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Conformal Equitorsion and Concircular Transformations in a Generalized Riemannian Space

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Abstract: In the beginning, the basic facts about a conformal transformations are exposed and then equitorsion conformal transformations are defined. For every five independent curvature tensors in Generalized Riemannian space, the above cited transformations are investigated and corresponding invariants-5 concircular tensors of concircular transformations are found.

Keywords: generalized Riemannian space; conformal; equitorsion and concircular transformations

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1. Introduction

In the sense of Eisenhart's definition [1], a generalized Riemannian space (GR_N) is a differentiable N-dimensional manifold that is endowed with *basic non-symmetric tensor* $(g_{ij} \neq g_{ji})$, where $det g_{ij} \neq 0$.

The symmetric part of g_{ij} is noted with $g_{\underline{ij}}$ and antisymmetric one with $g_{\underset{\vee}{ij}}$. The lowering and rising of indices in GR_N is defined by $g_{\underline{ij}}$ and $g^{\underline{ij}}$, respectively, where $g_{\underline{ij}} g^{\underline{ik}} = \delta^k_j$ $(det g_{\underline{ij}} \neq 0)$. *The Christoffel symbols* in GR_N are given in the next manner:

$$a)\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), \quad b)\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk} = \frac{1}{2}g^{\underline{ip}}(g_{jp,k} - g_{jk,p} + g_{pk,i}),$$
$$\text{where, e.g.,} \quad g_{ij,k} = \partial g_{ij}/\partial x^k. \tag{1}$$

Because of non-symmetry of the affine connection coefficients Γ^i_{jk} by indices j and k, there are four kinds of covariant differentiation in the space $\mathbb{GR}_N$. Namely, for a tensor a^i_j, these covariant derivatives are defined as:

$$a^i_{j\underset{\substack{1\\2\\3\\4}}{|}m} = a^i_{j,m} + \Gamma^i_{\substack{pm\\mp\\pm\\mp}} a^p_j - \Gamma^p_{\substack{jm\\mj\\mj\\jm}} a^i_p. \tag{2}$$

Yano in [2] investigates a conformal and concircular transformations in the R_N. In that case, of course, he considers one that is Riemannian curvature tensor. De and Mandal in [3] studied concircular curvature tensors as important tensors from the differential geometric point of view. In [4–11], Mikeš et al. have studied special kinds ot transformations in Riemannian space.

Minčić, in his doctoral dissertation (Novi Sad, 1975), obtained 12 curvature tensors, using non-symmetric connection. Among these 12 tensors, five of them are independent (se also [12–17]) and they are noted $\underset{1}{R}, \ldots, \underset{5}{R}$.

In [18], another combination of five independent curvature tensors is obtained, and they are denoted by $\underset{1}{K} \dots \underset{5}{K}$.

For five independent tensors $\underset{\theta}{K}$ in [19], the invariants $\underset{\theta}{Z}$ were found, which are different from the invariants $\underset{\theta}{\tilde{Z}}$ in the present paper (see Remark 3.1, at the end). Compare e.g., $\underset{1}{\tilde{Z}}$ from the present paper and $\underset{1}{Z}$ from [19], where $\underset{1}{R} = \underset{1}{K}$.

Investigation of various kinds of mappings in the settings of generalized Riemannian spaces is an active research topic, numerous results were obtained in the recent years; see, for instance [20–22]. Very recently, conformal and concircular diffeomorphisms of generalized Riemannian spaces have been studied by M. Z Petrović, M. S. Stanković and P. Peška [23].

2. Equitorsion Conformal Transformation in Generalized Riemannian Space

Consider a special transformation of the objects in GR_N.

Definition 1. *Conformal transformation is that one under which the basic tensor is changed according to the law*

$$\bar{g}_{ij}(x) = \rho^2(x) g_{ij}(x), \quad (g_{ij} \neq g_{ji}), \tag{3}$$

where $\rho(x) = \rho(x^1, \dots, x^N)$ is some differentiable function of coordinates in GR_N.

We see that g and $\bar{g}$ are considered in the common system of coordinates. The same is valid for the other geometric objects.

Furthermore, we have:

$$ds^2 = g_{ij} dx^i dx^j, \quad d\bar{s}^2 = \bar{g}_{ij} dx^i dx^j = \rho^2 g_{ij} dx^i dx^j,$$
$$d\bar{s}^2 = \rho^2 ds^2 \Leftrightarrow d\bar{s}/ds = \rho. \tag{4}$$

If the transformation (3) is effected, for the Christoffel symbols, it is obtained

$$\begin{aligned}\bar{\Gamma}_{i.jk} =& \frac{1}{2}(\bar{g}_{ji,k} - \bar{g}_{jk,i} + \bar{g}_{ik,j}) \\ =& \rho^2 [\frac{\rho_{,k}}{\rho} g_{ji} - \frac{\rho_{,i}}{\rho} g_{jk} + \frac{\rho_{,j}}{\rho} g_{ik} + \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j})].\end{aligned}$$

Denoting

$$(ln\rho)_{,i} = \frac{\partial(ln\rho)}{\partial x^i} = \frac{1}{\rho}\rho_{,i} = \rho_i, \tag{5}$$

the previous equation gives

$$\bar{\Gamma}_{i.jk} = \rho^2(\Gamma_{i.jk} + g_{ji}\rho_k - g_{jk}\rho_i + g_{ik}\rho_j). \tag{6}$$

For $\bar{\Gamma}^i_{jk}$, according to (1), we get

$$\bar{\Gamma}^i_{jk} = \frac{1}{2}\bar{g}^{\underline{ip}}(\bar{g}_{jp,k} - \bar{g}_{jk,p} + \bar{g}_{pk,j}). \tag{7}$$

Because the inverse matrix for $(g_{\underline{ij}})$ is the matrix $(g^{\underline{ij}})$, we get

$$\bar{g}^{\underline{ij}}(x) = [\rho(x)]^{-2} g^{\underline{ij}}(x) \tag{8}$$

and, based on (1), (6), (8),

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{\underline{jk}} + \xi^i_{jk}, \tag{9}$$

where

$$\xi^i_{jk} = g^{\underline{ip}}(g_{\underset{V}{jp}} \rho_k - g_{\underset{V}{jk}} \rho_p + g_{\underset{V}{pk}} \rho_j). \tag{10}$$

Denote

$$\rho^i = g^{\underline{ip}} \rho_p \underset{(5)}{=} g^{\underline{ip}} (ln\rho)_{,p}. \tag{11}$$

From (9), it is obtained: for the symmetric part of the connection

$$\bar{\Gamma}^i_{\underline{jk}} = \Gamma^i_{\underline{jk}} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{\underline{jk}}, \tag{12}$$

and for the torsion tensor (double skewsymmetric part of the connection)

$$\bar{T}^i_{jk} = 2\bar{\Gamma}^i_{\underset{V}{jk}} = T^i_{jk} + 2g^{\underline{ip}}(g_{\underset{V}{jp}} \rho_k - g_{\underset{V}{kp}} \rho_j + g_{\underset{V}{kj}} \rho_p) = T^i_{jk} + 2\xi^i_{jk}. \tag{13}$$

Definition 2. *An equitorsion conformal transformation of the connection in GR_N is that conformal transformation (3) on the base of which the torsion is not changed, i.e.,*

$$\bar{T}^i_{jk} = \bar{\Gamma}^i_{jk} - \bar{\Gamma}^i_{kj} = \Gamma^i_{jk} - \Gamma^i_{kj} = T^i_{jk}. \tag{14}$$

From (13), we conclude that

Theorem 1. *Necessary and sufficient condition for a conformal transformation of the connection to be equitorsion is*

$$\xi^i_{jk} = g^{\underline{ip}}(g_{\underset{V}{jp}} \rho_k - g_{\underset{V}{kp}} \rho_j + g_{\underset{V}{kj}} \rho_p) = 0. \tag{15}$$

3. Curvature Tensors in Equitorsion Conformal and Concircular Transformation in Generalized Riemannian Space

3.1. The First Curvature Tensor

The 1^{st} from the cited curvature tensors in GR_N is [12,13]

$$\underset{1}{R}{}^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}. \tag{16}$$

Based on (15), (9), we obtain

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{\underline{jk}}. \tag{17}$$

If by the transformation of the connection Γ into $\bar{\Gamma}$ we write

$$a)\ \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + P^i_{jk}, \quad b) P^i_{jk} = \delta^i_j \rho_k + \delta^i_k \rho_j - \rho^i g_{\underline{jk}} = P^i_{kj}, \tag{18}$$

we can consider how e.g., some curvature tensors from the above mentioned independent ones are transformed.

With respect to (18), for $\underset{1}{R}$, one obtains

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= \bar{\Gamma}^i_{jm,n} - \bar{\Gamma}^i_{jn,m} + \bar{\Gamma}^p_{jm}\bar{\Gamma}^i_{pn} - \bar{\Gamma}^p_{jn}\bar{\Gamma}^i_{pm} \\ &= \underset{1}{R}{}^i_{jmn} + P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{1}{|}m} + P^p_{jm}P^i_{pn} - P^p_{jn}P^i_{pm} + T^p_{mn}P^i_{jp},\end{aligned} \tag{19}$$

and substituting P from (18b):

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + \delta^i_j(\rho_{m\underset{1}{|}n} - \rho_{n\underset{1}{|}m} + T^p_{mn}\rho_p) + \delta^i_m(\rho_{j\underset{1}{|}n} + \rho_j\rho_n) \\ &- \delta^i_n(\rho_{j\underset{1}{|}m} - \rho_j\rho_m) + (\rho^i_{\underset{1}{|}m} - \rho^i\rho_m)g_{\underline{jn}} - (\rho^i_{\underset{1}{|}n} - \rho^i\rho_n)g_{\underline{jm}} \\ &+ \rho^p\rho_p(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) + T^i_{mn}\rho_j - T_{j.mn}\rho^i,\end{aligned} \tag{20}$$

where $\underset{1}{|}m$ denotes covariant derivative of the first kind on x^m. Because

$$\rho_{m\underset{1}{|}n} - \rho_{n\underset{1}{|}m} = -T^p_{mn}\rho_p, \tag{21}$$

the 2nd addend on the right side in (20) is 0. Introducing the notation

$$\underset{1}{\rho}_{ij} = \rho_{i\underset{1}{|}j} - \rho_i\rho_j + \frac{1}{2}g^{\underline{rs}}\rho_r\rho_s g_{\underline{ij}} = \rho_{i\underset{1}{|}j} - \rho_i\rho_j + \frac{1}{2}\rho_r\rho^r g_{\underline{ij}}, \tag{22}$$

we obtain

$$\underset{1}{\rho}_{mn} - \underset{1}{\rho}_{nm} \underset{(22)}{=} \rho_{m\underset{1}{|}n} - \rho_{n\underset{1}{|}m} = -T^p_{mn}\rho_p, \tag{23}$$

and, for $\underset{1}{\bar{R}}{}^i_{jmn}$,

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + \delta^i_m(\underset{1}{\rho}_{jn} - \frac{1}{2}g^{\underline{rs}}\rho_r\rho_s g_{\underline{jn}}) - \delta^i_n(\underset{1}{\rho}_{jm} - \frac{1}{2}g^{\underline{rs}}\rho_r\rho_s g_{\underline{jm}}) \\ &+ g^{\underline{ip}}g_{\underline{jn}}(\rho_{p\underset{1}{|}m} - \rho_p\rho_m) - g^{\underline{ip}}g_{\underline{jm}}(\rho_{p\underset{1}{|}n} - \rho_p\rho_n) + \rho_p\rho^p(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) \\ &+ A^i_{jmn}\end{aligned}$$

is obtained, where

$$A^i_{jmn} = T^i_{mn}\rho_j - T_{j.mn}\rho^i. \tag{24}$$

Furthermore,

$$\begin{aligned}\underset{1}{\bar{R}}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + \delta^i_m(\underset{1}{\rho}_{jn} - \frac{1}{2}g^{\underline{rs}}\rho_r\rho_s g_{\underline{jn}}) - \delta^i_n(\underset{1}{\rho}_{jm} - \frac{1}{2}g^{\underline{rs}}\rho_r\rho_s g_{\underline{jm}}) \\ &+ g^{\underline{ip}}g_{\underline{jn}}(\underset{1}{\rho}_{pm} - \frac{1}{2}g^{\underline{rs}}\rho_r\rho_s g_{\underline{pm}}) - g^{\underline{ip}}g_{\underline{jm}}(\underset{1}{\rho}_{pn} - \frac{1}{2}g^{\underline{rs}}\rho_r\rho_s g_{\underline{pn}}) + \rho_p\rho^p(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) \\ &+ A^i_{jmn},\end{aligned} \tag{25}$$

from where

$$\begin{aligned}\underset{1}{\bar{R}}{}^i{}_{jmn} &= \underset{1}{R}{}^i_{jmn} + \delta^i_m\underset{1}{\rho}_{jn} - \delta^i_n\underset{1}{\rho}_{jm} - \delta^i_m g_{\underline{jn}}\rho_p\rho^p + \delta^i_n g_{\underline{jm}}\rho_p\rho^p \\ &+ \rho^i_m g_{\underline{jn}} - \rho^i_n g_{\underline{jm}} + \rho_p\rho^p(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) + A^i_{jmn},\end{aligned}$$

and putting in order:

$$\underset{1}{\bar{R}}{}^{i}{}_{jmn} = \underset{1}{R}{}^{i}_{jmn} + \delta^i_m \underset{1}{\rho}_{jn} - \delta^i_n \underset{1}{\rho}_{jm} + \underset{1}{\rho}{}^i_m g_{\underline{jn}} - \underset{1}{\rho}{}^i_n g_{\underline{jm}} + A^i_{jmn}, \tag{26}$$

where A^i_{jmn} is given in (24). We are using the next definition from [2]

Definition 3. *If a conformal transformation in a Riemannian space R_N :*

$$\bar{g}_{ij} = \rho^2 g_{ij}, \quad (g_{ij} = g_{ji})$$

transforms every geodesic circle into geodesic circle, the function $\rho(x)$ satisfies the partial differential equation

$$\rho_{ij} = \Phi(x) g_{ij}(x), (g_{ij} = g_{ji}), \tag{27}$$

where

$$\rho_{ij} = \rho_{i;j} - \rho_i \rho_j + \frac{1}{2} \rho_p \rho^p g_{ij}, (g_{ij} = g_{ji}). \tag{28}$$

Such a transformation is called **a concircular transformation** *in R_N, and* **concircular geometry** *is geometry that treats the concircular transformations and the spaces that allow such kinds of transformations.*

In the GR_N, we consider transformations

$$\bar{g}_{ij} = \rho^2 g_{ij}, \quad (g_{ij} \neq g_{ji}) \tag{29}$$

where, based on (22), $\rho_{i\underset{1}{|}j} \neq \rho_{j\underset{1}{|}i}$ in GR_N. Now, we take

$$\underset{1}{\rho}_{\underline{ij}} = \underset{1}{\Phi}(x) g_{\underline{ij}}(x), \quad (g_{ij} \neq g_{ji}), \tag{30}$$

and such a transformation we name **a concircular transformation of the first kind** in GR_N.

We have to find the function $\underset{1}{\Phi}$. Substituting $\underset{1}{\rho}$ from (30) into (26), we get:

$$\underset{1}{\bar{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + 2\underset{1}{\Phi}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) + A^i_{jmn}. \tag{31}$$

If we effect the contraction with $i = n$, it follows that

$$\underset{1}{\bar{R}}_{jm} = \underset{1}{R}_{jm} + 2\underset{1}{\Phi}(\delta^i_m g_{\underline{ji}} - \delta^i_i g_{\underline{jm}}),$$

where $\underset{1}{R}_{jm} = \underset{1}{R}{}^i_{jmi}$, and so on, and we get:

$$\underset{1}{\bar{R}}_{jm} = \underset{1}{R}_{jm} + 2(1-N)\underset{1}{\Phi} g_{\underline{jm}} + A_{jm}.$$

By multiplying the corresponding sides of previous equation and the equation

$$\rho^2 \bar{g}^{\underline{jm}} = g^{\underline{jm}},$$

we obtain

$$\rho^2 \underset{1}{\bar{R}} = g^{\underline{jm}} \{ \underset{1}{R}_{jm} + 2(1-N)\underset{1}{\Phi} g_{\underline{jm}} + A_{jm} \},$$

where $\bar{\underset{1}{R}}_{jm}\bar{g}^{\underline{jm}} = \underset{1}{\bar{R}}$ and so on, while

$$A_{jm}g^{\underline{jm}} \underset{(24)}{=} A^i_{jmi}g^{\underline{jm}} = (T^i_{mi}\rho_j - T_{j.mi}\rho^i)g^{\underline{jm}} = 0,$$

and we get

$$\rho^2\underset{1}{\bar{R}} = \underset{1}{R} + \underset{1}{\Phi}[-2(N-1)N],$$

wherefrom it follows that

$$\underset{1}{\Phi}(x) = -\frac{\rho^2\underset{1}{\bar{R}} - \underset{1}{R}}{2(N-1)N}. \tag{32}$$

Substituting $\underset{1}{\Phi}$ into (31), we get

$$\underset{1}{\bar{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} - \frac{\rho^2\underset{1}{\bar{R}} - \underset{1}{R}}{(N-1)N}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) + A^i_{jmn} \tag{33}$$

and from here

$$\begin{aligned} &\underset{1}{\bar{R}}{}^i_{jmn} + \frac{\underset{1}{\bar{R}}(\rho^2\delta^i_m g_{jn} - \rho^2\delta^i_n g_{jm})}{(N-1)N} \\ =&\underset{1}{R}{}^i_{jmn} + \frac{\underset{1}{R}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})}{(N-1)N} + A^i_{jmn}. \end{aligned} \tag{34}$$

Taking into consideration that

$$\rho_i = \frac{1}{2N}[(\ln\bar{g})_{,i} - (\ln g)_{,i}] = \frac{1}{2N}(\bar{g}_i - g_i), \quad g = det(g_{ij}), \tag{35}$$

with respect to (24) and (35)

$$\begin{aligned} A^i_{jmn} &= T^i_{mn}\rho_j - T_{j.mn}\rho^i = T^i_{mn}\rho_j - T_{j.mn}g^{\underline{ip}}\rho_p \underset{(35)}{=} \frac{1}{2N}[T^i_{mn}(\bar{g}_j - g_j) - T_{j.mn}g^{\underline{ip}}(\bar{g}_p - g_p)] \\ &= \frac{1}{2N}[(\bar{T}^i_{mn}\bar{g}_j - \bar{T}_{j.mn}\bar{g}^{\underline{ip}}\bar{g}_p) - (T^i_{mn}g_j - T_{j.mn}g^{\underline{ip}}g_p)], \end{aligned} \tag{36}$$

where $T^i_{mn} = \bar{T}^i_{mn}$ (for the first addend) and $g^{\underline{ip}}g_{\underline{qj}} = \bar{g}^{\underline{ip}}\bar{g}_{\underline{qj}}$ (for the third addend). By substituting from (36) into (34) and because of

$$g^i_m = \bar{g}^i_m, \quad \rho^2 g_{jm} = \bar{g}_{jm}, \quad \bar{\delta}^i_m = \delta^i_m, \tag{37}$$

we obtain

$$\begin{aligned} &\underset{1}{\bar{R}}{}^i_{jmn} + \frac{\underset{1}{\bar{R}}(\bar{\delta}^i_m\bar{g}_{\underline{jn}} - \bar{\delta}^i_n\bar{g}_{\underline{jm}})}{(N-1)N} + \frac{1}{2N}(\bar{T}_{j.mn}\bar{g}^{\underline{ip}}\bar{g}_p - \bar{T}^i_{mn}\bar{g}_j), \\ =&\underset{1}{R}{}^i_{jmn} + \frac{\underset{1}{R}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})}{(N-1)N} + \frac{1}{2N}(T_{j.mn}g^{\underline{ip}}g_p - T^i_{mn}g_j). \end{aligned} \tag{38}$$

In that manner, we conclude that the following theorem is valid:

Theorem 2. *The tensor*

$$\underset{1}{\tilde{Z}}{}^{i}_{jmn} = \underset{1}{R}{}^{i}_{jmn} + \frac{\underset{1}{R}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})}{(N-1)N} + \frac{1}{2N}(T_{j.mn} g^{\underline{ip}} g_p - T^i_{mn} g_j) \tag{39}$$

is an invariant in the space GR_N, by an equitorsion concircular transformation i.e., according to (38):

$$\underset{1}{\bar{\tilde{Z}}}{}^{i}_{jmn} = \underset{1}{\tilde{Z}}{}^{i}_{jmn}, \tag{40}$$

where e.g., $g_j = (\ln g)_{,j} = \frac{\partial(\ln g)}{\partial x^j}$ and $\underset{1}{\tilde{Z}}{}^{i}_{jmn}$ is given by (39).

The tensor $\underset{1}{\tilde{Z}}{}^{i}_{jmn}$ is an **equitorsion concircular tensor of the first kind in GR_N.**

3.2. The Second Curvature Tensor

The tensor $\underset{2}{R}$ in GR_N is [12,17]

$$\underset{2}{R}{}^{i}_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp}, \tag{41}$$

and, for $\underset{2}{\bar{R}}{}^{i}_{jmn}$, by virtue of (18), it follows that

$$\underset{2}{\bar{R}}{}^{i}_{jmn} = \underset{2}{R}{}^{i}_{\ jmn} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{2}{|}m} + P^p_{mj}P^i_{mp} - P^p_{nj}P^i_{mp} - T^p_{mn}P^i_{pj}. \tag{42}$$

Substituting from (18) into the previous equation and arranging, one obtains

$$\begin{aligned}\underset{2}{\bar{R}}{}^{i}_{jmn} &= \underset{2}{R}{}^{i}_{mj,n} + \delta^i_j(\rho_{m\underset{2}{|}n} - \rho_{n\underset{2}{|}m} - T^p_{mn}\rho_p) + \delta^i_m(\rho_{j\underset{2}{|}n} - \rho_i\rho_n) - \delta^i_n(\rho_{j\underset{2}{|}m} - \rho_i\rho_m)\\ &+ (\rho^i_{\underset{2}{|}m} - \rho^i\rho_m)g_{\underline{jn}} - (\rho^i_{\underset{2}{|}n} - \rho^i\rho_n)g_{\underline{jm}} + \rho_p\rho^p(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) - T^i_{mn}\rho_j + T_{j.mn}\rho^i.\end{aligned} \tag{43}$$

The term in the 1*st* bracket on the right side is 0 because of

$$\rho_{m\underset{2}{|}n} - \rho_{n\underset{2}{|}m} = T^p_{mn}\rho_p. \tag{44}$$

If we introduce the denotation

$$\underset{2}{\rho}_{ij} = \rho_{i\underset{2}{|}j} - \rho_i\rho_j + \frac{1}{2}\rho_r\rho^r g_{\underline{ij}}, \tag{45}$$

we have

$$\underset{2}{\rho}_{mn} - \underset{2}{\rho}_{nm} = \rho_{m\underset{2}{|}n} - \rho_{n\underset{2}{|}m} = T^p_{mn}\rho_p$$

and, for $\underset{2}{\bar{R}}{}^{i}_{jmn}$ from (43)–(45), it follows that

$$\underset{2}{\bar{R}}{}^{i}_{jmn} = \underset{2}{R}{}^{i}_{jmn} + \delta^i_m \underset{2}{\rho}_{jn} - \delta^i_n \underset{2}{\rho}_{jm} + \underset{2}{\rho}{}^i_m g_{\underline{jn}} - \underset{2}{\rho}{}^i_n g_{\underline{jm}} - A^i_{jmn}, \tag{46}$$

where A^i_{jmn} is given (24). Furthermore, we use **the concircular transformation for** $\underset{2}{R}$

$$\underset{2}{\rho}_{\underline{ij}} = \underset{2}{\Phi}(x) g_{\underline{ij}}(x). \tag{47}$$

By substitution of $\underset{2}{\rho}_{ij}$ into (46), by procedure as for $\underset{1}{R}$, we obtain

$$\underset{2}{\Phi} = -\frac{\rho^2 \underset{2}{\bar{R}} - \underset{2}{R}}{2(N-1)N} \tag{48}$$

and at the end:

$$\underset{2}{\bar{R}}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} - \frac{\rho^2 \underset{2}{\bar{R}} - \underset{2}{R}}{(N-1)N} (\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) - A^i_{jmn}, \tag{49}$$

where A^i_{jmn} is given in (24).

Thus, we conclude that the next theorem is valid.

Theorem 3. *The tensor*

$$\begin{aligned} \underset{2}{\tilde{Z}}{}^i{}_{jmn} = & \underset{2}{R}{}^i_{jmn} + \frac{\underset{2}{R}(\delta^i_m g_{jn} - \delta^i_n g_{jm})}{(N-1)N} \\ & - \frac{1}{2N}(T_{j.mn} g^{\underline{ip}} g_p - T^i_{mn} g_j) \end{aligned} \tag{50}$$

is an invariant in GR_N with respect to an equitorsion concircular transformation, i.e., in force is

$$\underset{2}{\bar{\tilde{Z}}}{}^i_{jmn} = \underset{2}{\tilde{Z}}{}^i_{jmn}. \tag{51}$$

The tensor $\underset{2}{\tilde{Z}}$ is **an equitorsion concircular tensor of the** $2nd$ **kind at** GR_N *and e.g.,* $g_j = (\ln g)_{=} \frac{\partial(\ln g)}{\partial x^j}$.

3.3. The Third Curvature Tensor

The tensor $\underset{3}{R}$ in GR_N [12,14,17] is

$$\underset{3}{R}{}^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm} T^i_{pj}, \tag{52}$$

where T^i_{pj} is torsion tensor in local coordinates. For $\underset{3}{R}{}^i_{jmn}$ on the base of (18), it is obtained

$$\underset{3}{\bar{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^i_{np} P^p_{jm} - P^i_{pm} P^p_{nj} + T^i_{pj} P^p_{nm}, \tag{53}$$

where we take into consideration that P^i_{jp} is symmetric, with respect to (18).

By substituting from (18) into the previous equation and arranging, one obtains

$$\begin{aligned} \underset{3}{\bar{R}}{}^i_{jmn} = & \underset{3}{R}{}^i_{jmn} + \delta^i_m (\underset{2}{\rho}_{jn} - \frac{1}{2}\rho_p \rho^p g_{\underline{jn}}) - \delta^i_n (\underset{1}{\rho}_{jm} - \frac{1}{2}\rho_p \rho^p - g_{\underline{jm}}) \\ & + g^{\underline{ip}} g_{\underline{jn}} (\underset{1}{\rho}_{pm} - \frac{1}{2}\rho_r \rho^r g_{\underline{pm}}) - g^{\underline{ip}} g_{\underline{jm}} (\underset{2}{\rho}_{pn} - \frac{1}{2}\rho_r \rho^r g_{\underline{pm}}) \\ & + \rho_p \rho^p \delta^i_m g_{\underline{jn}} - \rho_p \rho^p \delta^i_n g_{\underline{jm}} + D^i_{jmn}, \end{aligned} \tag{54}$$

where
$$D^i_{jmn} = T^i_{jm}\rho_n + T^i_{nj}\rho_m + g^{\underline{ps}}g_{\underline{mn}}T^i_{jp}\rho_s. \tag{55}$$

From (55), it is obtained that

$$\bar{\underset{3}{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + \delta^i_m \underset{2}{\rho}_{jn} - \delta^i_n \underset{1}{\rho}_{jm} + \underset{1}{\rho}{}^i_m g_{\underline{jn}} - \underset{2}{\rho}{}^i_n g_{\underline{jm}} + D^i_{jmn}. \tag{56}$$

Consider, further, **the concircular transformation** for the tensor $\underset{3}{R}{}^i_{jmn}$ in the following manner. Taking

$$\underset{\theta}{\rho}_{\underline{ij}} = \underset{3}{\Phi}(x) g_{\underline{ij}}(x), \quad \theta = 1, 2, \tag{57}$$

we obtain from (56)

$$\bar{\underset{3}{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + 2\underset{3}{\Phi}\delta^i_m g_{\underline{jn}} - (\delta^i_n g_{\underline{jm}}) + D^i_{jmn}. \tag{58}$$

Putting $i = n$, we get

$$\bar{\underset{3}{R}}_{jm} = \underset{3}{R}_{jm} + 2\underset{3}{\Phi}\delta^i_m g_{\underline{ji}} - (\delta^i_i g_{\underline{jm}}) + D_{jm}, \tag{59}$$

and contracting with $\rho^2 \bar{g}^{jm} = g^{\underline{jm}}$ on the left and the right sides correspondingly in (59), we get

$$\rho^2 \bar{\underset{3}{R}} = \underset{3}{R} + 2\underset{3}{\Phi}(1 - N)N \tag{60}$$

because

$$\begin{aligned} D = D_{jm}g^{\underline{jm}} = D^i_{jmi}g^{\underline{jm}} &\underset{(55)}{=} (T^i_{jm}\rho_i + T^i_{i\ j}\rho_m + g^{\underline{ps}}g_{\underline{mi}}T^i_{jp}\rho_s)g^{\underline{jm}} \\ &= T^i_{jm}\rho_i g^{\underline{jm}} + 0 + g^{\underline{ps}}\delta^j_i T^i_{jp}\rho_s = T^i_{jm}\rho_i g^{\underline{jm}} \\ &= T_{i.jm}\rho^i g^{\underline{jm}} = -T_{j.im}\rho^i g^{\underline{jm}} = -T^m_{im}\rho^i = 0. \end{aligned} \tag{61}$$

By the further procedure as in the case of $\underset{1}{R}$, we obtain

$$\underset{3}{\Phi}(x) = -\frac{\rho^2 \bar{\underset{3}{R}} - \underset{3}{R}}{(N-1)N}. \tag{62}$$

Consider, further, the tensor D^i_{jmn}. By virtue of (35), one gets

$$D^i_{jmn} = \frac{1}{2N}[T^i_{jm}(\bar{g}_n - g_n) + T^i_{nj}(\bar{g}_m - g_m) + g^{\underline{ps}}g_{\underline{mn}}T^i_{jp}(\bar{g}_s - g_s)], \tag{63}$$

where the equitorsion is taken into consideration.

Substituting from (62), (63) into (58), it follows that

$$\begin{aligned} \bar{\underset{3}{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} &- \frac{\rho^2 \bar{\underset{3}{R}} - \underset{3}{R}}{(N-1)N}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) \\ &+ \frac{1}{2N}[T^i_{jm}(\bar{g}_n - g_n) + T^i_{nj}(\bar{g}_m - g_m) + g^{\underline{ps}}g_{\underline{mn}}T^i_{jp}(\bar{g}_s - g_s)]. \end{aligned} \tag{64}$$

from where we conclude that the next theorem is valid.

Theorem 4. *The tensor*

$$\underset{3}{\tilde{Z}}i_{jmn} = \underset{3}{R}{}^i_{jmn} + \frac{\underset{3}{R}(\delta^i_m g_{jn} - \delta^i_n g_{jm})}{(N-1)N} - \frac{1}{2N}(T^i_{jm} g_n + T^i_{nj} g_m + g^{\underline{ps}} g_{\underline{mn}} T^i_{jp} g_s) \tag{65}$$

is an invariant in GR_N with respect to an equitorsion concircular transformation, i.e., it is

$$\underset{3}{\bar{\tilde{Z}}}{}^i_{jmn} = \underset{3}{\tilde{Z}}{}^i_{jmn}. \tag{66}$$

The tensor $\underset{3}{\tilde{Z}}$ *is* **an equitorsion concircular tensor of the 3rd kind at** GR_N**.**

3.4. The Fourth Curvature Tensor

For the tensor $\underset{4}{R}$ in GR_N, we have [13,14,17]

$$\underset{4}{R}{}^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn} T^i_{pj}, \tag{67}$$

where T^i_{pj} is torsion tensor in local coordinates. For $\underset{4}{R}{}^i_{jmn}$ on the base of (18), it is obtained

$$\underset{4}{\bar{R}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^i_{np}P^p_{jm} - P^i_{pm}P^p_{nj} + T^i_{pj}P^p_{mn}. \tag{68}$$

From (53), (68), it follows that

$$\underset{4}{\bar{R}}{}^i_{jmn} - \underset{3}{\bar{R}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} - \underset{3}{R}{}^i_{jmn} + 2\underset{V}{P}{}^p_{mn} T^i_{pj} = \underset{4}{R}{}^i_{jmn} - \underset{3}{R}{}^i_{jmn}$$

because $\underset{V}{P}{}^p_{mn} = 0$. Thus, we have

$$\begin{aligned} \underset{4}{\bar{R}}{}^i_{jmn} - \underset{4}{R}{}^i_{jmn} &= \underset{3}{\bar{R}}{}^i_{jmn} - \underset{3}{R}{}^i_{jmn} \\ &\underset{(56)}{=} \delta^i_m \underset{2}{\rho}_{jn} - \delta^i_n \underset{1}{\rho}_{jm} + \underset{1}{\rho}{}^i_m g_{\underline{jn}} - \underset{2}{\rho}{}^i_n g_{\underline{jm}} + D^i_{jmn}, \end{aligned} \tag{69}$$

where D^i_{jmn} is given in (55). For the concircular transformation for the tensor $\underset{4}{R}{}^i_{jmn}$, we put

$$\underset{\theta}{\rho}_{\underline{ij}} = \underset{4}{\Phi}(x) g_{\underline{ij}}(x), \quad \theta = 1, 2,$$

and, by the same procedure as in the previous case, the next theorem is obtained.

Theorem 5. *The tensor*

$$\underset{4}{\tilde{Z}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} + \frac{\underset{4}{R}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})}{(N-1)N} - \frac{1}{2N}(T^i_{jm} g_n + T^i_{nj} g_m + g^{\underline{ps}} g_{\underline{mn}} T^i_{jp} g_s) \tag{70}$$

is an invariant in GR_N with respect to an equitorsion concircular transformation, i.e., in force is

$$\underset{4}{\bar{\tilde{Z}}}{}^i_{jmn} = \underset{4}{\tilde{Z}}{}^i_{jmn}. \tag{71}$$

The tensor $\underset{4}{\tilde{Z}}$ *is* **an equitorsion concircular tensor of the 4th kind at** GR_N**.**

3.5. The Fifth Curvature Tensor

Finally, consider the 5th curvature tensor $\underset{5}{R}{}^i_{jmn}$ in GR_N (in [12] $\underset{5}{R}$ is denoted with $\underset{2}{\tilde{R}}$). We have according to [12,17]

$$\underset{5}{R}{}^i_{jmn} = \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} \\ + \Gamma^p_{jm}\Gamma^i_{pn} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}), \tag{72}$$

which can be written in the form [17]:

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \frac{1}{2}(P^i_{jm\underset{3}{|}n} - P^i_{jn\underset{4}{|}m} + P^i_{mj\underset{4}{|}n} - P^i_{nj\underset{3}{|}m} \\ + P^p_{jm}P^i_{pn} - P^p_{jn}P^i_{mp} + P^p_{mj}P^i_{np} - P^p_{nj}P^i_{pm}), \tag{73}$$

where P^i_{jk} is given in (18). With substitution of P from (18) into (73), one obtains

$$\begin{aligned} \underset{5}{\bar{R}}{}^i_{jmn} &= \underset{5}{R}{}^i_{jmn} + \frac{1}{2}[\delta^i_j(\rho_{m\underset{2}{|}n} - \rho_{n\underset{1}{|}m} - \rho_{m\underset{1}{|}n} - \rho_{n\underset{2}{|}m}) \\ &+ \delta^i_m(\rho_{j\underset{2}{|}n} + \rho_{j\underset{1}{|}n} - 2\rho_j\rho_n + \rho_p\rho^p g_{\underline{jn}}) \\ &- \delta^i_n(\rho_{j\underset{1}{|}m} + \rho_{j\underset{2}{|}m} - 2\rho_j\rho_m + \rho_p\rho^p g_{\underline{jm}}) \\ &+ g_{\underline{jn}}(\rho^i_{\underset{1}{|}m} + \rho^i_{\underset{2}{|}m} - 2\rho^i\rho_m) - g_{\underline{jm}}(\rho^i_{\underset{1}{|}n} + \rho^i_{\underset{2}{|}n} - 2\rho^i\rho_n)]. \end{aligned} \tag{74}$$

Using (23) and (44) and introducing the denotation

$$\underset{5}{\rho}_{ij} = \frac{1}{2}(\rho_{i\underset{1}{|}j} + \rho_{i\underset{2}{|}j} - 2\rho_i\rho_j + \rho_p\rho^p g_{\underline{ij}}) = \frac{1}{2}(\underset{1}{\rho}_{ij} + \underset{2}{\rho}_{ij}) = \underset{5}{\rho}_{ji}, \tag{75}$$

Equation (74) obtains the form

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \delta^i_m\underset{5}{\rho}_{jn} - \delta^i_n\underset{5}{\rho}_{jm} \\ + \underset{5}{\rho}{}^i_m g_{\underline{jn}} - \underset{5}{\rho}{}^i_n g_{\underline{jm}} + \rho_p\rho^p(\delta^i_n g_{\underline{jm}} - \delta^i_m g_{\underline{jm}}). \tag{76}$$

Let us apply a **concircular transformation** for the tensor $\underset{5}{R}{}^i_{jmn}$. By virtue of (75), we put

$$\underset{5}{\rho}_{ij} = \underset{5}{\Phi}(x)g_{\underline{ij}}(x) = \underset{5}{\rho}_{ji}, \tag{77}$$

into (76) and we get

$$\underset{5}{\bar{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + 2\underset{5}{\Phi}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) - \rho_p\rho^p(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}). \tag{78}$$

Contracting by $i = n$, we obtain

$$\bar{\underset{5}{R}}_{jm} = \underset{5}{R}_{jm} + (1-N) g_{\underline{jm}} (2\underset{5}{\Phi} - \rho_p \rho^p).$$

Multiplying this equation with $\rho^2 \bar{g}^{jm} = g^{\underline{jm}}$, it follows that

$$2\underset{5}{\Phi} = \frac{\rho^2 \bar{\underset{5}{R}} - \underset{5}{R}}{(1-N)N} + \rho_p \rho^p$$

and substituting this value into (78), one gets that the following theorem is valid.

Theorem 6. *The tensor*

$$\tilde{\underset{5}{Z}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \frac{\underset{5}{R}(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})}{(N-1)N} \tag{79}$$

is an invariant in GR_N with respect to an equitorsion concircular transformation, i.e., in force is

$$\bar{\tilde{\underset{5}{Z}}}{}^i_{jmn} = \tilde{\underset{5}{Z}}{}^i_{jmn}. \tag{80}$$

The tensor $\tilde{\underset{5}{Z}}$ is **an equitorsion concircular tensor of the 5th kind at** GR_N**.**

Remark 1. *In [19], is $\underset{1}{K}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn}$, $\underset{3}{K}{}^i_{jmn} = \underset{3}{R}{}^i_{jm}$, while $\underset{\theta}{R}{}^i_{jm} \notin \{\underset{2}{K}{}^i_{jmn}, \underset{4}{K}{}^i_{jmn}, \underset{5}{K}{}^i_{jmn}\}, \theta = 2, 4, 5$. However, because of different procedures, it is $\tilde{\underset{\theta}{Z}}{}^i_{jmn} \notin \{\underset{1}{Z}{}^i_{jmn}, \ldots, \underset{5}{Z}{}^i_{jmn}\}, \theta = 1, \ldots, 5$, where $\underset{\theta}{Z}{}^i_{jmn}$ are from [19]. Thus, $\tilde{\underset{\theta}{Z}}{}^i_{jmn}$ are* **new invariants** *of the considered transformations.*

Remark 2. *In the case of $R_N(g_{ij} = g_{ji}, T^i_{jk} = 0)$, each of the obtained tensors $\tilde{\underset{\theta}{Z}}{}^i_{jmn}$ reduces to a known concircular tensor [2] $Z^i_{jmn} = R^i_{jmn} + \frac{R(\delta^i_m g_{jn} - \delta^i_n g_{jm})}{(N-1)N}$.*

4. Conclusions

Conformal equitorsion concircular transformations are investigated and corresponding invariants-5 concircular tensors of concircular transformations are found.

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References

1. Eisenhart, L.P. Generalized Riemann spaces. *Proc. Nat. Acad. Sci. USA* **1951**, *37*, 311–315. [CrossRef] [PubMed]
2. Yano, K. Concircular Geometry I. Concircular Transformations. *Proc. Imp. Acad. Tokyo* **1940**, *16*, 195–200. [CrossRef]
3. De, U.C.; Mandal, K. On K-contact Einstein Manifolds. *Novi Sad J. Math.* **2016**, *46*, 105–114. [CrossRef]
4. Berezovskii, V.E.; Hinterleitner, I.; Guseva, N.I.; Mikeš, J. Conformal Mappings of Riemannian Spaces onto Ricci Symmetric Spaces. *Math. Notes* **2018**, *103*, 304–307. [CrossRef]
5. Evtushik, L.E.; Hinterleitner, I.; Guseva, N.I.; Mikeš, J. Conformal mappings onto Einstein spaces. *Russ. Math.* **2016**, *60*, 5–9. [CrossRef]
6. Hinterleitner, I.; Guseva, N.; Mikeš, J. On conformal mappings onto compact Einstein. *Geom. Integr. Quant.* **2018**, *19*, 132–139. [CrossRef]

7. Mikesh, J. Equidistant Kahler spaces. *Math. Notes* **1985**, *38*, 855–858. [CrossRef]
8. Mikeš, J.; Chodorova, M. On concircular and torse-forming vector fields on compact manifolds. *Acta Math. Acad. Paedagog. Nyházi.* **2010**, *26*, 329–335.
9. Mikesh, J.; Gavril'chenko, M.L.; Gladysheva, E.I. On conformal mappings on to Einsteinian spaces. *Vestnik Moskovskogo Universiteta. Ser. 1 Matematika Mekhanika* **1994**, *3*, 13–17.
10. Mikeš, J.; Hinterleitner, I.; Guseva, N. There are no conformal Einstein rescalings of pseudo-Riemannian Einstein spaces with n complete light-like geodesics. *Mathematics* **2019**, *7*, 801. [CrossRef]
11. Shandra, I.G.; Mikeš, J. Geodesic mappings of Vn(K)-spaces and concircular vector. *Mathematics* **2019**, *7*, 692. [CrossRef]
12. Minčić, S.M. Ricci identities in the space of non-symmetric affine connexion. *Matematički Vesnik* **1973**, *10*, 161–172.
13. Minčić, S.M. New commutation formulas in the non-symmetric affine connexion space. *Publ. Inst. Math.* **1977**, *22*, 189–199.
14. Minčić, S.M. Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion. *Coll. Math. Soc. János Bolyai, 31. Dif. geom., Budapest (Hungary)* **1979**, 445–460.
15. Minčić, S.M. Frenet formulas for curves in a generalized Riemannian space. *Zbor. rad. Fil. Fak (Niš) Ser. Mat.* **1989**, *3*, 73–82.
16. Minčić, S.M.; Stanković, M.S. Equitorsion geodesic mappings of generalized Riemannian Spaces. *Publ. Inst. Math.* **1997**, *6*, 97–104.
17. Minčić, S.M.; Stanković, M.S.; Velimirović, L.S. *Generalized Riemannian Spaces and Spaces of Non-symmetric Affine Connection*; University of Niš: Niš, Serbia, 2013; ISBN 978-86-83481-90-3.
18. Zlatanović, M. New projective tensors for equitorsion geodesic mappings. *Math. Lett.* **2012**, *25*, 890–897. [CrossRef]
19. Zlatanović, M.; Hinterleitner, I.; Najdanović, M. On Equitorsion Concircular Tensors of Generalized Riemannian Spaces. *Filomat* **2014**, *28*, 464–471. [CrossRef]
20. Petrović, M.Z. Canonical almost geodesic mappings of type $\underset{\theta}{\pi}2(0,F)$, $\theta \in \{1,2\}$, between generalized parabolic Kähler manifolds. *Miskolc Math. Notes* **2018**, *19*, 469–482. [CrossRef]
21. Petrović, M.Z. Special almost geodesic mappings of the second type between generalized Riemannian spaces. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 707–727. [CrossRef]
22. Petrović, M.Z.; Velimirović, L.S. Generalized Kahler spaces in Eisenhart's sense admitting a holomorphically projective mapping. *Mediterr. J. Math.* **2018**, *15*, 150. [CrossRef]
23. Petrović, M.Z.; Stanković, M.S.; Peška, P. On Conformal and Concircular Diffeomorphisms of Eisenhart's Generalized Riemannian Spaces. *Mathematics* **2019**, *7*, 626, doi:10.3390/math7070626. [CrossRef]