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# Fixed-Point Results for a Generalized Almost $(s,q)-$Jaggi $F$-Contraction-Type on $b-$Metric-Like Spaces

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**Abstract:** The purpose of this article is to present a new generalized almost $(s,q)-$Jaggi $F-$contraction-type and a generalized almost $(s,q)-$Jaggi $F-$Suzuki contraction-type and some results in related fixed point on it in the context of $b-$metric-like spaces are discussed. Also, we support our theoretical results with non-trivial examples. Finally, applications to find a solution for the electric circuit equation and second-order differential equations are presented and an strong example is given here to support the first application.

**Keywords:** electric circuit equations; wardoski contraction; almost $(s,q)-$Jaggi-type; $b-$metric-like spaces; second-order differential equations

**MSC:** 47H10; 54H25

## 1. Introduction

Mathematical models can take many forms, including dynamical systems, statistical models, differential equations and game theoretic models and real world problems. In various branches of mathematics, The existence of solution for these matters has been checked, for example, differential equations, integral equations, functional analysis, etc. Fixed point technique is one of these methods to find the solution of these problems. So this technique has many applications not only is limited to mathematics but also occurs in various sciences, such that, economics, biology, chemistry, computer science, physics, etc. More clearly, for example, In economics, this technique is applied to find the solution of the equilibrium problem in game theory.

Problems in the nonlinear analysis are solved by a popular tool called Banach contraction principle. This principle appeared in Banach's thesis [1], where it was used in proving the existence and uniqueness of solution of integral equations, it stated as: A nonlinear self mapping $\Gamma$ on a metric space $(\Omega, d)$ is called a Banach contraction if there exists $\delta \in [0,1)$ such that

$$d(\Gamma\kappa, \Gamma\mu) \leq \delta d(\kappa, \mu), \forall \kappa, \mu \in \Omega. \tag{1}$$

Notice that the contractive condition (1) is satisfied for all $\kappa, \mu \in \Omega$ which forces the mapping $\Gamma$ to be continuous, while it is not applicable in case of discontinuity. In view of the applicability of contraction principle this is the major draw-back of this principle. Many authors attempted to overcome this drawback (see, for example [2–4]).

In 1989, one of the interesting generalizations of this basic principle was given by Bakhtin [5] (and also Czerwik [6], 1993) by introducing the concept of $b-$metric spaces. For fixed point results in $b-$metric spaces. See [7–14].

In 2010, the concept of a $b-$metric-like initiated by Alghamdi et al. [9] as an extension of a $b-$metric. They studied some related fixed point consequences concerning with this space. Recently, many contributions on fixed points results via certain contractive conditions in mentioned spaces are made (for example, see [15–20]).

In 2012, a new contraction called $F-$ contraction-type is presented by Wardowski [21], where $F : R^+ \to R$. By this style, recent fixed point results and strong examples to obtain a different type of contractions are discussed.

**Definition 1** ([21]). *A mapping $\Gamma : \Omega \longrightarrow \Omega$ defined on a metric space $(\Omega, d)$, is called an $F-$contraction if there is $F \in \Sigma$ and $\tau > 0$ such that*

$$d(\Gamma\kappa, \Gamma\mu) > 0 \text{ implies } \tau + F(d(\Gamma\kappa, \Gamma\mu)) \leq F(d(\kappa, \mu)), \ \forall \kappa, \mu \in \Omega, \tag{2}$$

*where $\Sigma$ is the set of functions $F : (0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:*
*$(\Im_1)$ $F$ is strictly increasing, i.e., for all $a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$;*
*$(\Im_2)$ For every sequence $\{a_n\}_{n\in N}$ of positive numbers, $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} F(a_n) = -\infty$;*
*$(\Im_3)$ There exists $\mu \in (0, 1)$ such that $\lim_{a\to 0^+} a^\mu F(a) = 0$.*

The following functions $F_a : (0, +\infty) \longrightarrow \mathbb{R}$ for $a \in \{1, 2, 3, 4\}$, are the elements of $\Sigma$. Furthermore, substituting these functions in (2), Wardowski obtained the following contractions:

$$(1)\ F_1(\theta) = \ln(\theta), \qquad \frac{d(\Gamma\kappa,\Gamma\mu)}{d(\kappa,\mu)} \leq e^{-\tau},$$

$$(2)\ F_2(\theta) = \ln(\theta) + \theta, \qquad \frac{d(\Gamma\kappa,\Gamma\mu)}{d(\kappa,\mu)} \leq e^{-\tau+d(\kappa,\mu)-d(\Gamma\kappa,\Gamma\mu)},$$

$$(3)\ F_3(\theta) = \frac{-1}{\sqrt{\theta}}, \qquad \frac{d(\Gamma\kappa,\Gamma\mu)}{d(\kappa,\mu)} \leq \frac{1}{\left(1+\tau\sqrt{d(\kappa,\mu)}\right)^2}$$

$$(4)\ F_4(\theta) = \ln(\theta^2 + \theta), \qquad \frac{d(\Gamma\kappa,\Gamma\mu)(1+d(\Gamma\kappa,\Gamma\mu))}{d(\kappa,\mu)(1+d(\kappa,\mu))} \leq e^{-\tau}.$$

for all $\kappa, \mu \in \Omega$ with $\theta > 0$ and $\Gamma\kappa \neq \Gamma\mu$.

**Remark 1.** *It follows from (2) that*

$$d(\Gamma\kappa, \Gamma\mu) < d(\kappa, \mu), \text{ for all } \kappa, \mu \in \Omega,$$

*this means that $\Gamma$ is contractive with $\Gamma\kappa \neq \Gamma\mu$. Hence, if the mapping is $F-$contraction, then it continuous.*

**Remark 2** ([22]). *For $p > 1$ and $\theta > 0$, the function $F(\theta) = \frac{-1}{\sqrt[p]{\theta}}$ belong to $\Sigma$.*

In a different way to generalize the Banach contraction principle, Wardowski [21] established the following theorem:

**Theorem 1** ([21]). *Suppose that $(\Omega, d)$ is a complete metric space and $\Gamma$ is a self-mapping on it satisfying the condition (2). Then there exists a unique fixed point $\kappa^*$ of $\Gamma$. As well as, the sequence $\{\Gamma^n\kappa_\circ\}_{n\in\mathbb{N}}$ is convergent to $\kappa^*$, for any $\kappa_\circ \in \Omega$.*

The Wardowski-contraction is extended by many authors such as Abbas et al. [23] to give certain fixed point results, Batra et al. [24,25], to generalize it on graphs and alter distances, and Cosentino and Vetro [26] to introduce some fixed point consequences for Hardy-Rogers-type self-mappings in ordered and complete metric spaces.

In 2014, some fixed point consequences proved via the notion of an $F-$Suzuki contraction by Piri and Kumam [27]. This concept is stated as follows:

**Definition 2** ([27]). *Let $(\Omega, d)$ be a complete metric space and a mapping $\Gamma : \Omega \to \Omega$ is called $F-$Suzuki contraction if there exists $F \in \Sigma$, and $\tau > 0$ such that*

$$\frac{1}{2}d(\kappa, \Gamma\kappa) < d(\kappa, \mu) \Rightarrow \tau + F(d(\Gamma\kappa, \Gamma\mu)) \leq F(d(\kappa, \mu)), \forall \kappa, \mu \in \Omega,$$

*with $\Gamma\kappa \neq \Gamma\mu$.*

In 1975, Jaggi [28] defined the concept of a generalized Banach contraction principle as follows:

**Definition 3.** *Let $(\Omega, d)$ be a complete metric space. A continuous self-mapping $\Gamma$ on a set $\Omega$ is called Jaggi contraction-type if*

$$d(\Gamma\kappa, \Gamma\mu) \leq \alpha \frac{d(\kappa, \Gamma\kappa).d(\mu, \Gamma\mu)}{d(\kappa, \mu)} + \beta d(\kappa, \mu),$$

*for all $\kappa, \mu \in \Omega$, $\kappa \neq \mu$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.*

Recently, the same author [29], extended his above result on $b-$metric-like spaces as follows:

**Definition 4.** *Let $(\Omega, \varpi)$ be a $b-$metric-like space with parameter $s \geq 1$. A nonlinear self-mapping $\Gamma$ on a set $\Omega$ is called $(s, q)-$Jaggi contraction type if it satisfies the following condition*

$$s^q \varpi(\Gamma\kappa, \Gamma\mu) \leq \alpha \frac{\varpi(\kappa, \Gamma\kappa).\varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)} + \beta \varpi(\kappa, \mu),$$

*for all $\kappa, \mu \in \Omega$, whenever $\varpi(\kappa, \mu) \neq 0$, where $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and for some $q \geq 1$.*

In addition, Berinde [30] introduced the notion of almost contraction by generalized the Zamfirescu fixed point theorem, his result incorporated as follows:

**Definition 5.** *Let $\Gamma$ be a nonlinear self-mapping on a complete metric space $(\Omega, d)$. Then it called Ciric almost contraction, if there exists $\delta \in [0, 1)$ and $\exists L \geq 0$ such that*

$$d(\Gamma\kappa, \Gamma\mu) \leq \delta d(\kappa, \mu) + L d(\mu, \Gamma\kappa), \tag{3}$$

*for all $\kappa, \mu \in \Omega$, $\kappa \neq \mu$.*

After that, the same author [31] extended the contraction (3) and obtained some related fixed point results on complete metric spaces as follows:

**Theorem 2.** *Let $(\Omega, d)$ be a complete metric space and a self-mapping $\Gamma$ on the set $\Omega$ be a Ciric almost contraction, if there exist $\alpha \in [0, 1)$ and $\exists L \geq 0$ such that*

$$d(\Gamma\kappa, \Gamma\mu) \leq \alpha M(\kappa, \mu) + L d(\mu, \Gamma\kappa) \text{ for all } \kappa, \mu \in \Omega,$$

*where $M(\kappa, \mu) = \max\{d(\kappa, \mu), d(\kappa, \Gamma\kappa), d(\mu, \Gamma\mu), d(\kappa, \Gamma\mu), d(\mu, \Gamma\kappa)\}$. Then,*

*i. there is a non-empty fixed point of the mapping $\Gamma$, i.e., $Fix(\Gamma) \neq \emptyset$;*

*ii. for any $\kappa_\circ = \kappa \in \Omega$, $n \geq 0$ the Picard iteration $\kappa_{n+1} = \Gamma\kappa_n$ converges to $\kappa^* \in Fix(\Gamma)$;*

*iii. The following estimate holds*

$$d(\kappa_n, \kappa^*) \leq \frac{\alpha^n}{(1-\alpha)^2} d(\kappa, \Gamma\kappa),$$

*for all* $n \geq 1$.

Inspired by Definitions 1, 4 and 5, we introduce a new generalized $(s,q)-$Jaggi $F-$contraction-type on the context of $b-$metric-like spaces as the following:

**Definition 6.** *Let* $\Gamma$ *be a self-mapping on a* $b-$*metric-like space* $(\Omega, \varpi)$ *with parameter* $s \geq 1$. *Then the mapping* $\Gamma$ *is said to be generalized* $(s,q)-$*Jaggi F contraction-type if there is* $F \in \Sigma$ *and* $\tau > 0$ *such that*

$$\varpi(\Gamma\kappa, \Gamma\mu) > 0 \Rightarrow \tau + F\left(s^q \varpi(\Gamma\kappa, \Gamma\mu)\right) \leq F\left(\alpha \frac{\varpi(\kappa, \Gamma\kappa).\varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)} + \beta\varpi(\kappa, \mu) + \gamma\varpi(\mu, \Gamma\kappa)\right), \quad (4)$$

*for all* $\kappa, \mu \in \Omega$ *and* $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + 2\gamma s < 1$, *and for some* $q > 1$.

To support our definition, we state the following example:

**Example 1.** *Let* $\Omega = [0, +\infty)$ *and* $\varpi(\kappa, \mu) = \kappa^2 + \mu^2 + |\kappa - \mu|^2 \; \forall \kappa, \mu \in \Omega$. *It's obvious that* $\varpi$ *is a* $b-$*metric like on* $\Omega$, *with coefficient* $s = 2$. *Define a nonlinear self-mapping* $\Gamma : \Omega \to \Omega$ *by* $\Gamma\kappa = \frac{1}{16}\ln(1 + \frac{\kappa}{2})$, *for all* $\kappa \in \Omega$, *and the function* $F(\theta) = \ln(\theta)$. *Consider the constants* $q = 2$, $\tau = \ln(2)$, $\alpha = \gamma = 0$ *and* $\beta = \frac{1}{4}$. *So* $\alpha + \beta + 2s\gamma = \frac{1}{4} < 1$. *Since* $t \geq \ln(1 + t)$ *for each* $t \in [0, \infty)$, *for all* $\kappa, \mu \in \Omega$, *we have*

$$\begin{aligned}
\tau + F(s^q \varpi(\Gamma\kappa, \Gamma\mu)) &= \tau + F(s^2 \varpi(\Gamma\kappa, \Gamma\mu)) = \tau + F\left(s^2\left(T^2\kappa + T^2\mu - |T\kappa - T\mu|^2\right)\right) \\
&= \ln(2) + F\left(4\left(\left(\frac{\ln(1+\frac{\kappa}{2})}{16}\right)^2 + \left(\frac{\ln(1+\frac{\mu}{2})}{16}\right)^2 + \left|\frac{\ln(1+\frac{\kappa}{2})}{16} - \frac{\ln(1+\frac{\mu}{2})}{16}\right|^2\right)\right) \\
&\leq \ln(2) + F\left(\frac{1}{16}\left(\frac{\kappa^2}{4} + \frac{\mu^2}{4} + \left|\frac{\kappa}{4} - \frac{\mu}{4}\right|^2\right)\right) \\
&= \ln(2) + \ln\left(\frac{1}{16}\left(\frac{\kappa^2}{4} + \frac{\mu^2}{4} + \left|\frac{\kappa}{4} - \frac{\mu}{4}\right|^2\right)\right) \\
&\leq \ln\left(\frac{1}{4}\left(\kappa^2 + \mu^2 + |\kappa - \mu|^2\right)\right) \\
&= F\left(\alpha \frac{\varpi(\kappa, \Gamma\kappa).\varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)} + \beta\varpi(\kappa, \mu) + \gamma\varpi(\mu, \Gamma\kappa)\right).
\end{aligned}$$

*Therefore the mapping* $\Gamma$ *is a generalized almost* $(s,q)-$*Jaggi* $F-$*contraction-type.*

In this article, we present some related fixed point results for a generalized almost $(s,q)-$Jaggi $F$-contraction-type and generalized almost $(s,q)-$Jaggi $F-$Suzuki contraction-type on $b-$metric-like spaces. Also, we give some examples to illustrate these main results. Moreover, applications to find solutions of electric circuit equations and second-order differential equations are discussed and we justify the first application with an example.

## 2. Preliminaries and Known Results

In the context of this paper, we will use the following notations: $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{Q}$ denotes the set of positive integers, real numbers, nonnegative real numbers and rational numbers, respectively. We begin this part with backgrounds about metric-like and $b-$metric-like spaces.

**Definition 7** ([9]). *Let* $\Omega$ *be a nonempty set. A mapping* $\omega : \Omega \times \Omega \to \mathbb{R}^+$ *is said to be dislocated (metric-like) if the following three conditions hold for all* $\kappa, \mu, \tau \in \Omega$ :
$(\omega_1)$ $\omega(\kappa, \mu) = 0 \Rightarrow \kappa = \mu$;
$(\omega_2)$ $\omega(\kappa, \mu) = \omega(\mu, \kappa)$;

$(\omega_3)$ $\omega(\kappa,\tau) \leq \omega(\kappa,\mu) + \omega(\mu,\tau)$.
*In this case, the pair* $(\Omega,\omega)$ *is called a dislocated (metric-like) space.*

**Definition 8** ([32]). *A* $b-$*dislocated on a nonempty set* $\Omega$ *is a function* $\varpi : \Omega \times \Omega \to \mathbb{R}^+$ *such that for all* $\kappa,\mu,\tau \in \Omega$ *and a constant* $s \geq 1$*, the following three conditions are satisfied:*
$(\varpi_1)$ $\varpi(\kappa,\mu) = 0 \Rightarrow \kappa = \mu$;
$(\varpi_2)$ $\varpi(\kappa,\mu) = \varpi(\mu,\kappa)$;
$(\varpi_3)$ $\varpi(\kappa,\tau) \leq s[\varpi(\kappa,\mu) + \varpi(\mu,\tau)]$.

*In this case, the pair* $(\Omega,\varpi)$ *is called a* $b-$*dislocated (metric-like) space (with constant* $s$*).*

It should be noted that the class of $b-$metric-like spaces is larger than the class of metric-like spaces, since a $b-$metric-like is a metric-like with $s = 1$.

For new examples in metric-like and $b-$metric-like spaces (see [33,34]).

A $b-$metric-like on $\Omega$ satisfies all of the conditions of a metric except that $\varpi(\kappa,\mu)$ may be positive for $\kappa \in \Omega$, so each $b-$metric-like $\varpi$ on $\Omega$ generates a topology $\Re_\varpi$ on $\Omega$ whose base is the family of open $\varpi-$balls

$$\Re_\varpi(\kappa,\epsilon) = \{\mu \in \Omega : |\varpi(\kappa,\mu) - \varpi(\kappa,\kappa)| < \frac{\epsilon}{s}\},$$

for all $\kappa \in \Omega$, $s \geq 1$ and $\epsilon > 0$.
According to a topology $\Re_\varpi$, we can present the following results:

**Definition 9.** *Let* $(\Omega,\varpi)$ *be a* $b-$*metric-like space and* $\chi$ *be a subset of* $\Omega$*. We say that* $\chi$ *is a* $\varpi-$*open subset of* $\Omega$*, if for all* $\kappa \in \Omega$ *there exists* $\epsilon > 0$ *such that* $\Re_\varpi(\kappa,\epsilon) \subseteq \chi$*. Also* $\sigma \subseteq \Omega$ *is a* $\varpi-$*closed subset of* $\Omega$ *if* $\Omega\backslash\chi$ *is a* $\varpi-$*open subset in* $\Omega$*.*

**Lemma 1.** *Let* $(\Omega,\varpi)$ *be a* $b-$*metric-like space and* $\sigma$ *be a* $\varpi-$*closed subset of* $\Omega$*. Let* $\{\kappa_n\}$ *be a sequence in* $\sigma$ *such that* $\lim_{n\to\infty} \kappa_n = \kappa$*. Then* $\kappa \in \sigma$*.*

**Proof.** Let $\kappa \notin \sigma$ by Definition 9, $(\Omega\backslash\sigma)$ is a $\varpi-$open set. Then there exists $\epsilon > 0$ such that $\Re_\varpi(\kappa,\epsilon) \subseteq \Omega\backslash\sigma$. On the other hand, we have $\lim_{n\to\infty} |\varpi(\kappa_n,\kappa) - \varpi(\kappa,\kappa)| = 0$ since $\lim_{n\to\infty} \kappa_n = \kappa$. Hence, there exists $\iota_\circ \in N$ such that

$$|\varpi(\kappa_n,\kappa) - \varpi(\kappa,\kappa)| < \frac{\epsilon}{s},$$

for all $\iota \geq \iota_\circ$. So, we conclude that $\{\kappa_n\} \subseteq \Re_\varpi(\kappa,\epsilon) \subseteq \Omega\backslash\sigma$ for all $\iota \geq \iota_\circ$. This is a contradiction since $\{\kappa_n\} \subseteq \sigma$ for all $\iota_\circ \in N$. □

In a $b-$metric-like space $(\Omega,\varpi)$, if $\kappa,\mu \in \Omega$ and $\varpi(\kappa,\mu) = 0$, then $\kappa = \mu$, but the converse is not true in general.

**Example 2.** *Let* $\Omega = \{0,1,2,3,4\}$ *and let*

$$\varpi(\kappa,\mu) = \begin{cases} 5, & \kappa = \mu = 0, \\ \frac{1}{5}, & \text{otherwise.} \end{cases}$$

*Then* $(\Omega,\varpi)$ *is a* $b-$*metric-like space with the constant* $s = 5$*.*

**Definition 10.** *Let* $\{\kappa_n\}$ *be a sequence on a* $b-$*metric-like space* $(\Omega,\varpi)$ *with a coefficient* $s$*. Then*
1. *If* $\lim_{m,n\to\infty} \varpi(\kappa_n,\kappa) = \varpi(\kappa,\kappa)$*, then the sequence* $\{\kappa_n\}$ *is said to be convergent to* $\kappa$*.* $\{\kappa_n\}$ *is said to be a Cauchy sequence if* $\lim_{m,n\to\infty} \varpi(\kappa_m,\kappa_n)$ *exists and is finite. The pair* $(\Omega,\varpi)$ *is said to be a complete* $b-$*metric-like space if for every Cauchy sequence* $\{\kappa_n\}$ *in* $\Omega$*, there exists a* $\kappa \in \Omega$*, such that*

$$\lim_{m,n\to\infty} \varpi(\kappa_m, \kappa_n) = \varpi(\kappa, \kappa) = \lim_{n\to\infty} \varpi(\kappa_n, \kappa).$$

2. $\{\kappa_n\}$ *in* $\Omega$ *is called a* $0-\varpi-$*Cauchy sequence if* $\lim_{m,n\to\infty} \varpi(\kappa_m, \kappa_n) = 0$. *The space* $(\Omega, \varpi)$ *is said to be* $0-\varpi-$*complete if every* $0-\varpi-$*Cauchy sequence in* $\Omega$ *converges with respect to* $\Re_\varpi$ *to a point* $\kappa \in \Omega$ *such that* $\varpi(\kappa, \kappa) = 0$.
3. *A nonlinear mapping* $\Gamma$ *is continuous on the set* $\Omega$, *if the following limits*

$$\lim_{n\to\infty} \varpi(\kappa_n, \kappa), \quad \lim_{n\to\infty} \varpi(\Gamma\kappa_n, \Gamma\kappa).$$

*Existing and equal.*

The following example elucidates every $\varpi-$complete $b-$metric-like space is $0-\varpi-$complete but the converse is not true.

**Example 3.** *Let* $\Omega = [0, 1) \cap \mathbb{Q}$ *and* $\varpi : \Omega \times \Omega \to [0, +\infty)$ *be a function defined by*

$$\varpi(\kappa, \mu) = \begin{cases} 2\kappa & \text{if } \kappa = \mu, \\ (\max\{\kappa, \mu\})^2 & \text{otherwise.} \end{cases}$$

$\forall \kappa, \mu \in \Omega$. *Then* $(\Omega, \varpi)$ *is a* $b-$*metric like spaces with a coefficient* $s = 2$. *Also, if we take a Cauchy sequence* $\{\kappa_n\} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$, *then* $\lim_{m,n\to\infty} \varpi(\kappa_m, \kappa_n) = 0$. *So* $\{\kappa_n\}$ *is a* $0-\varpi-$*Cauchy sequence converges to a point* $0 \in \Omega$. *Therefore the pair* $(\Omega, \varpi)$ *is a* $0-\varpi-$*complete* $b-$*metric-like space, while, if we consider* $\{\kappa_n\} = \left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$, *then* $\lim_{m,n\to\infty} \varpi(\kappa_m, \kappa_n)$ *exists and is finite but converges to a point* $2 \notin \Omega$, *so, the pair* $(\Omega, \varpi)$ *is not a* $\varpi-$*complete* $b-$*metric-like space.*

**Remark 3.** *In a* $b-$*metric-like space the limit of a sequence need not be unique and a convergent sequence need not be a Cauchy sequence.*

To show this remark, we gave the following example:

**Example 4.** *Let* $\Omega = [0, +\infty)$. *Define a function* $\varpi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *by* $\varpi(\kappa, \mu) = (\max\{\kappa, \mu\})^2$. *Then* $(\Omega, \varpi)$ *is a* $b-$*metric-like space with a coefficient* $s = 2$. *Suppose that*

$$\{\kappa_n\} = \begin{cases} 0 & \text{when} \quad n \text{ is odd} \\ 1 & \text{when} \quad n \text{ is even} \end{cases}.$$

*For* $\kappa \in \Omega$, $\lim_{n\to\infty} \varpi(\kappa_n, \kappa) = \lim_{n\to\infty} (\max\{\kappa_n, \kappa\})^2 = \kappa^2 = \varpi(\kappa, \kappa)$. *Therefore, it is a convergent sequence and* $\kappa_n \to \kappa$. *Now if we take* $\kappa = 0$, *therefore,*
*if n is an odd number, we have* $\lim_{n\to\infty} \varpi(\kappa_n, \kappa) = \lim_{n\to\infty} (\max\{0, 0\})^2 = 0$,
*if n is an even number, we get* $\lim_{n\to\infty} \varpi(\kappa_n, \kappa) = \lim_{n\to\infty} (\max\{1, 0\})^2 = 1$.
*That is, the sequence has not limit although it has two subsequences (for odd n and for even n) both having a limit with both limits being distinct.*

## 3. New Fixed Point Results

This section is devoted to present some new fixed point results for a generalized almost $(s, q)-$Jaggi $F-$contraction-type and almost $(s, q)-$Jaggi $F-$Suzuki-type contraction on the context of $b-$ metric-like spaces.

We begin with the first main result.

**Theorem 3.** *Let $(\Omega, \varpi)$ be a $0-\varpi-$complete $b-$metric-like space with a coefficient $s \geq 1$ and $\Gamma$ be a self mapping satisfying a generalized almost $(s,q)-$Jaggi $F-$contraction-type (4). Then, $\Gamma$ has a unique fixed point whenever $F$ or $\Gamma$ is continuous.*

**Proof.** Let $\kappa_\circ$ be an arbitrary point of $\Omega$. Define a sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ by $\kappa_{n+1} = \Gamma\kappa_n$. If $\Gamma\kappa_\circ = \kappa_\circ$, then the proof is finished. Again, if there exists $i_\circ \in \{1, 2, ..\}$ the right hand side of (4) is 0 for $\kappa = \kappa_{i_\circ - 1}$ and $\mu = \kappa_{i_\circ}$ so the proof is stopped. So, without loss of generality we may assume that $\kappa_{n+1} \neq \kappa_n$ for all $n \geq 1$ and $\varpi_n = \varpi(\kappa_{n+1}, \kappa_n)$. Then $\varpi_n > 0$. On the other hand, $\Gamma$ is a generalized almost $(s,q)-$Jaggi $F-$contraction-type, hence we get

$$\begin{aligned}
\tau + F(\varpi_n) &\leq \tau + F(s^q\varpi_n) = \tau + F(s^q\varpi(\kappa_{n+1}, \kappa_n)) \\
&= \tau + F(s^q\varpi(\Gamma\kappa_n, \Gamma\kappa_{n-1})) \\
&\leq F\left(\alpha\frac{\varpi(\kappa_n, \Gamma\kappa_n).\varpi(\kappa_{n-1}, \Gamma\kappa_{n-1})}{\varpi(\kappa_n, \kappa_{n-1})} + \beta\varpi(\kappa_n, \kappa_{n-1}) + \gamma\varpi(\kappa_{n-1}, \Gamma\kappa_n)\right) \\
&= F\left(\alpha\varpi(\kappa_n, \kappa_{n+1}) + \beta\varpi(\kappa_n, \kappa_{n-1}) + \gamma\varpi(\kappa_{n-1}, \kappa_{n+1})\right). \qquad (5)
\end{aligned}$$

By condition $(\varpi_3)$, we have

$$\varpi(\kappa_{n-1}, \kappa_{n+1}) \leq s[\varpi(\kappa_{n-1}, \kappa_n) + \varpi(\kappa_n, \kappa_{n+1})]. \qquad (6)$$

Applying (6) in (5), one can write

$$\tau + F(\varpi_n) \leq F\left(\alpha\varpi(\kappa_n, \kappa_{n+1}) + \beta\varpi(\kappa_n, \kappa_{n-1}) + \gamma s\varpi(\kappa_{n-1}, \kappa_n) + \gamma s\varpi(\kappa_n, \kappa_{n+1})\right).$$

Since $F$ is strictly increasing, then

$$\varpi_n < \alpha\varpi_n + \beta\varpi_{n-1} + \gamma s\varpi_{n-1} + \gamma s\varpi_n,$$

this leads to

$$(1 - \alpha - \gamma s)\varpi_n < (\beta + \gamma s)\varpi_{n-1} \text{ for all } n \geq 1.$$

Since $\alpha + \beta + 2\gamma s < 1$, we deduce that $1 - \alpha - \gamma s > 0$, and thus

$$\varpi_n < \frac{\beta + \gamma s}{1 - \alpha - \gamma s}\varpi_{n-1} < \varpi_{n-1}.$$

Consequently,

$$\tau + F(\varpi_n) \leq F(\varpi_{n-1}).$$

By the same method, we can prove that

$$\begin{aligned}
F(\varpi_n) &\leq F(\varpi_{n-1}) - \tau \\
&\leq F(\varpi_{n-2}) - 2\tau \\
&\quad\vdots \\
&\leq F(\varpi_\circ) - n\tau \text{ for all } n \geq 1. \qquad (7)
\end{aligned}$$

Passing the limit as $n \to \infty$ in (7), we can get

$$\lim_{n\to\infty} F(\varpi_n) = -\infty.$$

So, by $(\Im_2)$, we obtain

$$\lim_{n\to\infty} \varpi_n = 0. \qquad (8)$$

Apply $(\Im_3)$, there exists $\lambda \in (0,1)$ such that

$$\lim_{n\to\infty} \varpi_n^{\lambda} F(\varpi_n) = 0. \tag{9}$$

By (7), for all $n \geq 1$, yields

$$\varpi_n^{\lambda} \left(F(\varpi_n) - F(\varpi_\circ)\right) \leq -n\tau\varpi_n^{\lambda} \leq 0. \tag{10}$$

Considering (8), (9) and passing $n \to \infty$ in (10), we get

$$\lim_{n\to\infty} n\varpi_n^{\lambda} = 0. \tag{11}$$

By (11), there exists $n_1 \in \mathbb{N}$ such that $n\varpi_n^{\lambda} \leq 1$ for all $n \geq n_1$, or

$$\varpi_n \leq \frac{1}{n^{\frac{1}{\lambda}}} \; \forall n \geq n_1. \tag{12}$$

Now, we shall prove that $\{\kappa_n\}$ is $0-\varpi-$Cauchy sequence, let $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. By (12) and the assumption $(\varpi_3)$, we have

$$\begin{aligned}
\varpi(\kappa_n, \kappa_m) &\leq s\varpi(\kappa_n, \kappa_{n+1}) + s\varpi(\kappa_{n+1}, \kappa_m) \\
&\leq s\varpi(\kappa_n, \kappa_{n+1}) + s^2\varpi(\kappa_{n+1}, \kappa_{n+2}) + s^2\varpi(\kappa_{n+2}, \kappa_m) \\
&\quad \vdots \\
&\leq s\varpi(\kappa_n, \kappa_{n+1}) + s^2\varpi(\kappa_{n+1}, \kappa_{n+2}) + s^3\varpi(\kappa_{n+2}, \kappa_{n+3}) + \ldots \\
&\quad + s^{m-n-1}\varpi(\kappa_{m-2}, \kappa_{m-1}) + s^{m-n}\varpi(\kappa_{m-1}, \kappa_m) \\
&= \sum_{i=n}^{m-1} s^{i-n+1}\varpi_i \\
&\leq \sum_{i=n}^{\infty} s^{i-n+1}\left(\frac{1}{i^{\frac{1}{\lambda}}}\right).
\end{aligned}$$

Since the series $\sum\limits_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}$ is converges, as $i \to \infty$ and since multiply a scalar number in a convergent series gives a convergent series, so, $\varpi(\kappa_n, \kappa_m) \to 0$. Therefore $\{\kappa_n\}$ is $0-\varpi-$ Cauchy sequence in $(\Omega, \varpi)$. Since $\Omega$ is $0-\varpi-$complete $b-$metric-like space, there exists $\kappa^* \in \Omega$ such that $\kappa_n \to \kappa^*$ or equivalently,

$$\lim_{n,m\to\infty} \varpi(\kappa_n, \kappa_m) = \lim_{n\to\infty} \varpi(\kappa_n, \kappa^*) = \varpi(\kappa^*, \kappa^*) = 0. \tag{13}$$

If $\Gamma$ is $\varpi$ continuous, it follows from (12) that

$$\lim_{n\to\infty} \varpi(\Gamma\kappa_n, \Gamma\kappa^*) = \lim_{n\to\infty} \varpi(\kappa_{n+1}, \Gamma\kappa^*) = \varpi(\kappa^*, \Gamma\kappa^*) = 0.$$

this implies that

$$\kappa^* = \Gamma\kappa^*.$$

Furthermore, suppose that $F$ is continuous, we prove that $\kappa^*$ is a fixed point of $\Gamma$ by contrary, suppose $\kappa^* \neq \Gamma\kappa^*$, so there exist an $n_\circ \in \mathbb{N}$ and a subsequence $\{\kappa_{n_i}\}$ of $\{\kappa_n\}$ such that $\varpi(\kappa_{n_i+1}, \Gamma\kappa^*) > 0$ for all $n_i \geq n_\circ$ (otherwise, there exists $n_1 \in N$ such that $\kappa_n = \Gamma\kappa^* \; \forall n \geq n_1$, which implies that $\kappa_n \to \kappa^*$. That is a contradiction, with $\kappa^* \neq \Gamma\kappa^*$). Since $\varpi(\kappa_{n_i+1}, \Gamma\kappa^*) > 0 \; \forall n_i \geq n_\circ$, then by (4), we have

$$\begin{aligned} & \tau + F(s^q \varpi(\kappa_{n_i+1}, \Gamma\kappa^*)) \\ = \; & \tau + F(s^q \varpi(\Gamma\kappa_{n_i}, \Gamma\kappa^*)) \\ \leq \; & F\left(\alpha \frac{\varpi(\kappa_{n_i}, \Gamma\kappa_{n_i}).\varpi(\kappa^*, \Gamma\kappa^*)}{\varpi(\kappa_{n_i}, \kappa^*)} + \beta\varpi(\kappa_{n_i}, \kappa^*) + \gamma\varpi(\kappa^*, \Gamma\kappa_{n_i})\right) \\ = \; & F\left(\alpha \frac{\varpi(\kappa_{n_i}, \kappa_{n_i+1}).\varpi(\kappa^*, \Gamma\kappa^*)}{\varpi(\kappa_{n_i}, \kappa^*)} + \beta\varpi(\kappa_{n_i}, \kappa^*) + \gamma\varpi(\kappa^*, \kappa_{n_i+1})\right). \end{aligned} \tag{14}$$

Letting $n \to \infty$ in (14) and since $F$ is continuous, we can get

$$\begin{aligned} \tau + F(s^q \varpi(\kappa^*, \Gamma\kappa^*)) &\leq F(\alpha\varpi(\kappa^*, \Gamma\kappa^*)) \\ &< F(\varpi(\kappa^*, \Gamma\kappa^*)), \end{aligned}$$

the above inequality say that $s^q < 1$ for some $q > 1$. This a contradiction. Hence $\kappa^* = \Gamma\kappa^*$.

For uniqueness. Suppose that $\kappa_1^*$ and $\kappa_2^*$ are two distinct fixed points of a mapping $\Gamma$, hence $F(\varpi(\Gamma\kappa_1^*, \Gamma\kappa_2^*)) = \varpi(\kappa_1^*, \kappa_2^*) > 0$, which implies by (4) that

$$\begin{aligned} F(s^q \varpi(\kappa_1^*, \kappa_2^*)) &= F(s^q \varpi(\Gamma\kappa_1^*, \Gamma\kappa_2^*)) \\ &< \tau + F(s^q \varpi(\Gamma\kappa_1^*, \Gamma\kappa_2^*)) \\ &\leq F\left(\alpha \frac{\varpi(\kappa_1^*, \Gamma\kappa_1^*).\varpi(\kappa_2^*, \Gamma\kappa_2^*)}{\varpi(\kappa_1^*, \kappa_2^*)} + \beta\varpi(\kappa_1^*, \kappa_2^*) + \gamma\varpi(\kappa_2^*, \Gamma\kappa_1^*)\right) \\ &= F((\beta + \gamma)\varpi(\kappa_1^*, \kappa_2^*)) \\ &< F(\varpi(\kappa_1^*, \kappa_2^*)), \end{aligned}$$

a contradiction again. Hence, the fixed point is unique. The proof is finished. □

**Remark 4.** *In the real, we can obtain some classical results of our new contraction (4) if we take the following considerations on a complete metric space* $(\Omega, d)$.

- *Put* $\alpha = \gamma = 0$ *and* $\beta = 1$, *we have Wardowski contraction [21].*
- *Take* $\alpha = \gamma = 0$, $\beta \in [0, 1)$, $F(\theta) = \ln(\theta)$ *with* $\theta > 0$, *we get Banach contraction [1].*
- *Consider* $\gamma = 0$ *and* $F(\theta) = \ln(\theta)$ *with* $\theta > 0$, *we have Jaggi-contraction [29].*
- *Let* $\gamma = 0$, $F(\theta) = \ln(\theta)$ *with* $\theta > 0$, *we have Jaggi-contraction [28].*
- *Set* $\alpha = 0$, $\beta \in [0, 1)$, $\gamma \geq 0$, $F(\theta) = \ln(\theta)$ *with* $\theta > 0$, *we get Ciric almost contraction [30].*

Now, we present the following example to discuss the validity results of Theorem 3.

**Example 5.** *Let* $\Omega = [0, 1) \cap \mathbb{Q}$ *and* $\varpi : \Omega \times \Omega \to \mathbb{R}^+$ *be a function defined by*

$$\varpi(\kappa, \mu) = \begin{cases} 2\sqrt[3]{\kappa}, & \text{if } \kappa = \mu \\ \left(\max\{2\sqrt[3]{\kappa}, 2\sqrt[3]{\mu}\}\right)^2 & \text{otherwise} \end{cases}$$

*for all* $\kappa, \mu \in \Omega$. *Suppose that* $\{\kappa_n\} \in \Omega$. *If* $\lim_{n\to\infty} \varpi(\kappa_n, \kappa) = (\kappa, \kappa) = 0$, *then* $\lim_{n\to\infty} \varpi(\kappa_n, \mu) = (\kappa, \mu)$ *for all* $\mu \in \Omega$. *By the condition* $(\varpi_3)$, *we can write*

$$\varpi(\kappa_n, \kappa_m) \leq 2(\varpi(\kappa_n, \kappa) + \varpi(\kappa, \kappa_m)). \tag{15}$$

*Passing limit as* $n, m \to \infty$ *in (15), we obtain* $\lim_{n,m\to\infty} \varpi(\kappa_n, \kappa_m) = 0$. *Thus* $(\Omega, \varpi)$ *is* $0 - \omega -$*complete* $b -$*metric-like space with a coefficient* $s = 2$. *Note, here* $(\Omega, \varpi)$ *is not a complete* $b -$ *metric like space. Indeed,*

*consider the sequence* $\{\kappa_n\} = \frac{1}{n}$ *for* $n \in \mathbb{N}$ *in* $\Omega$. *then* $\lim_{n\to\infty} \kappa_n = 0$. *Let* $\lim_{n\to\infty} \varpi(\kappa_n, \kappa) = \varpi(0, \kappa)$ *for all* $\kappa \in \Omega$.

*If* $\kappa = 0$, *then* $\varpi(0, \kappa) = 0$. *So* $\lim_{n,m\to\infty} \varpi(\kappa_n, \kappa_m) = \varpi(0, \kappa) = \varpi(\kappa_n, \kappa)$.

*If* $\kappa \neq 0$, *then* $\varpi(0, \kappa) = 2\sqrt[3]{\kappa}$. *So* $\lim_{n,m\to\infty} \varpi(\kappa_n, \kappa_m) \neq \varpi(0, \kappa) = \varpi(\kappa_n, \kappa)$.

*Therefore,* $(\Omega, \varpi)$ *is a* $0 - \varpi-$*complete* $b-$*metric-like space, which is not a* $\varpi-$*complete* $b-$*metric-like space. Define a nonlinear mapping* $\Gamma$ *by* $\Gamma\kappa = \frac{\kappa}{512}$. *Take* $F(\theta) = \ln(\theta) + \theta$, $q = 3$ *and* $\tau = \ln(2)$. *We shall prove that* $\Gamma$ *satisfy the condition (4) with* $\alpha = \gamma = 0$ *and* $\beta = \frac{1}{4}$. *So* $\alpha + \beta + 2\gamma s = \frac{1}{4} < 1$. *Then for* $\kappa < \mu$,

$$\varpi(\Gamma\kappa, \Gamma\mu) = \varpi(\frac{\kappa}{512}, \frac{\mu}{512}) = \left(\frac{2\sqrt[3]{\mu}}{8}\right)^2 = \frac{\left(\sqrt[3]{\mu}\right)^2}{16} > 0,$$

*by simple calculations, we can get*

$$\begin{array}{ll} \varpi(\kappa, \Gamma\kappa) = \varpi(\kappa, \frac{\kappa}{512}) = 4\left(\sqrt[3]{\kappa}\right)^2, & \varpi(\mu, \Gamma\mu) = \varpi(\mu, \frac{\mu}{512}) = 4\left(\sqrt[3]{\mu}\right)^2, \\ \varpi(\kappa, \mu) = 4\left(\sqrt[3]{\mu}\right)^2, & \varpi(\mu, \Gamma\kappa) = \varpi(\mu, \frac{\kappa}{512}) = 4\left(\sqrt[3]{\mu}\right)^2, \end{array}$$

*and*

$$\begin{aligned} & \alpha\frac{\varpi(\kappa, \Gamma\kappa).\varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)} + \beta\varpi(\kappa, \mu) + \gamma\varpi(\mu, \Gamma\kappa) \\ = & \left(\alpha \times 4\left(\sqrt[3]{\kappa}\right)^2\right) + \left(\beta \times 4\left(\sqrt[3]{\mu}\right)^2\right) + \left(\gamma \times 4\left(\sqrt[3]{\mu}\right)^2\right) = \left(\sqrt[3]{\mu}\right)^2. \end{aligned}$$

*Hence*

$$\begin{aligned} \tau + F(s^q(\varpi(\Gamma\kappa, \Gamma\mu)) &= \ln(2) + F(8 \times \frac{\left(\sqrt[3]{\mu}\right)^2}{16}) \\ &= \ln(2) + F\left(\frac{\left(\sqrt[3]{\mu}\right)^2}{2}\right) \\ &\leq \ln(2) + \ln\left(\frac{\left(\sqrt[3]{\mu}\right)^2}{2}\right) + \frac{\left(\sqrt[3]{\mu}\right)^2}{2} \\ &\leq \ln\left(\left(\sqrt[3]{\mu}\right)^2\right) + \left(\sqrt[3]{\mu}\right)^2 \\ &= F\left(\left(\sqrt[3]{\mu}\right)^2\right) \\ &= F(\alpha\frac{\varpi(\kappa, \Gamma\kappa) \times \varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)} + \beta\varpi(\kappa, \mu) + \gamma\varpi(\mu, \Gamma\kappa)). \end{aligned}$$

*By the same manner, for* $\kappa = \mu \neq 0$, *one gets that*

$$\begin{aligned} \tau + F(s^q(\varpi(\Gamma\kappa, \Gamma\mu)) &= \ln(2) + F(8 \times \frac{\left(\sqrt[3]{\kappa}\right)^2}{16}) \\ &= \ln(2) + F\left(\frac{\left(\sqrt[3]{\kappa}\right)^2}{2}\right) \\ &\leq \ln(2) + \ln\left(\frac{\left(\sqrt[3]{\kappa}\right)^2}{2}\right) + \frac{\left(\sqrt[3]{\kappa}\right)^2}{2} \\ &\leq \ln\left(\left(\sqrt[3]{\kappa}\right)^2\right) + \left(\sqrt[3]{\kappa}\right)^2 \\ &= F\left(\left(\sqrt[3]{\kappa}\right)^2\right) \\ &= F(\alpha\frac{\varpi(\kappa, \Gamma\kappa).\varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)} + \beta\varpi(\kappa, \mu) + \gamma\varpi(\mu, \Gamma\kappa)). \end{aligned}$$

*So, all required hypotheses of Theorem 3 are verified and the point* $0 \in \Omega$ *is a unique fixed point of* $\Gamma$.

The second result of this section is to introduce the notion of a generalized almost $(s,q)-$Jaggi $F-$Suzuki contraction-type in the context of $b-$metric-like spaces and study some related fixed point results in this direction.

**Definition 11.** *Let* $\Gamma$ *be a self-mapping on a* $b-$*metric-like space* $(\Omega, \varpi)$ *with parameter* $s \geq 1$. *Then the mapping* $\Gamma$ *is said to be a generalized almost* $(s,q)-$*Jaggi* $F-$*Suzuki contraction-type if there exists* $F \in \Sigma$ *and* $\tau \in (0, +\infty)$ *such that*

$$\tfrac{1}{2s}\varpi(\kappa, \Gamma\kappa) < \varpi(\kappa, \mu) \Rightarrow \tau + F\left(s^q \varpi(\Gamma\kappa, \Gamma\mu)\right) \leq F\left(\alpha \tfrac{\varpi(\kappa,\Gamma\kappa).\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)} + \beta\varpi(\kappa,\mu) + \gamma\varpi(\mu,\Gamma\kappa)\right), \tag{16}$$

*for all* $\kappa, \mu \in \Omega$ *and* $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + 2\gamma s < 1$, *for some* $q > 1$ *and satisfying* $\varpi(\Gamma\kappa, \Gamma\mu) > 0$.

**Theorem 4.** *Let* $(\Omega, \varpi)$ *be a* $0 - \varpi-$*complete* $b-$*metric-like space with a coefficient* $s \geq 1$ *and* $\Gamma$ *be a self mapping satisfying a generalized almost* $(s,q)-$*Jaggi-type* $F-$*Suzuki contraction (16).Then,* $\Gamma$ *has a unique fixed point whenever* $F$ *or* $\Gamma$ *is continuous.*

**Proof.** Let $\kappa_\circ \in \Omega$ and $\{\kappa_n\}_{n=1}^{\infty}$ defined by $\kappa_{n+1} = \Gamma\kappa_n = \Gamma^{n+1}\kappa_\circ$. If there exists $n \in \mathbb{N}$ such that $\varpi(\kappa_n, \Gamma\kappa_n) = 0$, thus the proof is completed. So, suppose that $0 < \varpi(\kappa_n, \Gamma\kappa_n) = \varpi(\kappa_n, \kappa_{n+1}) = \varpi_n$, therefore for all $n \in \mathbb{N}$

$$\frac{1}{2s}\varpi(\kappa_n, \Gamma\kappa_n) < \varpi(\kappa_n, \Gamma\kappa_n),$$

yields

$$\begin{aligned}
\tau + F\left(\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n)\right) &\leq \tau + F\left(s^q\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n)\right) = \tau + F\left(s^q\varpi(\Gamma\kappa_n, \Gamma(\Gamma\kappa_n))\right) \\
&\leq F\left(\alpha\frac{\varpi(\kappa_n,\Gamma\kappa_n).\varpi(\Gamma\kappa_n,\Gamma^2\kappa_n)}{\varpi(\kappa_n,\Gamma\kappa_n)} + \beta\varpi(\kappa_n,\Gamma\kappa_n) + \gamma\varpi(\Gamma\kappa_n,\Gamma\kappa_n)\right) \\
&= F\left(\alpha\varpi(\Gamma\kappa_n,\Gamma^2\kappa_n) + \beta\varpi(\kappa_n,\Gamma\kappa_n) + \gamma\varpi(\Gamma\kappa_n,\Gamma\kappa_n)\right) \\
&\leq F\left(\alpha\varpi(\Gamma\kappa_n,\Gamma^2\kappa_n) + \beta\varpi(\kappa_n,\Gamma\kappa_n) + 2s\gamma\varpi(\kappa_n,\Gamma\kappa_n)\right).
\end{aligned}$$

Since $F$ is strictly increasing, then

$$\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n) < \alpha\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n) + \beta\varpi(\kappa_n, \Gamma\kappa_n) + 2s\gamma\varpi(\kappa_n, \Gamma\kappa_n). \tag{17}$$

Since $1 - \alpha > 0$, and $\alpha + \beta + 2\gamma s < 1$ then we can write

$$\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n) < \left(\frac{\beta + 2s\gamma}{1-\alpha}\right)\varpi(\kappa_n, \Gamma\kappa_n) < \varpi(\kappa_n, \Gamma\kappa_n). \tag{18}$$

Consequently,

$$\tau + F(\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n)) \leq F(\varpi(\kappa_n, \Gamma\kappa_n)),$$

or,

$$F(\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n)) \leq F(\varpi(\kappa_n, \Gamma\kappa_n)) - \tau,$$

By the same method, we can deduce that

$$\begin{aligned} F(\varpi_n) &= F(\varpi(\kappa_n, \Gamma\kappa_n)) = F(\varpi(\Gamma\kappa_{n-1}, \Gamma^2\kappa_{n-1})) \\ &\leq F(\varpi(\kappa_{n-1}, \Gamma\kappa_{n-1})) - \tau \\ &\leq F(\varpi(\kappa_{n-2}, \Gamma\kappa_{n-2})) - 2\tau \\ &\quad . \\ &\quad . \\ &\leq F(\varpi(\kappa_\circ, \Gamma\kappa_\circ)) - n\tau \text{ for all } n \geq 1. \end{aligned}$$

By the same manner of Theorem 3, we deduce that $\{\kappa_n\}$ is $0-\varpi-$Cauchy sequence in $(\Omega, \varpi)$. Since $\Omega$ is $0-\varpi-$complete $b-$metric-like space, there exists $\kappa^* \in \Omega$ such that $\kappa_n \to \kappa^*$ or equivalently,

$$\lim_{n,m\to\infty} \varpi(\kappa_n, \kappa_m) = \lim_{n\to\infty} \varpi(\kappa_n, \kappa^*) = \varpi(\kappa^*, \kappa^*) = 0. \tag{19}$$

Now, we shall prove that

$$\frac{1}{2s}\varpi(\kappa_n, \Gamma\kappa_n) < \varpi(\kappa_n, \kappa^*) \text{ or } \frac{1}{2s}\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n) < \varpi(\Gamma\kappa_n, \kappa^*). \tag{20}$$

Assuming the opposite, that there is $m \in \mathbb{N}$ such that

$$\frac{1}{2s}\varpi(\kappa_m, \Gamma\kappa_m) \geq \varpi(\kappa_m, \kappa^*) \text{ or } \frac{1}{2s}\varpi(\Gamma\kappa_m, \Gamma^2\kappa_m) \geq \varpi(\Gamma\kappa_m, \kappa^*). \tag{21}$$

Hence

$$2s\varpi(\kappa_m, \kappa^*) \leq \varpi(\kappa_m, \Gamma\kappa_m) \leq s[\varpi(\kappa_m, \kappa^*) + \varpi(\kappa^*, \Gamma\kappa_m)],$$

which leads to

$$s\varpi(\kappa_m, \kappa^*) \leq s\varpi(\kappa^*, \Gamma\kappa_m),$$

or

$$\varpi(\kappa_m, \kappa^*) \leq \varpi(\kappa^*, \Gamma\kappa_m), \tag{22}$$

From (21) and (22), we have

$$\varpi(\kappa_m, \kappa^*) \leq \varpi(\kappa^*, \Gamma\kappa_m) \leq \frac{1}{2s}\varpi(\Gamma\kappa_m, \Gamma^2\kappa_m).$$

Since $\frac{1}{2s}\varpi(\kappa_m, \Gamma\kappa_m) < \varpi(\kappa_m, \Gamma\kappa_m)$ and using (16), we can get

$$\begin{aligned} \tau + F\left(\varpi(\Gamma\kappa_m, \Gamma^2\kappa_m)\right) &\leq \tau + F\left(s^q\varpi(\Gamma\kappa_m, \Gamma^2\kappa_m)\right) \\ &\leq F\left(\alpha\frac{\varpi(\kappa_m, \Gamma\kappa_m)}{\varpi(\kappa_m, \Gamma\kappa_m)} + \beta\varpi(\kappa_m, \Gamma\kappa_m) + \gamma\varpi(\Gamma\kappa_m, \Gamma\kappa_m)\right) \\ &= F\left(\alpha\varpi(\Gamma\kappa_m, \Gamma^2\kappa_m) + \beta\varpi(\kappa_m, \Gamma\kappa_m) + \gamma\varpi(\Gamma\kappa_m, \Gamma\kappa_m)\right) \\ &\leq F\left(\alpha\varpi(\Gamma\kappa_m, \Gamma^2\kappa_m) + \beta\varpi(\kappa_m, \Gamma\kappa_m) + 2s\gamma\varpi(\kappa_m, \Gamma\kappa_m)\right). \end{aligned}$$

Replace $n$ with $m$ in the inequalities (17) and (18) and apply the same above steps, we can write

$$\varpi(\Gamma\kappa_m, \Gamma^2\kappa_m) < \varpi(\kappa_m, \Gamma\kappa_m).$$

It follows from (21) and (22), that

$$\begin{aligned}\varpi(\Gamma\kappa_m,\Gamma^2\kappa_m) &< \varpi(\kappa_m,\Gamma\kappa_m)\\ &\leq s[\varpi(\kappa_m,\kappa^*)+\varpi(\kappa^*,\Gamma\kappa_m)]\\ &\leq s[\varpi(\kappa^*,\Gamma\kappa_m)+\frac{1}{2s}\varpi(\Gamma\kappa_m,\Gamma^2\kappa_m)]\\ &\leq s[\frac{1}{2s}\varpi(\Gamma\kappa_m,\Gamma^2\kappa_m)+\frac{1}{2s}\varpi(\Gamma\kappa_m,\Gamma^2\kappa_m)]\\ &= \varpi(\Gamma\kappa_m,\Gamma^2\kappa_m).\end{aligned}$$

A contradiction, so (20) holds for all $n\in\mathbb{N}$, this leads to

$$\begin{aligned}\tau+F\left(2s^2\varpi(\Gamma\kappa_n,\Gamma\kappa^*)\right) &\leq \tau+F\left(2s^q\varpi(\Gamma\kappa_n,\Gamma\kappa^*)\right)\\ &\leq F\left(\alpha\frac{\varpi(\kappa_n,\Gamma\kappa_n).\varpi(\kappa^*,\Gamma\kappa^*)}{\varpi(\kappa_n,\kappa^*)}+\beta\varpi(\kappa_n,\kappa^*)+\gamma\varpi(\kappa^*,\Gamma\kappa_n)\right)\\ &< F\left(\alpha\frac{2s\varpi(\kappa_n,\kappa^*).\varpi(\kappa^*,\Gamma\kappa^*)}{\varpi(\kappa_n,\kappa^*)}+\beta\varpi(\kappa_n,\kappa^*)+\gamma\varpi(\kappa^*,\Gamma\kappa_n)\right)\\ &= F\left(2s\alpha\varpi(\kappa^*,\Gamma\kappa^*)+\beta\varpi(\kappa_n,\kappa^*)+\gamma\varpi(\kappa^*,\Gamma\kappa_n)\right)\\ &\leq F\left(\begin{array}{c}2s^2\alpha\varpi(\kappa^*,\kappa_n)+2s^3\alpha\varpi(\kappa_n,\Gamma\kappa_n)+2s^3\alpha\varpi(\Gamma\kappa_n,\Gamma\kappa^*)\\ +\beta\varpi(\kappa_n,\kappa^*)+\gamma s\varpi(\kappa^*,\kappa_n)+\gamma s\varpi(\kappa_n,\Gamma\kappa_n)\end{array}\right),\end{aligned}$$

a gain, since $\varpi(\kappa_n,\Gamma\kappa_n)<2s\varpi(\kappa_n,\kappa^*)$, $\tau>0$ and $F$ is strictly increasing, this yields,

$$2s^2\varpi(\Gamma\kappa_n,\Gamma\kappa^*)<(2s^2\alpha+4s^4\alpha+\beta+\gamma s+2s^2\gamma)\varpi(\kappa_n,\kappa^*)+2s^3\alpha\varpi(\Gamma\kappa_n,\Gamma\kappa^*),$$

or,

$$(1-2\alpha s)\,\varpi(\Gamma\kappa_n,\Gamma\kappa^*)<(\alpha+2s^2\alpha+\beta+2\gamma)\varpi(\kappa_n,\kappa^*),$$

Thus,

$$\begin{aligned}\tau+F((1-2\alpha s)\,\varpi(\Gamma\kappa_n,\Gamma\kappa^*)) &\leq F((\alpha+2s^2\alpha+\beta+2\gamma)\varpi(\kappa_n,\kappa^*))\\ &= F(\alpha+2s^2\alpha+\beta+2\gamma)\varpi(\Gamma\kappa_{n+1},\kappa^*)).\end{aligned}\tag{23}$$

Passing the limit as $n\to\infty$ in (23), using $(\Im_2)$ and (19) we can get

$$\lim_{n\to\infty}\varpi(\Gamma\kappa_n,\Gamma\kappa^*)=0.$$

Therefore, $\varpi(\kappa^*,\Gamma\kappa^*)=\lim_{n\to\infty}\varpi(\kappa_n,\Gamma\kappa^*)=\lim_{n\to\infty}\varpi(\Gamma\kappa_{n+1},\Gamma\kappa^*)=0$.
Similarly, by (20) for all $n\in\mathbb{N}$, one can write

$$\begin{aligned}\tau+F\left(2s^2\varpi(\Gamma^2\kappa_n,\Gamma\kappa^*)\right) &\leq \tau+F\left(2s^q\varpi(\Gamma^2\kappa_n,\Gamma\kappa^*)\right)\\ &\leq F\left(\alpha\frac{\varpi(\Gamma\kappa_n,\Gamma^2\kappa_n).\varpi(\kappa^*,\Gamma\kappa^*)}{\varpi(\Gamma\kappa_n,\kappa^*)}+\beta\varpi(\Gamma\kappa_n,\kappa^*)+\gamma\varpi(\kappa^*,\Gamma^2\kappa_n)\right)\\ &< F\left(\alpha\frac{2s\varpi(\Gamma\kappa_n,\kappa^*).\varpi(\kappa^*,\Gamma\kappa^*)}{\varpi(\Gamma\kappa_n,\kappa^*)}+\beta\varpi(\Gamma\kappa_n,\kappa^*)+\gamma\varpi(\kappa^*,\Gamma^2\kappa_n)\right)\\ &= F\left(2s\alpha\varpi(\kappa^*,\Gamma\kappa^*)+\beta\varpi(\Gamma\kappa_n,\kappa^*)+\gamma\varpi(\kappa^*,\Gamma^2\kappa_n)\right)\\ &\leq F\left(\begin{array}{c}2s^2\alpha\varpi(\kappa^*,\Gamma\kappa_n)+2s^3\alpha\varpi(\Gamma\kappa_n,\Gamma^2\kappa_n)+2s^3\alpha\varpi(\Gamma^2\kappa_n,\Gamma\kappa^*)\\ +\beta\varpi(\Gamma\kappa_n,\kappa^*)+\gamma s\varpi(\kappa^*,\Gamma\kappa_n)+\gamma s\varpi(\Gamma\kappa_n,\Gamma^2\kappa_n)\end{array}\right),\end{aligned}$$

since $\varpi(\Gamma\kappa_n, \Gamma^2\kappa_n) < 2s\varpi(\Gamma\kappa_n, \kappa^*)$, $\tau > 0$ and $F$ is strictly increasing, this leads to

$$2s^2\varpi(\Gamma^2\kappa_n, \Gamma\kappa^*) < (2s^2\alpha + 4s^4\alpha + \beta + \gamma s + 2\gamma s^2)\varpi(\kappa^*, \Gamma\kappa_n) + 2s^3\alpha\varpi(\Gamma^2\kappa_n, \Gamma\kappa^*),$$

or,

$$(1 - 2\alpha s)\varpi(\Gamma^2\kappa_n, \Gamma\kappa^*) < (\alpha + 2s^2\alpha + \beta + 2\gamma)\varpi(\kappa^*, \Gamma\kappa_n).$$

Thus,

$$\begin{aligned} \tau + F((1 - 2\alpha s)\,\varpi(\Gamma^2\kappa_n, \Gamma\kappa^*)) &\leq F((\alpha + 2s^2\alpha + \beta + 2\gamma)\varpi(\kappa^*, \Gamma\kappa_n)) \\ &= F(\alpha + 2s^2\alpha + \beta + 2\gamma)\varpi(\kappa_{n+1}, \kappa^*)). \end{aligned} \tag{24}$$

Taking the limit as $n \to \infty$ in (24), using $(\Im_2)$ and (19) we can get

$$\lim_{n\to\infty} \varpi(\Gamma^2\kappa_n, \Gamma\kappa^*) = 0.$$

Therefore, $\varpi(\kappa^*, \Gamma\kappa^*) = \lim_{n\to\infty} \varpi(\kappa_{n+2}, \Gamma\kappa^*) = \lim_{n\to\infty} \varpi(\Gamma^2\kappa_n, \Gamma\kappa^*) = 0$. Hence, $\kappa^* = \Gamma\kappa^*$.

The uniqueness follows immediately from the proof of Theorem 3, and this completes the proof. □

**Example 6.** *By taking all assumptions of Example 5, if $\kappa < \mu$, we have*

$$\frac{1}{2s}\varpi(\kappa, \Gamma\kappa) = \frac{1}{4} \times 4\left(\sqrt[3]{\kappa}\right)^2 = \left(\sqrt[3]{\kappa}\right)^2 < 4\left(\sqrt[3]{\mu}\right)^2 = \varpi(\kappa, \mu),$$

*also, if $\kappa = \mu \neq 0$, we deduce that*

$$\frac{1}{2s}\varpi(\kappa, \Gamma\kappa) = \frac{1}{4} \times 4\left(\sqrt[3]{\kappa}\right)^2 = \left(\sqrt[3]{\kappa}\right)^2 < 4\left(\sqrt[3]{\kappa}\right)^2 = \varpi(\kappa, \mu).$$

*Therefore all required hypotheses of Theorem 4 are satisfied and a mapping $\Gamma$ has a unique fixed point $0 \in \Omega$.*

## 4. Solution of Electric Circuit Equation

Fixed point theory is involved in physical applications especially the solution of the the electric circuit equation, which was presented in [35,36]. The authors applied their theorems obtained to solve this equation under $F-$contraction mapping. In this part, we present the solution of electric circuit equation, which is in the form of second-order differential equation. It contains a resistor $R$, an electromotive force $E$, a capacitor $C$, an inductor $L$ and a voltage $V$ in series as Figure 1.

If the rate of change of charge $q$ with respect to time $t$ denoted by the current $I$, i.e., $I = \frac{dq}{dt}$. We get the following relations:

- $V = IR$;
- $V = \frac{q}{C}$;
- $V = L\frac{dI}{dt}$.

The sum of these voltage drops is equal to the supplied voltage (law of Kirchhoff voltage), i.e.,

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t),$$

or

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t),\ q(0) = 0,\ q'(0) = 0, \tag{25}$$

where $C = \frac{4L}{R^2}$ and $\tau = \frac{R}{2L}$, this case is said to be the resonance solution in a Physics context. Then, the Green function associated with (25) is given by

$$\Lambda(t,\varrho) = \begin{cases} -\varrho e^{-\tau(\varrho - t)} \text{ if } & 0 \leq \varrho \leq t \leq 1 \\ -te^{-\tau(\varrho - t)} \text{ if } & 0 \leq t \leq \varrho \leq 1 \end{cases}.$$

Using Green function, problem (25) is equivalent to the following nonlinear integral equation

$$\kappa(t) = \int_0^t \Lambda(t,\varrho)\chi(\varrho,\kappa(\varrho))d\varrho, \tag{26}$$

where $t \in [0,1]$.

Let $\Omega = C([0,1])$ be the set of all continuous functions defined on $[0,1]$, endowed with

$$\varpi(\kappa,\mu) = (\|\kappa\|_\infty + \|\mu\|_\infty)^m \text{ for all } \kappa,\mu \in \Omega,$$

where $\|\kappa\|_\infty = \sup_{t\in[0,1]}\{|\kappa(t)|\, e^{-2t\tau m}\}$ and $m > 1$. It is clear that $(\Omega,\varpi)$ is a complete $b-$metric-like space with parameter $s = 2^{m-1}$.

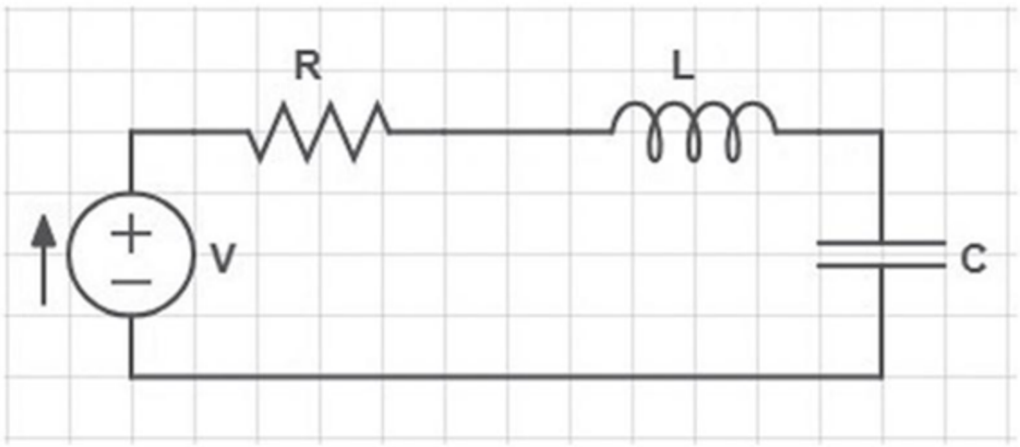


**Figure 1.** Electric circuit.

Now, we state and prove the main theorem of this section.

**Theorem 5.** *Let $\Gamma$ be a nonlinear self mapping on $\Omega$ of a $b-$metric-like space $(\Omega,\varpi)$, such that the following conditions hold*
*(i) $\Lambda : [0,1]\times[0,1] \to [0,\infty)$ is a continuous function;*
*(ii) $\chi : [0,1]\times\mathbb{R} \to \mathbb{R}$, where $\chi(\varrho,.)$ is monotone nondecreasing mapping for all $\varrho \in [0,1]$;*
*(iii) there exists a constant $\tau \in \mathbb{R}^+$ such that for all $(t,\varrho) \in [0,1]^2$ and $\kappa,\mu \in \mathbb{R}^+$,*

$$|\chi(t,\kappa) + \chi(t,\mu)| \leq \tau^2 \left(\frac{e^{-\tau}}{s^2}\right)^{\frac{1}{m}} B^{\frac{1}{m}}(\kappa,\mu),$$

*where*

$$B(\kappa,\mu) = \alpha\frac{\varpi(\kappa,\Gamma\kappa).\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)} + \beta\varpi(\kappa,\mu) + \gamma\varpi(\mu,\Gamma\kappa),$$

*for all, $t \in [0,1]$ and $\alpha,\beta,\gamma \geq 0$ such that $\alpha + \beta + 2\gamma s < 1$. Then the Equation (25) has a unique solution.*

**Proof.** Define a nonlinear self-mapping $\Gamma : \Omega \to \Omega$ by

$$\Gamma\kappa(t) = \int_0^t \Lambda(t,\varrho)\chi(\varrho,\kappa(\varrho))d\varrho.$$

It is clear that if $\kappa^*$ is a fixed point of the mapping $\Gamma$, then it a solution of the problem (26). Suppose that $\kappa, \mu \in \Omega$, we can get $\varpi(\Gamma\kappa(t), \Gamma\mu(t)) > 0$,

$$\begin{aligned}
& s^2\left(|\Gamma\kappa(t)| + |\Gamma\mu(t)|\right)^m \\
= \; & s^2\left(\left|\int_0^t \Lambda(t,\varrho)\chi(\varrho,\kappa(\varrho))d\varrho\right| + \left|\int_0^t \Lambda(t,\varrho)\chi(\varrho,\mu(\varrho))d\varrho\right|\right)^m \\
\leq \; & s^2\left(\int_0^t |\Lambda(t,\varrho)\chi(\varrho,\kappa(\varrho))|\, d\varrho + \int_0^t |\Lambda(t,\varrho)\chi(\varrho,\mu(\varrho))d\varrho|\right)^m \\
= \; & s^2\left(\int_0^t \Lambda(t,\varrho)\left(|\chi(\varrho,\kappa(\varrho))| + |\chi(\varrho,\mu(\varrho))|\right) d\varrho\right)^m \\
\leq \; & s^2\left(\int_0^t \Lambda(t,\varrho)\tau^2\left(\frac{e^{-\tau}}{s^2}\right)^{\frac{1}{m}} B^{\frac{1}{m}}(\kappa,\mu)d\varrho\right)^m \\
= \; & s^2\left(\int_0^t \Lambda(t,\varrho)\tau^2 e^{2\tau\varrho}e^{-2\tau\varrho}\left(\frac{e^{-\tau}}{s^2}\right)^{\frac{1}{m}} B^{\frac{1}{m}}(\kappa,\mu)d\varrho\right)^m \\
\leq \; & e^{-\tau}\left\|B(\kappa,\mu)\right\|_\infty \left(\tau^2\int_0^t e^{2\tau\varrho}\Lambda(t,\varrho)d\varrho\right)^m \\
\leq \; & e^{-\tau}\left\|B(\kappa,\mu)\right\|_\infty \left(e^{2\tau t}\left[1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}\right]\right)^m,
\end{aligned}$$

so we have

$$s^2\left(|\Gamma\kappa(t)| + |\Gamma\mu(t)|\right)^m .e^{-2t\tau m} \leq e^{-\tau}\left\|B(\kappa,\mu)\right\|_\infty \left[1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}\right]^m,$$

which leads to,

$$s^2\left(\|\Gamma\kappa(t)\|_\infty + \|\Gamma\mu(t)\|_\infty\right)^m \leq e^{-\tau}\left\|B(\kappa,\mu)\right\|_\infty \left[1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}\right]^m,$$

since $1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t} \leq 1$, we obtain that

$$s^2\varpi(\Gamma\kappa(t), \Gamma\mu(t)) \leq e^{-\tau}\left\|B(\kappa,\mu)\right\|_\infty.$$

Taking $F(\theta) = \ln(\theta)$, for all $\theta > 0$, which is $F \in \Sigma$, we obtain

$$\ln(s^2\varpi(\Gamma\kappa(t), \Gamma\mu(t))) \leq \ln(e^{-\tau}\left\|B(\kappa,\mu)\right\|_\infty),$$

or

$$\tau + \ln(s^2\varpi(\Gamma\kappa(t), \Gamma\mu(t))) \leq \ln(\|B(\kappa,\mu)\|_\infty).$$

Equivalently

$$\tau + F(s^q \varpi(\Gamma\kappa(\rho), \Gamma\mu(\rho))) \leq F(\alpha \frac{\varpi(\kappa, \Gamma\kappa).\varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)} + \beta\varpi(\kappa, \mu) + \gamma\varpi(\mu, \Gamma\kappa)).$$

By Theorem 3 and taking the coefficient $q = 2$, we deduce that $\Gamma$ has a fixed point, which is a solution of the differential equation arising in the electric circuit equation. This finished the proof. □

The following example satisfy all required hypotheses of Theorem 5.

**Example 7.** *Consider the following nonlinear integral equation*

$$\kappa(t) = \frac{\tau^2 e^{-4t\tau}}{50s^3} \int_0^1 \varrho^3 \kappa(\varrho) d\varrho, \ \varrho \in [0, 1]. \tag{27}$$

*Then it has a solution in* $\Omega$.

**Proof.** Let $\Gamma : \Omega \to \Omega$ be defined by $\Gamma\kappa(t) = \frac{\tau^2 e^{-4t\tau}}{50s^3} \int_0^1 \varrho^3 \kappa(\varrho) d\varrho$. By specifying $\Lambda(t, \varrho) = \frac{\varrho^3}{5}$, $\chi(t, \kappa) = \frac{\tau^2 e^{-4t\tau} \kappa(\varrho)}{10s^3}$ in Theorem 5, it follows that:
(i) the function $\Lambda(t, \varrho)$ is continuous on $[0, 1] \times [0, 1]$,
(ii) $\chi(\varrho, \kappa(\varrho))$ is monotone increasing on $[0, 1] \times \mathbb{R}$ for all $\varrho \in [0, 1]$,
(iii) By taking $m = 2$ and $\tau = \ln(4)$, hence $s = 2$, so, for all $(t, \varrho) \in [0, 1] \times [0, 1]$ and $\kappa, \mu \in \mathbb{R}^+$, we obtain that

$$\begin{aligned}
|\chi(t, \kappa) + \chi(t, \mu)| &= \frac{\tau^2 e^{-4t\tau}}{10s^3} |\kappa(\varrho) + \mu(\varrho)| \leq \frac{\tau^2 e^{-4t\tau}}{20s^2} (|\kappa(\varrho)| + |\mu(\varrho)|) \\
&\leq \frac{\tau^2 e^{-4t\tau}}{4s^2} (|\kappa(\varrho)| + |\mu(\varrho)|)^2 \\
&= \left(\frac{\tau}{s}\right)^2 (\frac{1}{e^{\ln 4}.2^2})^{\frac{1}{2}} \left([|\kappa(\varrho)| + |\mu(\varrho)|] e^{-4t\tau}\right)^2 \\
&= \left(\frac{\tau}{s}\right)^2 (\frac{1}{e^{\tau}.s^2})^{\frac{1}{m}} \left([|\kappa(\varrho)| + |\mu(\varrho)|] e^{-2t\tau m}\right)^m \\
&= \left(\frac{\tau}{s}\right)^2 (\frac{1}{e^{\tau}.s^2})^{\frac{1}{m}} (\|\kappa(\varrho)\|_\infty + \|\mu(\varrho)\|_\infty)^m \\
&= \left(\frac{\tau}{s}\right)^2 (\frac{1}{e^{\tau}.s^2})^{\frac{1}{m}} \varpi(\kappa, \mu) \\
&\leq \tau^2 \left(\frac{e^{-\tau}}{s^2}\right)^{\frac{1}{m}} B^{\frac{1}{m}}(\kappa, \mu).
\end{aligned}$$

Finally

$$\sup_{t,\varrho \in [0,1]} \int_0^1 \Lambda(t, \varrho) d\varrho = \sup_{t \in [0,1]} \int_0^1 \frac{\varrho^3}{5} d\varrho \leq \sup_{\rho \in [0,1]} \frac{1}{20} = \frac{1}{20} < 1.$$

Therefore, all conditions of Theorem 5 are satisfied, therefore a mapping $\Gamma$ has a fixed point in $\Omega$, which is a solution to the problem (27). □

## 5. Solution of Second-Order Differential Equations

In this part, we shall apply the previous theoretical results of Theorem 3 to study the existence and uniqueness of solutions for the following second-order differential equation:

$$\begin{cases} \kappa''(t) = -\chi(t, \kappa(t)), & t \in [0,1] \\ \kappa(0) = \kappa(1) = 0 \end{cases}, \tag{28}$$

where $\chi : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous functions.

The problem (28) is equivalent to the following integral equation:

$$\kappa(t) = \int_0^1 \Lambda(t, \varrho)\chi(t, \kappa(\varrho))d\varrho, \ \forall t \in [0,1] \tag{29}$$

where $\Lambda$ is the Green function defined by

$$\Lambda(\rho, \varrho) = \begin{cases} t(1-\varrho) \text{ if } & 0 \le t \le \varrho \le 1 \\ \varrho(1-t) \text{ if } & 0 \le \varrho \le t \le 1 \end{cases},$$

and $\chi$ be a function as in Theorem 5. Hence if $\kappa \in C^2([0,1], \mathbb{R})$, then $\kappa$ is a solution of (28) if and only if $\kappa$ is a solution of (29).

Let $\Omega = C([0,1], \mathbb{R})$ be the set of all continuous functions defined on $[0,1]$, endowed with the same distance of the above section. Then $(\Omega, \varpi)$ is a complete $b-$ metric-like space with parameter $s = 2^{m-1}$.

Now, we introduce the main theorem of this part.

**Theorem 6.** *Let $\Gamma$ be a nonlinear self mapping on $\Omega$ of a $b-$metric-like space $(\Omega, \varpi)$, such that there exists monotone nondecreasing mapping $\chi : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that*

$$|\chi(\varrho, \kappa) + \chi(\varrho, \mu)| \le \left(\frac{7\beta e^{-\tau}}{s^2}\right)^{\frac{1}{m}} (|\kappa| + |\mu|),$$

*for all $\varrho \in [0,1]$, $\kappa, \mu \in \mathbb{R}$, $0 \le \beta < \frac{1}{8}$ and for $m > 1$.*

*Then the problem (28) has a unique solution $\kappa \in C([0,1], \mathbb{R})$, provided that the conditions (i) and (ii) of Theorem 5 are satisfied.*

**Proof.** Let us define a nonlinear self-mapping $\Gamma$ on a set $\Omega$ by

$$\Gamma\kappa(t) = \int_0^1 \Lambda(t, \varrho)\chi(t, \kappa(\varrho))d\varrho,$$

for all $t \in [0,1]$ and $\kappa \in \Omega$. The solution of the problem (28) is equivalent to find a fixed point $\kappa$ of $\Gamma$ on $\Omega$. Suppose that $\kappa, \mu \in \Omega$, we have $\varpi(\Gamma\kappa, \Gamma\mu) > 0$,

$$\begin{aligned}
s^2\left(|\Gamma\kappa(t)|+|\Gamma\mu(t)|\right)^m &= s^2\left(\left|\int_0^1\Lambda(t,\varrho)\chi(t,\kappa(\varrho))d\varrho\right|+\left|\int_0^1\Lambda(t,\varrho)\chi(t,\mu(\varrho))d\varrho\right|\right)^m\\
&\leq s^2\left(\int_0^1|\Lambda(t,\varrho)\chi(t,\kappa(\varrho))|\,d\varrho+\left|\int_0^1|\Lambda(t,\varrho)\chi(t,\mu(\varrho))|\,d\varrho\right|\right)^m\\
&= s^2\left(\int_0^1\Lambda(t,\varrho)\,|\chi(t,\kappa(\varrho))+\chi(t,\mu(\varrho))|\,d\varrho\right)^m\\
&\leq s^2\left(\int_0^1\Lambda(t,\varrho)\left(\frac{7\beta e^{-\tau}}{s^2}\right)^{\frac{1}{m}}(|\kappa|+|\mu|)d\varrho\right)^m\\
&= 7\beta e^{-\tau}\left(\int_0^1\Lambda(t,\varrho)e^{2t\tau}e^{-2t\tau}\,(|\kappa|+|\mu|)\,d\varrho\right)^m\\
&\leq 7\beta e^{-\tau}\left(\|\kappa\|_\infty+\|\mu\|_\infty\right)^m e^{2t\tau m}\left(\int_0^1\Lambda(t,\varrho)d\varrho\right)^m\\
&= \frac{7}{7^m}\beta e^{-\tau}e^{2t\tau m}\left(\|\kappa\|_\infty+\|\mu\|_\infty\right)^m\\
&\leq \beta e^{-\tau}e^{2t\tau m}\left(\|\kappa\|_\infty+\|\mu\|_\infty\right)^m.
\end{aligned}$$

For instance above, for all $t\in[0,1]$, we can get $\int_0^1\Lambda(t,\varrho)d\varrho=\frac{1}{2}(1-t)$ and thus, we choose $\sup_{\varrho\in[0,1]}\int_0^1\Lambda(t,\varrho)d\varrho=\frac{1}{7}$. Hence

$$s^2\left(\Gamma\kappa(t)+|\Gamma\mu(t)|\right)^m.e^{-2t\tau m}\leq e^{-\tau}\beta\left(\|\kappa\|_\infty+\|\mu\|_\infty\right)^m,$$

or

$$s^2\left(\|\Gamma\kappa\|_\infty+\|\Gamma\mu\|_\infty\right)^m\leq e^{-\tau}\beta\left(\|\kappa\|_\infty+\|\mu\|_\infty\right)^m, \tag{30}$$

Let $\alpha=\gamma=0$, hence $\alpha+\beta+2s\gamma<\frac{1}{8}$. It follows from (30) that

$$s^2\varpi(\Gamma\kappa,\Gamma\mu)\leq\left(e^{-\tau}\right)[\beta\varpi(\kappa,\mu)]. \tag{31}$$

Taking the function $F(\theta)=\ln(\theta)$ in (31), such that $F\in\Sigma$, we can obtain

$$\begin{aligned}
\tau+F(s^2\varpi(\Gamma\kappa,\Gamma\mu)) &\leq F(\beta\varpi(\kappa,\mu))\\
&= F(\alpha\frac{\varpi(\kappa,\Gamma\kappa).\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)}+\beta\varpi(\kappa,\mu)+\gamma\varpi(\mu,\Gamma\kappa)).
\end{aligned}$$

Hence all requirements of Theorem 3 are holds by taking the coefficient $q=2$, therefore $\Gamma$ has a fixed point $\kappa\in\Omega$, that is, (28) has a unique solution $\kappa\in C^2([0,1],\mathbb{R})$. □

## 6. Question

It was proved in [22] that if $F(\theta)=\frac{-1}{\sqrt[p]{\theta}}$, where $p>1$ and $\theta>0$, then $F\in\Sigma$. The question that arises here, what are the properties of the contraction mapping under this function?

## 7. Conclusions

The paper generalizes known contraction conditions and the obtained fixed point results, generalized several results known before such as Banach contraction [1], Jaggi-contraction [28], [29], and Ciric almost contraction [30]. Furthermore, as it has been observed in studies, fixed point results in $b-$metric-like spaces can be derived from the results of ordinary and $b-$metric spaces under some suitable conditions. We have applied our results to get the existence of a solution for electric circuit equation and second-order differential equation.

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