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A Lyapunov-Type Inequality for Partial Differential Equation Involving the Mixed Caputo Derivative

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Abstract: In this work, we derive a Lyapunov-type inequality for a partial differential equation on a rectangular domain with the mixed Caputo derivative subject to Dirichlet-type boundary conditions. The obtained inequality provides a necessary condition for the existence of nontrivial solutions to the considered problem and an example is given to illustrate it. Moreover, we present some applications to demonstrate the effectiveness of the new results.

Keywords: Lyapunov-type inequality; mixed Riemann–Liouville integral; mixed Caputo derivative; Green's function; boundary value problem

1. Introduction

In this paper, we focus on the representation of the Lyapunov-type inequality for the following boundary value problem:

$$\begin{cases} {}^{C}D_0^r u(x,y) + q(x,y)u(x,y) = 0, \quad (x,y) \in (0,a) \times (0,b), \\ u(0,y) = 0, u(a,y) = 0, \quad y \in [0,b], \\ u(x,0) = 0, u(x,b) = 0, \quad x \in [0,a], \end{cases} \tag{1}$$

where $a, b > 0$, $r = (r_1, r_2)$, $1 < r_1, r_2 < 2$, ${}^{C}D_0^r$ is the mixed Caputo derivative of order r and $q : [0,a] \times [0,b] \to \mathbb{R}$ is a given Lebesgue integrable function. To do that, we convert problem (1) into an integral equation. With the help of the properties of its Green function, we establish a new Lyapunov-type inequality, which provides a necessary condition for the existence of nontrivial solutions to problem (1). Furthermore, an example is given to illustrate it. In the end, we apply the obtained inequality to prove the uniqueness of solutions for the nonhomogenous boundary value problem and derive an estimation related to the eigenvalue of the corresponding equation.

The well-known Lyapunov inequality [1] states the fact that if the boundary value problem

$$\begin{cases} u''(t) + q(t)u(t) = 0, \ \ t \in (a,b), \\ u(a) = u(b) = 0 \end{cases} \tag{2}$$

has a nontrival solution $u \in C^2[a,b]$, then the inequality

$$\int_a^b |q(t)|dt > \frac{4}{b-a} \tag{3}$$

holds, where q is a real-valued continuous function. The Lyapunov inequality (3) was regarded as a very important and useful tool in the study of differential equations, especially in the aspect of stability theory, oscillation theory, intervals of disconjugacy, and eigenvalue problems [2–5]. Subsequently, there were many improvements and extensions of the inequality (3) related to integer-order derivative, see for instance [6–11] and the references therein. Since fractional calculus (see for example [12–14]) is more effective and powerful in describing practical phenomena than integer-order calculus, more and more researchers pay more attention to this subject. Recently, many results in connection with the representations of Lyapunov-type inequalities for fractional boundary value problem were presented. The first work in this direction is Ferreira's study [15], in which the author obtained a Lyapunov-type inequality for a fractional differential equation with Riemann–Liouville derivative and applied the inequality to deduce a criterion for the nonexistence of real zeros of a certain Mittag-Leffler function. Next, the same author obtained a new Lyapunov-type inequality and used that result to get an interval where a certain Mittag-Leffler function has no real zeros in [16], where u'' is replaced by the Caputo fractional derivative ${}^{C}D^{\alpha}_{a^+}$, $1 < \alpha < 2$. Relatively new conclusions related to Lyapunov-type inequalities for fractional differential equations with various kinds of boundary conditions were given, refer to [17–23] and the references therein.

In [12–14], the mixed fractional integrals and derivatives of order $r = (r_1, r_2)$ were defined. After that, hyperbolic partial differential equations and inclusions of fractional order have been intensely studied by many researchers, see for instance [24–30] and the references therein. These papers mainly studied the existence of solutions for initial value problems of partial differential equations with the mixed fractional derivatives, and few scholars studied boundary value problem for the corresponding equations. Most of previous results related to Lyapunov-type inequality were discussed for ordinary differential equations. There are few papers [7–10,23], related to Lyapunov-type inequalities for partial differential equations, in particular, with fractional partial differential equations. As far as we know, few papers deal with Lyapunov-type inequalities for partial differential equations with the mixed fractional derivatives.

Motivated by the above-cited excellent works, we present a Lyapunov-type inequality for problem (1) in this paper. The obtained inequality provides a necessary condition for the existence of nontrivial solutions to the considered problem and an example is given to illustrate it. We present two applications to demonstrate the effectiveness of the new Lyapunov-type inequality. One application is making use of the obtained inequality to prove the uniqueness of solutions for the corresponding nonhomogenous boundary value problem. The other application is that we derive an estimation related to the eigenvalue of the corresponding equation by using our obtained Lyapunov-type inequality. Furthermore, the obtained inequality generalizes some existing results in the literature.

The paper is organized as follows. In Section 2, we provide some notations, definitions, and preliminary results related to the mixed fractional integral and derivatives. In Section 3, the Green's function of boundary value problem (1) is given and a Lyapunov-type inequality for problem (1) is established by using the properties of its Green's function. Moreover, we give an example to illustrate that the obtained inequality provides a necessary condition for the existence of nontrivial solutions to problem (1). Some applications are presented to demonstrate the effectiveness of the new results in Section 4.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary results, which will be used throughout the article. Let $P = (0, a] \times (0, b]$, $\overline{P} = [0, a] \times [0, b]$, and $r = (r_1, r_2)$, where $0 < a, b < \infty$. Let $L(P)$ be the space of Lebesgue-integrable functions $f : P \to \mathbb{R}$. $C(\overline{P})$ denotes the Banach space of all continuous functions from $\overline{P}$ to $\mathbb{R}$ with the norm

$$\| f \| = \max_{(x,y)\in\overline{P}} |f(x, y)|.$$

By $AC(\overline{P})$, we denote the space of absolutely continuous functions on $\overline{P}$. $AC^2(\overline{P}) = \{f : \overline{P} \to \mathbb{R} \ and \ D_{xy} f(x, y) \in AC(\overline{P}), D_{xy} = \frac{\partial^2}{\partial x \partial y}\}$.

Definition 1 ([14])**.** *For $f(x,y) \in L(P)$, $r_1, r_2 > 0$, the expression*

$$I_0^r f(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-t)^{r_1-1}(y-s)^{r_2-1} f(t,s)dtds$$

is called the left mixed Riemann–Liouville integral of order r of $f(x,y)$.

In particular,

$$I_0^n f(x,y) = \frac{1}{((n-1)!)^2} \int_0^x \int_0^y (x-t)^{n-1}(y-s)^{n-1} f(t,s)dtds, \quad n \geq 1,$$

$$I_0^0 f(x,y) = f(x,y).$$

Lemma 1 ([14])**.** *Let $r = (r_1, r_2)$, $q = (q_1, q_2)$, $0 < r_1, r_2 \leq 1$, $q_1, q_2 > 0$. For $f(x,y) \in L(P)$, then*

$$I_0^q I_0^r f(x,y) = I_0^{q+r} f(x,y).$$

Definition 2 ([14])**.** *For $0 < r_1, r_2 < 1$, the expression*

$$D_0^r f(x,y) = D_{xy} I_0^{1-r} f(x,y)$$

is called the left mixed Riemann–Liouville derivative of order r of $f(x,y)$, where $1 - r = (1 - r_1, 1 - r_2)$.

In particular,

$$D_0^1 f(x,y) = D_{xy} f(x,y) = \frac{\partial^2 f(x,y)}{\partial x \partial y}.$$

Lemma 2 ([24])**.** *If $f(x,y) \in L(P)$ and $0 < r_1, r_2 \leq 1$, then*

$$D_0^r I_0^r f(x,y) = f(x,y)$$

holds for almost all $(x,y) \in P$.

Definition 3 ([26])**.** *For $0 < r_1, r_2 < 1$, the expression*

$${}^C D_0^r f(x,y) = I_0^{1-r}(D_{xy} f(x,y)) = \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y (x-t)^{-r_1}(y-s)^{-r_2} D_{ts} f(t,s)dtds$$

is called the mixed Caputo derivative of order r of $f(x,y)$.

In particular,

$${}^C D_0^1 f(x,y) = D_{xy} f(x,y) = \frac{\partial^2 f(x,y)}{\partial x \partial y}.$$

Lemma 3 ([26])**.** *Let $0 < r_1, r_2 \leq 1$. If $f(x,y) \in AC(\overline{P})$, then*

$$D_0^r (f(x,y) - \lambda(x,y)) = {}^C D_0^r f(x,y)$$

holds for almost all $(x,y) \in \overline{P}$, where $\lambda(x,y) = f(x,0) + f(0,y) - f(0,0)$.

Lemma 4. *Let $0 < r_1, r_2 \leq 1$. If $f(x,y) \in C(\overline{P})$ and $I_0^r f(x,y) \in AC(\overline{P})$, then*

$${}^C D_0^r I_0^r f(x,y) = f(x,y).$$

Proof. Since $f(x,y) \in C(\overline{P})$, it exists a constant $M > 0$ such that

$$|f(x,y)| \leq M.$$

Therefore,

$$\begin{aligned} |f_r(x,y)| &= |I_0^r f(x,y)| \\ &= \left| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-t)^{r_1-1}(y-s)^{r_2-1} f(t,s)dtds \right| \\ &\leq \frac{Mx^{r_1}y^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}. \end{aligned} \tag{4}$$

It follows from (4) and $I_0^r f(x,y) \in C(\overline{P})$ that

$$f_r(x,0) = f_r(0,y) = 0, \quad x \in [0,a], \quad y \in [0,b]. \tag{5}$$

By virtue of Lemmas 2 and 3 and (5), we get

$$\begin{aligned} {}^C D_0^r I_0^r f(x,y) &= D_0^r (I_0^r f(x,y) - f_r(x,0) - f_r(0,y) + f_r(0,0)) \\ &= D_0^r I_0^r f(x,y) \\ &= f(x,y). \end{aligned}$$

The proof is completed. □

Now, we pass to mixed Caputo derivative of large order $r = (r_1, r_2) \in (1,2] \times (1,2]$. In the higher order case, we can generalize to the following.

Definition 4. *Let $1 < r_1, r_2 < 2$. The expression*

$${}^C D_0^r f(x,y) = I_0^{2-r}(D_{xy}^2 f(x,y)) = \frac{1}{\Gamma(2-r_1)\Gamma(2-r_2)} \int_0^x \int_0^y (x-t)^{1-r_1}(y-s)^{1-r_2} D_{ts}^2 f(t,s)dtds$$

is called the mixed Caputo derivative of order r of $f(x,y)$, where $D_{xy}^2 = D_{xy}D_{xy}$.

Lemma 5. *For $f(x,y) \in AC^2(\overline{P})$, then*

$$I_0^2 D_{xy}^2 f(x,y) = f(x,y) - \gamma(x,y),$$

where

$$\begin{aligned} \gamma(x,y) = &- xy\frac{\partial^2 f(0,0)}{\partial x \partial y} + y\left(\frac{\partial f(x,0)}{\partial y} - \frac{\partial f(0,0)}{\partial y}\right) + x\left(\frac{\partial f(0,y)}{\partial x} - \frac{\partial f(0,0)}{\partial x}\right) \\ &+ f(x,0) + f(0,y) - f(0,0). \end{aligned}$$

Proof. Since $I_0^1 D_{xy} f(x,y) = f(x,y) - [f(x,0) + f(0,y) - f(0,0)]$, we get

$$\begin{aligned} I_0^1 D_{xy}^2 f(x,y) &= I_0^1 D_{xy}(D_{xy} f(x,y)) \\ &= D_{xy} f(x,y) - [(D_{xy} f(x,y))|_{x=0} + (D_{xy} f(x,y))|_{y=0} - (D_{xy} f(x,y))|_{x=0,y=0}]. \end{aligned}$$

Therefore,

$$\begin{aligned} I_0^2 D_{xy}^2 f(x,y) =& I_0^1 (I_0^1 D_{xy}^2 f(x,y)) \\ =& I_0^1 (D_{xy} f(x,y) - [(D_{xy} f(x,y))|_{x=0} + (D_{xy} f(x,y))|_{y=0} - (D_{xy} f(x,y))|_{x=0,y=0}]) \end{aligned}$$

$$
\begin{aligned}
&= f(x,y) - \left[-xy\frac{\partial^2 f(0,0)}{\partial x \partial y} + y\left(\frac{\partial f(x,0)}{\partial y} - \frac{\partial f(0,0)}{\partial y}\right) + x\left(\frac{\partial f(0,y)}{\partial x} - \frac{\partial f(0,0)}{\partial x}\right) \right. \\
&\quad \left. + f(x,0) + f(0,y) - f(0,0) \right] \\
&= f(x,y) - \gamma(x,y).
\end{aligned}
$$

The proof is completed. □

Lemma 6. *Let $1 < r_1, r_2 < 2$. For almost all $(x,y) \in \overline{P}$, then*
(1)${}^C D_0^r I_0^r f(x,y) = f(x,y)$, if $f(x,y) \in C(\overline{P})$ and $I_0^{r-1} f(x,y) \in AC(\overline{P})$;
(2)$I_0^r({}^C D_0^r f(x,y)) = f(x,y) - \gamma(x,y)$, if $f(x,y) \in AC^2(\overline{P})$, where $\gamma(x,y)$ is given by Lemma 5.

Proof. (1) According to Definition 4 and Lemmas 1 and 4, we have

$$
\begin{aligned}
{}^C D_0^r I_0^r f(x,y) &= I_0^{2-r} D_{xy}^2 I_0^r f(x,y) = I_0^{2-r} D_{xy}^2 I_0^1 I_0^{r-1} f(x,y) \\
&= I_0^{2-r} D_{xy} I_0^{r-1} f(x,y) = {}^C D_0^{r-1} I_0^{r-1} f(x,y) \\
&= f(x,y).
\end{aligned}
$$

(2) Using Definition 4 and Lemmas 1 and 5, we get

$$
I_0^r({}^C D_0^r f(x,y)) = I_0^r I_0^{2-r} D_{xy}^2 f(x,y) = I_0^2 D_{xy}^2 f(x,y) = f(x,y) - \gamma(x,y).
$$

The proof is completed. □

3. A Lyapunov-Type Inequality for Problem (1)

In order to obtain the Lyapunov-type inequality for problem (1), we first give an expression for the Green's function of the the boundary value problem (1) and its properties. Then, a Lyapunov-type inequality for problem (1) is presented by making use of the properties of the obtained Green's function.

Lemma 7. *Assume that $q \in L(\overline{P})$. A function u is a solution of problem (1), then it satisfies the integral equation*

$$
u(x,y) = -\int_0^a \int_0^b G(x,y,s,t) q(s,t) u(s,t) ds dt, \tag{6}
$$

where the Green function $G(x,y,s,t)$ is given by

$$
G(x,y,s,t) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} H(x,s) K(y,t), \tag{7}
$$

where $H(x,s)$ and $K(y,t)$ are given by

$$
H(x,s) = \begin{cases} \frac{x}{a}(a-s)^{r_1-1} - (x-s)^{r_1-1}, & 0 \le s \le x \le a, \\ \frac{x}{a}(a-s)^{r_1-1}, & 0 \le x \le s \le a, \end{cases} \tag{8}
$$

$$
K(y,t) = \begin{cases} \frac{y}{b}(b-t)^{r_2-1} - (y-t)^{r_2-1}, & 0 \le t \le y \le b, \\ \frac{y}{b}(b-t)^{r_2-1}, & 0 \le y \le t \le b. \end{cases} \tag{9}
$$

Proof. If $u(x,y)$ is a solution of (1), applying the integral operator I_0^r to (1) and making use of Lemma 6, we have

$$
u(x,y) = \gamma(x,y) - I_0^r(q(x,y)u(x,y)). \tag{10}
$$

Let
$$\frac{\partial u(x,0)}{\partial y} = c(x), \frac{\partial u(0,y)}{\partial x} = d(y). \tag{11}$$

By virtue of boundary value conditions $u(0,y) = u(x,0) = u(a,y) = u(x,b) = 0$, we get

$$\frac{\partial^2 u(0,0)}{\partial x \partial y} = d'(0), \quad \frac{\partial u(0,0)}{\partial y} = \frac{\partial u(0,y)}{\partial y}\Big|_{y=0} = 0, \quad \frac{\partial u(0,0)}{\partial x} = \frac{\partial u(x,0)}{\partial x}\Big|_{x=0} = 0, \tag{12}$$

$$c(a) = \lim_{y\to 0^+} \frac{u(a,y) - u(a,0)}{y} = 0 \tag{13}$$

and
$$d(b) = \lim_{x\to 0^+} \frac{u(x,b) - u(0,b)}{x} = 0. \tag{14}$$

Applying (11), (12), and $u(x,0) = u(0,y) = 0$ to (10), we have

$$u(x,y) = -xyd'(0) + yc(x) + xd(y) - I_0^r(q(x,y)u(x,y)). \tag{15}$$

Let $x = a$ and $y = b$ in (15) at the same time, we can calculate

$$d'(0) = -\frac{1}{ab} I_0^r(q(a,b)u(a,b)). \tag{16}$$

Furthermore, let $x = a$ and $y = b$ in (15) respectively, we can obtain

$$d(y) = yd'(0) + \frac{1}{a} I_0^r(q(a,y)u(a,y)) \tag{17}$$

and
$$c(x) = xd'(0) + \frac{1}{b} I_0^r(q(x,b)u(x,b)). \tag{18}$$

Applying (16)–(18) into (15), we have

$$\begin{aligned}
u(x,y) =& -\frac{xy}{ab} I_0^r(q(a,b)u(a,b)) + \frac{y}{b} I_0^r(q(x,b)u(x,b)) + \frac{x}{a} I_0^r(q(a,y)u(a,y)) - I_0^r(q(x,y)u(x,y)) \\
=& -\frac{1}{\Gamma(r_1)\Gamma(r_2)} \left[\frac{xy}{ab} \int_0^a \int_0^b (a-s)^{r_1-1}(b-t)^{r_2-1} q(s,t)u(s,t)dsdt \right. \\
& - \frac{y}{b} \int_0^x \int_0^b (x-s)^{r_1-1}(b-t)^{r_2-1} q(s,t)u(s,t)dsdt \\
& - \frac{x}{a} \int_0^a \int_0^y (a-s)^{r_1-1}(y-t)^{r_2-1} q(s,t)u(s,t)dsdt \\
& \left. + \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} q(s,t)u(s,t)dsdt \right] \\
=& -\frac{1}{\Gamma(r_1)\Gamma(r_2)} \left[\int_0^x \int_0^y \left[\frac{x}{a}(a-s)^{r_1-1} - (x-s)^{r_1-1} \right] \left[\frac{y}{b}(b-t)^{r_2-1} - (y-t)^{r_2-1} \right] q(s,t)u(s,t)dsdt \right. \\
& + \int_0^x \int_y^b \frac{y}{b}(b-t)^{r_2-1} \left[\frac{x}{a}(a-s)^{r_1-1} - (x-s)^{r_1-1} \right] q(s,t)u(s,t)dsdt \\
& + \int_x^a \int_0^y \frac{x}{a}(a-s)^{r_1-1} \left[\frac{y}{b}(b-t)^{r_2-1} - (y-t)^{r_2-1} \right] q(s,t)u(s,t)dsdt \\
& \left. + \int_x^a \int_y^b \frac{xy}{ab}(a-s)^{r_1-1}(b-t)^{r_2-1} q(s,t)u(s,t)dsdt \right] \\
=& -\int_0^a \int_0^b G(x,y,s,t)q(s,t)u(s,t)dsdt,
\end{aligned}$$

where $G(x, y, s, t)$ is given by (7)–(9).

The proof is completed. □

Lemma 8. *The Green function G given by (7) satisfies*

$$\max_{\substack{x,s\in[0,a]\\ y,t\in[0,b]}} |G(x,y,s,t)| = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \max\{v(r_1), p(r_1)\} \max\{v(r_2), p(r_2)\} a^{r_1-1} b^{r_2-1}, \tag{19}$$

where

$$v(r) = \frac{1}{r}\left(1 - \frac{1}{r}\right)^{r-1} > 0, \quad p(r) = (2-r)\left(\frac{1}{r-1}\right)^{\frac{r-1}{r-2}} > 0. \tag{20}$$

Proof. It follows from (7) that

$$\max_{\substack{x,s\in[0,a]\\ y,t\in[0,b]}} |G(x,y,s,t)| = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \max_{x,s\in[0,a]} |H(x,s)| \max_{y,t\in[0,b]} |K(y,t)|. \tag{21}$$

In the case $0 \le s \le x \le a$, for fixed $x \in [0, a]$, we have

$$\frac{\partial H(x,s)}{\partial s} = (1-r_1)\left[\frac{x}{a}(a-s)^{r_1-2} - (x-s)^{r_1-2}\right].$$

Obviously, $\frac{\partial H(x,s)}{\partial s} \ge 0$. Hence, we establish that $H(x, s)$ is increasing in s. Therefore,

$$\max_{s\in[0,a]} H(x,s) = H(x,x) = \frac{x}{a}(a-x)^{r_1-1} \ge 0,$$
$$\min_{s\in[0,a]} H(x,s) = H(x,0) = x(a^{r_1-2} - x^{r_1-2}) \le 0.$$

Let $m(x) = \frac{x}{a}(a-x)^{r_1-1}$. From $m'(x) = \frac{(a-x)^{r_1-2}(a-r_1x)}{a} = 0$, we get that $x = \frac{a}{r_1}$. Moreover, we obtain that

$$\max_{x,s\in[0,a]} H(x,s) = \max_{x\in[0,a]} m(x) = m\left(\frac{a}{r_1}\right) = v(r_1)a^{r_1-1}. \tag{22}$$

Let $n(x) = x(a^{r_1-2} - x^{r_1-2})$. From $n'(x) = a^{r_1-2} - (r_1-1)x^{r_1-2} = 0$, we get that $x = \frac{a}{(r_1-1)^{\frac{1}{r_1-2}}}$. Due to $n(x) < 0$, we obtain

$$\min_{x,s\in[0,a]} H(x,s) = n\left(\frac{a}{(r_1-1)^{\frac{1}{r_1-2}}}\right) = -p(r_1)a^{r_1-1}. \tag{23}$$

By virtue of (22) and (23), we conclude that

$$\max_{x,s\in[0,a]} |H(x,s)| = \max\{v(r_1), p(r_1)\}a^{r_1-1}, \qquad 0 \le s \le x \le a. \tag{24}$$

In the case $0 \le x \le s \le a$, obviously, $H(x, s)$ is decreasing in s. Therefore, with the help of (22), we have

$$\max_{x,s\in[0,a]} H(x,s) = \max_{x\in[0,a]} H(x,x) = v(r_1)a^{r_1-1}, \tag{25}$$

$$\min_{x,s\in[0,a]} H(x,s) = \min_{x\in[0,a]} H(x,a) = 0. \tag{26}$$

From (25) and (26), we deduce that

$$\max_{x,s\in[0,a]} |H(x,s)| = v(r_1)a^{r_1-1}, \qquad 0 \le x \le s \le a. \tag{27}$$

By means of (24) and (27), we have

$$\max_{x,s\in[0,a]} |H(x,s)| = \max\{v(r_1), p(r_1)\}a^{r_1-1}. \tag{28}$$

Analogously, we can obtain the fact that

$$\max_{y,t\in[0,b]} |K(y,t)| = \max\{v(r_2), p(r_2)\}b^{r_2-1}. \tag{29}$$

In conclusion, (19) is obtained with the help of (21), (28), and (29). The proof is completed. □

Our main aim is the following Lyapunov-type inequality.

Theorem 1. *If a function $u(x,y) \in C(\overline{P}) \cap C^2(P)$ is a nontrivial solution to problem (1), then*

$$\int_0^a \int_0^b |q(x,y)|dxdy > \frac{\Gamma(r_1)\Gamma(r_2)}{\max\{v(r_1), p(r_1)\}\max\{v(r_2), p(r_2)\}a^{r_1-1}b^{r_2-1}}. \tag{30}$$

Proof. It follows from Lemma 7 that a solution to problem (1) satisfies the integral equation

$$u(x,y) = -\int_0^a \int_0^b G(x,y,s,t)q(s,t)u(s,t)dsdt.$$

Hence,

$$\begin{aligned} \| u \| &= \max_{(x,y)\in\overline{P}} |u(x,y)| \\ &= \max_{(x,y)\in\overline{P}} \left| \int_0^a \int_0^b G(x,y,s,t)q(s,t)u(s,t)dsdt \right|. \end{aligned}$$

With the help of Lemma 8, we have

$$\| u \| < \frac{1}{\Gamma(r_1)\Gamma(r_2)} \max\{v(r_1), p(r_1)\}\max\{v(r_2), p(r_2)\}a^{r_1-1}b^{r_2-1} \int_0^a \int_0^b |q(s,t)|dsdt \, \| u \| . \tag{31}$$

Since $u(x,y)$ is a nontrivial solution, (31) is equivalent to

$$\Gamma(r_1)\Gamma(r_2) < \max\{v(r_1), p(r_1)\}\max\{v(r_2), p(r_2)\}a^{r_1-1}b^{r_2-1} \int_0^a \int_0^b |q(s,t)|dsdt,$$

from which inequality (30) follows.

The proof is completed. □

Remark 1. *Theorem 1 provides a necessary condition for the existence of nontrivial solutions to the considered problem. That is to say, if*

$$\int_0^a \int_0^b |q(x,y)|dxdy < \frac{\Gamma(r_1)\Gamma(r_2)}{\max\{v(r_1), p(r_1)\}\max\{v(r_2), p(r_2)\}a^{r_1-1}b^{r_2-1}}, \tag{32}$$

then problem (1) has only zero solution in $C(\overline{P}) \cap C^2(P)$.

Example 1. *Consider the following boundary value problem:*

$$\begin{cases} {}^{C}D_0^r u(x,y) + x^{-\frac{1}{3}} y^{-\frac{1}{2}} u(x,y) = 0, & (x,y) \in P = (0,1) \times (0,1), \\ u(0,y) = 0, u(1,y) = 0, & y \in [0,1], \\ u(x,0) = 0, u(x,1) = 0, & x \in [0,1], \end{cases} \tag{33}$$

where $r = (r_1, r_2) = (\frac{3}{2}, \frac{3}{2})$, ${}^{C}D_0^r$ *is the mixed Caputo derivative of* r, *and* $q(x,y) = x^{-\frac{1}{3}} y^{-\frac{1}{2}}$ *is a Lebesgue function on* $\overline{P}$. *Since* $v(1.5) = \frac{2\sqrt{3}}{9}$, $p(1.5) = \frac{1}{4}$, *and* $\Gamma(1.5) = \frac{\sqrt{\pi}}{2}$, *we can calculate*

$$\frac{\Gamma(1.5)\Gamma(1.5)}{\max\{v(1.5), p(1.5)\} \max\{v(1.5), p(1.5)\}} = \frac{27\pi}{16}.$$

However, by simple calculation, we get

$$\int_0^1 \int_0^1 q(x,y) dxdy = \int_0^1 \int_0^1 x^{-\frac{1}{3}} y^{-\frac{1}{2}} dxdy = 3 < \frac{27\pi}{16}.$$

According to Remark 1, problem (33) has only trivial solution $u(x,y) = 0$ *in* $C(\overline{P}) \cap C^2(P)$.

4. Applications

In this section, some applications of the obtained Lyapunov-type inequality (30) in Section 3 are presented.

One application is making use of inequality (30) to prove the uniqueness of solutions for the corresponding nonhomogenous boundary value problem. Consider the following nonhomogenous boundary value problem:

$$\begin{cases} {}^{C}D_0^r u(x,y) + q(x,y) u(x,y) = w(x,y), & (x,y) \in P = (0,a) \times (0,b), \\ u(0,y) = 0, u(a,y) = 0, & y \in [0,b], \\ u(x,0) = 0, u(x,b) = 0, & x \in [0,a]. \end{cases} \tag{34}$$

Corollary 1. *If the solution to problem (34) exists, and*

$$\int_0^a \int_0^b |q(x,y)| dxdy < \frac{\Gamma(r_1)\Gamma(r_2)}{\max\{v(r_1), p(r_1)\} \max\{v(r_2), p(r_2)\} a^{r_1 - 1} b^{r_2 - 1}} \tag{35}$$

holds, then problem (34) has a unique solution.

Proof. Assume that $u_1(x,y)$, $u_2(x,y)$ are both solutions to problem (34), then $u(x,y) = u_1(x,y) - u_2(x,y)$ is a solution of the corresponding homogenous boundary value problem. By virtue of Remark 1, the corresponding homogeneous boundary value problem has only zero solution in $C(\overline{P}) \cap C^2(P)$. Therefore, problem (34) has a unique solution. □

The other application is that we derive an estimation related to the eigenvalue of the corresponding equation by using our obtained Lyapunov-type inequality (30). For given $\lambda \in \mathbb{R}$, we consider the following boundary value problem

$$\begin{cases} {}^{C}D_0^r u(x,y) + \lambda u(x,y) = 0, & (x,y) \in P = (0,a) \times (0,b), \\ u(0,y) = 0, u(a,y) = 0, & y \in [0,b], \\ u(x,0) = 0, u(x,b) = 0, & x \in [0,a], \end{cases} \tag{36}$$

where ${}^{C}D_0^r$ is the mixed Caputo derivative of order r and $a, b > 0$, $r = (r_1, r_2)$, $1 < r_1, r_2 < 2$. If problem (36) admits a nontrivial solution $u_\lambda \in C(\overline{P}) \cap C^2(P)$, we say that λ is an eigenvalue of problem (36).

Corollary 2. *If λ is an eigenvalue of problem (36), then*

$$|\lambda| > \frac{\Gamma(r_1)\Gamma(r_2)}{\max\{v(r_1), p(r_1)\} \max\{v(r_2), p(r_2)\} a^{r_1-2} b^{r_2-2}}. \tag{37}$$

Proof. Since λ is an eigenvalue of problem (36), it means that problem (36) has a nontrivial solution u_λ. According to Theorem 1, we have

$$\int_0^a \int_0^b |\lambda| dxdy > \frac{\Gamma(r_1)\Gamma(r_2)}{\max\{v(r_1), p(r_1)\} \max\{v(r_2), p(r_2)\} a^{r_1-1} b^{r_2-1}}.$$

Therefore,

$$|\lambda| > \frac{\Gamma(r_1)\Gamma(r_2)}{\max\{v(r_1), p(r_1)\} \max\{v(r_2), p(r_2)\} a^{r_1-2} b^{r_2-2}},$$

which is the desired result.

The proof is completed. □

5. Conclusions

In this article, we consider a partial differential equation on a rectangular domain with the mixed Caputo derivative subject to Dirichlet-type boundary conditions. A new Lyapunov-type inequality for the considered problem is derived. The obtained inequality provides a necessary condition for the existence of nontrivial solutions. Our approach is based on converting the boundary value problem into an integral equation and then finding the maximum value of its Green's function. We give two applications related to our obtained inequality. The new results generalize some existing results in the literature. We expect that the proposed approaches and the obtained results in this paper can be adapted to study other fractional boundary value problems.

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