}$$

Figure 15. The sketch of f' in Lemma 10.

Then, $\omega(f') = \omega(f) - 12 + 10 < \omega(f)$, a contradiction with f being a γ_{r3}-function. $\square$

Lemma 11. *Let f be a $\gamma_{r3}(C_4 \Box C_m)$-function. If $S_1 = \{j \mid |f(v_{i,j})| = |f(v_{i+2,j})| = |f(v_{i+1,j+1})| = 1, |f(v_{i+1,j})| = |f(v_{i+3,j})| = |f(v_{i,j+1})| = |f(v_{i+2,j+1})| = |f(v_{i+3,j+1})| = 0, 0 \leq j \leq m-1\}$, then $|S_1| = 0$.*

Proof. By contrast, suppose $|S_1| \geq 1$. Without loss of generality, we may assume that $|f(v_{0,s})| = 1$; it follows that $|f(v_{0,s+2})| \geq 1$, $|f(v_{2,s+2})| \geq 1$ and $|f(v_{3,s+2})| \geq 3$ (*say*, $\{1\} \subseteq f(v_{2,s+2})$).

We can construct a function f' such that $\omega(f') \leq \omega(f)$. Figure 16 shows the sketch of f'.

$$f'(v_{i,t}) = \begin{cases} \emptyset, & i = 3 \wedge t = s+2, \\ \{3\}, & i = 3 \wedge t = s+1, \\ \{2\} \cup f(v_{i,t}), & i = 3 \wedge t = s+3, \\ f(v_{i,t}), & \textit{otherwise}. \end{cases}$$

$$\begin{array}{ccccc}
s & s+1 & s+2 & s+3 & s+4 \\
f(v_{0,s}) & f(v_{0,s+1}) & f(v_{0,s+2}) & f(v_{0,s+3}) & f(v_{0,s+4}) \\
f(v_{1,s}) & f(v_{1,s+1}) & f(v_{1,s+2}) & f(v_{1,s+3}) & f(v_{1,s+4}) \\
f(v_{2,s}) & f(v_{2,s+1}) & \{1\} \subseteq f(v_{2,s+2}) & f(v_{2,s+3}) & f(v_{2,s+4}) \\
f(v_{3,s}) & \{3\} & \emptyset & \{2\} \cup f(v_{3,s+3}) & f(v_{3,s+4})
\end{array}$$

Figure 16. The sketch of f' in Lemma 11.

Then, $\omega(f') = \omega(f) - 3 + 2 < \omega(f)$, a contradiction with f being a γ_{r3}-function. □

Lemma 12. *There is a $\gamma_{r3}(C_4 \square C_m)$-function f such that $|S_2| = 0$, where $S_2 = \{j \mid |f(v_{i,j})| = |f(v_{i+2,j})| = |f(v_{i,j+1})| = 1$ and $|f(v_{i+1,j})| = |f(v_{i+1,j+1})| = |f(v_{i+3,j})| = |f(v_{i+3,j+1})| = |f(v_{i+2,j+1})| = 0, 0 \leq j \leq m-1\}$.*

Proof. By contrast, suppose $|S_2| \geq 1$ for all $\gamma_{r3}(C_4 \square C_m)$-functions, that is the minimum $|S_2|$ is one. Let f be a $\gamma_{r3}(C_4 \square C_m)$-function such that $|S_2| = 1$; we denote $|S_2|_f = 1$. Let s be the smallest positive integer such that $s \in S_2$ $(0 \leq s \leq m-1)$. Without loss of generality, we may assume that $|f(v_{0,s})| = 1$; it follows that $|f(v_{1,s+2})| \geq 2$, $|f(v_{2,s+2})| \geq 2$, and $|f(v_{3,s+2})| \geq 2$ (*say*, $\{1\} \subseteq f(v_{3,s+2})$).

We can construct a function f' as follows satisfying $\omega(f') = \omega(f)$ and $|S_2|_{f'} < |S_2|_f = 1$ (see Figure 17 for the sketch of f'). Thus, there is a contradiction with the minimum $|S_2|$ being one.

$$f'(v_{i,t}) = \begin{cases} \emptyset, & i = 2 \wedge t = s+2, \\ \{3\}, & i = 2 \wedge t = s+1, \\ \{2\} \cup f(v_{i,t}), & i = 2 \wedge t = s+3, \\ f(v_{i,t}), & \text{otherwise}. \end{cases}$$

$$\begin{array}{ccccc}
s & s+1 & s+2 & s+3 & s+4 \\
f(v_{0,s}) & f(v_{0,s+1}) & f(v_{0,s+2}) & f(v_{0,s+3}) & f(v_{0,s+4}) \\
f(v_{1,s}) & f(v_{1,s+1}) & f(v_{1,s+2}) & f(v_{1,s+3}) & f(v_{1,s+4}) \\
f(v_{2,s}) & \{3\} & \emptyset & \{2\} \cup f(v_{2,s+3}) & f(v_{2,s+4}) \\
f(v_{3,s}) & f(v_{3,s+1}) & \{1\} \subseteq f(v_{3,s+2}) & f(v_{3,s+3}) & f(v_{3,s+4})
\end{array}$$

Figure 17. The sketch of f' in Lemma 12.

□

Lemma 13. *Let f be a $\gamma_{r3}(C_4 \square C_m)$-function with $|S_1| = |S_2| = 0$, then $\gamma_{r3}(C_4 \square C_m) \geq 2m$ $(m \geq 5)$, where S_1 and S_2 are defined as in Lemmas 11 and 12.*

Proof. By Lemma 10, $\omega(f_j) \geq 1$ for $0 \leq j \leq m-1$.

Case 1. If $\omega(f_j) = 1$, by Lemma 9 (2), $(\omega(f_{j-1}) + \omega(f_j)) + (\omega(f_j) + \omega(f_{j+1})) \geq 1 \times 2 + 7 > 8$.

Case 2. If $\omega(f_j) = 2$, then there are three subcases.

Case 2.1. There is one vertex v with $|f(v)| = 2$. Let $|f(v_{0,j})| = 2$, then $|f(v_{1,j})| = |f(v_{2,j})| = |f(v_{3,j})| = 0$. It follows that $|f(v_{1,j-1})| + |f(v_{1,j+1})| \geq 1$, $|f(v_{2,j-1})| + |f(v_{2,j+1})| \geq 3$, $|f(v_{3,j-1})| + |f(v_{3,j+1})| \geq 1$. Hence, $(\omega(f_{j-1}) + \omega(f_j)) + (\omega(f_j) + \omega(f_{j+1})) \geq 2 \times 2 + 5 > 8$.

Case 2.2. There are two vertices with a weight of one, and they are neighbors. Let $|f(v_{0,j})| = |f(v_{1,j})| = 1$, then $|f(v_{2,j})| = |f(v_{3,j})| = 0$. It follows that $|f(v_{2,j-1})| + |f(v_{2,j+1})| \geq 2$, $|f(v_{3,j-1})| + |f(v_{3,j+1})| \geq 2$. Hence, $(\omega(f_{j-1}) + \omega(f_j)) + (\omega(f_j) + \omega(f_{j+1})) \geq 2 \times 2 + 4 = 8$.

Case 2.3. There are two vertices with a weight of one, and they are not neighbors. Let $|f(v_{i,j})| = |f(v_{i+2,j})| = 1$. By Lemmas 11 and 12, we can get $\omega(f_{j-1}) \geq 2$ and $\omega(f_{j+1}) \geq 2$. Hence, $(\omega(f_{j-1}) + \omega(f_j)) + (\omega(f_j) + \omega(f_{j+1})) \geq 2 + 2 \times 2 + 2 = 8$.

Case 3. If $\omega(f_j) \geq 3$, $(\omega(f_{j-1}) + \omega(f_j)) + (\omega(f_j) + \omega(f_{j+1})) \geq 1 + 2 \times 3 + 1 = 8$,

Thus,

$$\begin{aligned} 4\omega(f) &= 4\sum\nolimits_{0 \leq j \leq m-1} \omega(f_j) \\ &= \sum\nolimits_{0 \leq j \leq m-1} (\omega(f_{j-1}) + \omega(f_j)) + (\omega(f_j) + \omega(f_{j+1})) \\ &\geq 8m. \end{aligned}$$

That is, $\omega(f) \geq 2m$. □

Theorem 8. *For $m \geq 4$, $\gamma_{r3}(C_4 \square C_m) = 2m$.*

Proof. By Theorem 3 and Lemma 13, it has $\gamma_{r3}(C_4 \square C_m) = 2m$ $(m \geq 5)$; together with Theorem 7, we have $\gamma_{r3}(C_4 \square C_m) = 2m$ $(m \geq 4)$. □

5. Conclusions

In this paper, we investigate the 3-rainbow domination number of Cartesian products of cycles $C_n \square C_m$. We determine the exact values of the 3-rainbow domination number of $C_3 \square C_m$ and $C_4 \square C_m$, i.e., $\gamma_{r3}(C_3 \square C_m) = \lceil \frac{3m+\alpha}{2} \rceil$, $\alpha = 2$ for $m \equiv 2, 10 (\mathrm{mod}\ 12)$, and $\alpha = 0$ for $m \not\equiv 2, 10 (\mathrm{mod}\ 12)$, $\gamma_{r3}(C_4 \square C_m) = 2m$. For $n \geq 5$, by Lemma 1 and Theorem 3, we present a better bound on the 3-rainbow domination number of $C_n \square C_m$, that is $\frac{3mn}{7} \leq \gamma_{r3}(C_n \square C_m) \leq \frac{mn}{2}$.

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