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Slant Curves in Contact Lorentzian Manifolds with CR Structures

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Abstract: In this paper, we first find the properties of the generalized Tanaka–Webster connection in a contact Lorentzian manifold. Next, we find that a necessary and sufficient condition for the $\hat{\nabla}$-geodesic is a magnetic curve (for ∇) along slant curves. Finally, we prove that when $c \leq 0$, there does not exist a non-geodesic slant Frenet curve satisfying the $\hat{\nabla}$-Jacobi equations for the $\hat{\nabla}$-geodesic vector fields in M. Thus, we construct the explicit parametric equations of pseudo-Hermitian pseudo-helices in Lorentzian space forms $M^3_1(\hat{H})$ for $\hat{H} = 2c > 0$.

Keywords: slant curves; Jacobi equation; CR structure; Lorentzian Sasakian space forms

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1. Introduction

The notion of slant curves was introduced in [1] for a contact Riemannian three-manifold, that is, a curve in a contact three-manifold is said to be *slant* if its tangent vector field has a constant angle with the Reeb vector field. In [2], we showed that proper biharmonic curves are helices in three-dimensional Sasakian space forms of constant holomorphic sectional curvature $\widetilde{H}(= 2c - 3)$. In particular, if $\widetilde{H} \neq 1$, then it is a slant helix; that is, a helix such that $\eta(\gamma') = \cos\alpha_0$ is a constant, with $\kappa^2 + \tau^2 = 1 + (\widetilde{H} - 1)\sin^2\alpha_0$. In [3], we studied slant curves satisfying $\hat{\nabla}$-Jacobi equations for a $\hat{\nabla}$-geodesic vector field in Sasakian space forms with respect to the Tanaka–Webster connection $\hat{\nabla}$. In [4], we showed that proper biharmonic Frenet curves are pseudo-helices in three-dimensional Lorentzian Sasakian space forms of constant holomorphic sectional curvature $H(= 2c + 3)$. In particular, if $H \neq -1$, then it is a slant pseudo-helix; that is, a pseudo-helix such that $\eta(\gamma')$ is a constant, with $\kappa^2 - \tau^2 = -1 + (H + 1)(1 + \varepsilon_1 a^2)$ for $a = \eta(\gamma')$.

In this paper, we study the slant curves in Lorentzian Sasakian space forms of constant holomorphic sectional curvature $\hat{H} = 2c$ for the Tanaka–Webster connection $\hat{\nabla}$.

D. Perrone [5,6] showed that the notion of non-degenerate almost CR structures is equivalent to the notion of contact pseudo-metric structures. Thus, he defined the generalized Tanaka–Webster connection $\hat{\nabla}$ in a contact pseudo-metric manifold.

In Section 3, we find the properties of the Tanaka–Webster connection in a contact Lorentzian manifold. In Section 4.1, we find that a necessary and sufficient condition for a $\hat{\nabla}$-geodesic is a magnetic curve (for ∇) along slant curves.

Next, we investigate the $\hat{\nabla}$-Jacobi equation for a $\hat{\nabla}$-geodesic vector field in contact Lorentzian manifolds:

$$\begin{cases} \hat{\nabla}_{\gamma'}\gamma' = \hat{\sigma}(\gamma), \\ \hat{\nabla}^2_{\gamma'}\hat{\sigma}(\gamma) - \hat{\nabla}_{\gamma'}\hat{T}(\hat{\sigma}(\gamma), \gamma') - \hat{R}(\hat{\sigma}(\gamma), \gamma')\gamma' = 0, \end{cases} \tag{1}$$

where the torsion $\hat{T}(X, Y) = [X, Y] - \hat{\nabla}_X Y + \hat{\nabla}_Y X$ and pseudo-Hermitian curvature $\hat{R}(X, Y) = \hat{\nabla}_{[X,Y]} - [\hat{\nabla}_X, \hat{\nabla}_Y]$. Then, in Section 4.2, we prove that when $c \leq 0$, there does not exist a non-geodesic slant Frenet curve satisfying the $\hat{\nabla}$-Jacobi equations for the $\hat{\nabla}$-geodesic vector fields in M. Thus, we obtain the explicit parametric equations satisfying (1) in Lorentzian space forms $M^3_1(\hat{H})$ for $\hat{H} = 2c > 0$.

2. Preliminaries

2.1. Contact Lorentzian Manifold

An *almost contact structure* (φ, ξ, η) on a $(2n+1)$-dimensional differentiable manifold M has a tensor field φ of $(1,1)$, a global vector field ξ, and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2}$$

$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0. \tag{3}$$

If a $(2n+1)$-dimensional smooth manifold M with almost contact structure (φ, ξ, η) admits a compatible Lorentzian metric such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{4}$$

then we say that M has an almost contact Lorentzian structure (η, ξ, φ, g). Setting $Y = \xi$, we have

$$\eta(X) = -g(X, \xi). \tag{5}$$

Next, if the compatible Lorentzian metric g satisfies

$$d\eta(X, Y) = g(X, \varphi Y), \tag{6}$$

then η is a contact form on M, ξ is the associated Reeb vector field, g is an associated metric, and $(M, \varphi, \xi, \eta, g)$ is called a *contact Lorentzian manifold*.

For a contact Lorentzian manifold M, one may naturally define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{\mathrm{d}}{\mathrm{d}t}) = (\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}),$$

where X is a vector field tangent to M, t is the coordinate of $\mathbb{R}$, and f is a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, then the contact Lorentzian manifold M is called *normal* or *Sasakian*. It is known that a contact Lorentzian manifold M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

Proposition 1 ([7,8])**.** *An almost contact Lorentzian manifold $(M^{2n+1}, \eta, \xi, \varphi, g)$ is Sasakian if and only if*

$$(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X. \tag{7}$$

Using similar arguments and computations to those of [9], we obtain:

Proposition 2 ([7,8])**.** *Let $(M^{2n+1}, \eta, \xi, \varphi, g)$ be a contact Lorentzian manifold. Then*

$$\nabla_X \xi = \varphi X - \varphi h X, \tag{8}$$

where $h = \frac{1}{2} L_\xi \varphi$.

If ξ is a killing vector field with respect to the Lorentzian metric g, that is, M^{2n+1} is a *K-contact Lornetzian manifold*. Then

$$\nabla_X \xi = \varphi X. \tag{9}$$

Proposition 3. *Let $\{T, N, B\}$ be orthonormal Frame fields in a Lorentzian three-manifold. Then*

$$T \wedge_L N = \varepsilon_3 B, \quad N \wedge_L B = \varepsilon_1 T, \quad B \wedge_L T = \varepsilon_2 N.$$

2.2. Lorentzian Bianchi–Cartan–Vranceanu Model Space

The one-parameter family of Riemannian three-manifolds $\{\mathcal{M}^3(\widetilde{H})\}_{\widetilde{H}\in\mathbb{R}}$ is classically known by L. Bianchi [10], E. Cartan [11], and G. Vranceanu [12] . The model $\mathcal{M}^3(\widetilde{H})$ of the Sasakian three-space form is called the *Bianchi–Cartan–Vranceanu model* of the three-dimensional Sasakian space form. Cartan classified all three-dimensional spaces with four-dimensional isometry groups in [11]. Thus, he proved that they are all homogeneous. Moreover, parallel surfaces in Bianchi–Cartan–Vranceanu spaces are classified in [13].

On the other hand, G. Calvaruso [7] proved that there is a one-to-one correspondence between homogeneous contact Riemannian three-manifolds and homogeneous contact Lorentzian three-manifolds.

Now, we construct a *Lorentzian Bianchi–Cartan–Vranceanu model* of three-dimensional Lorentzian Sasakian space forms.

Let c be a real number, and set

$$\mathcal{D} = \left\{ (x, y, z) \in \mathbb{R}^3(x, y, z) \mid 1 + \frac{c}{2}(x^2 + y^2) > 0 \right\}.$$

Note that $\mathcal{D}$ is the whole $\mathbb{R}^3(x, y, z)$ for $c \geq 0$. In the region $\mathcal{D}$, we take the contact form

$$\eta = dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)}.$$

Then, the Reeb vector field of η is $\xi = \frac{\partial}{\partial z}$.
Next, we equip $\mathcal{D}$ with the Lorentzian metric g_c as follows:

$$g_c = \frac{dx^2 + dy^2}{\{1 + \frac{c}{2}(x^2 + y^2)\}^2} - \left(dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)} \right)^2.$$

We take the following orthonormal frame field on $(\mathcal{D}, g_c)$:

$$u_1 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ u_2 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ u_3 = \frac{\partial}{\partial z}.$$

Then, the endomorphism field φ is defined by

$$\varphi u_1 = u_2,\ \varphi u_2 = -u_1,\ \varphi u_3 = 0.$$

The Levi–Civita connection ∇ of this Lorentzian three-manifold is described as

$$\begin{gathered}
\nabla_{u_1} u_1 = c\, y u_2,\ \ \nabla_{u_1} u_2 = -c\, y u_1 + u_3,\ \ \nabla_{u_1} u_3 = u_2, \\
\nabla_{u_2} u_1 = -c\, x u_2 - u_3,\ \ \nabla_{u_2} u_2 = c\, x u_1,\ \ \nabla_{u_2} u_3 = -u_1, \qquad (10) \\
\nabla_{u_3} u_1 = u_2,\ \ \nabla_{u_3} u_2 = -u_1,\ \ \nabla_{u_3} u_3 = 0. \\
[u_1, u_2] = -c\, y u_1 + c\, x u_2 + 2u_3,\ \ [u_2, u_3] = [u_3, u_1] = 0.
\end{gathered}$$

The contact form η on $\mathcal{D}$ satisfies

$$d\eta(X, Y) = g(X, \varphi Y). \qquad (11)$$

Moreover, the structure $(\varphi, \xi, \eta, g_c)$ is Sasakian. The curvature tensor $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ on $(M^3, \eta, \xi, \varphi, g_c)$ is given by

$$R(u_1,u_2)u_2 = -(2c+3)u_1, \quad R(u_1,u_3)u_3 = -u_1,$$

$$R(u_2,u_1)u_1 = -(2c+3)u_2, \quad R(u_2,u_3)u_3 = -u_2,$$

$$R(u_3,u_1)u_1 = u_3, \quad R(u_3,u_2)u_2 = u_3.$$

The sectional curvature ([7]) is given by

$$K(\xi, u_i) = -R(\xi, u_i, \xi, u_i) = -1, \; for \; i = 1,2,$$

and

$$K(u_1,u_2) = R(u_1,u_2,u_1,u_2) = 2c+3.$$

Hence, $(\mathcal{D}, g_c)$ is of constant holomorphic sectional curvature $H = 2c+3$.

Hereafter, we denote this model $(\mathcal{D}, g_c)$ of a Lorentzian Sasakian space form by $\mathcal{M}_1^3(H)$.

The harmonic maps $\phi : (M^m, g) \to (N^n, h)$ between two pseudo-Riemannian manifolds as critical points of the energy $E(\phi) = \int_M |d\phi|^2 dv$. The *tension field* τ_ϕ is defined by

$$\tau_\phi = trace \nabla^\phi d\phi = \Sigma_{i=1}^m \varepsilon_i (\nabla^\phi_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)),$$

where ∇^ϕ and $\{e_i\}$ denote the induced connection by ϕ on the bundle $\phi^* TN^n$. A smooth map ϕ is called a *harmonic map* if its tension field vanishes.

Next, the bienergy $E_2(\phi)$ of a map ϕ is defined by $E_2(\phi) = \int_M |\tau_\phi|^2 dv$,; ϕ is biharmonic if it is a critical point of the bienergy. Harmonic maps are clearly biharmonic. Non-harmonic biharmonic maps are called *proper* biharmonic maps. We define the *bitension field* $\tau_2(\phi)$ by

$$\tau_2(\phi) := \Sigma_{i=1}^m \varepsilon_i ((\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i} e_i})\tau_\phi - R^N(\tau_\phi, d\phi(e_i))d\phi(e_i)),$$

where R^N is the curvature tensor of N^n and is defined by $R^N(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ (see [14]).

We now restrict our attention to isometric immersions $\gamma : I \to (M, g)$ from an interval I to a pseudo-Riemannian manifold. The image $C = \gamma(I)$ is the trace of a curve in M, and γ is a parametrization of C by arc length. In this case, the tension field becomes $\tau_\gamma = \varepsilon_1 \nabla_{\gamma'} \gamma'$ and the biharmonic equation reduces to

$$\tau_2(\gamma) = \varepsilon_1 (\nabla^2_{\gamma'} \tau_\gamma - R(\tau_\gamma, \gamma')\gamma') = 0.$$

Note that $C = \gamma(I)$ is part of a geodesic of M if and only if γ is harmonic. Moreover, from the biharmonic equation, if γ is harmonic, geodesics are a subclass of biharmonic curves.

In [4], we showed that proper biharmonic Frenet curves are pseudo-helices in three-dimensional Lorentzian Sasakian space forms of constant holomorphic sectional curvature $H(= 2c+3)$. In particular, in [15], we studied proper biharmonic spacelike curves in Lorentzian Heisenberg space.

3. Almost CR Manifold

We recall the notions of CR structure and pseudo-Hermitian geometry.

Let $(M, \mathcal{H}(M), J, \theta)$ be a non-degenerate almost CR manifold. If we extend J to an endomorphism φ of the tangent bundle by $\varphi \mid_{\mathcal{H}(M)} = J$ and $\varphi(P) = 0$, where P is the Reeb vector field of θ, then $\varphi^2 = -I + \theta \otimes P$. The *Webster metric* g_θ is given by

$$g_\theta(X,Y) = (d\theta)(X, JY), \quad g_\theta(X,P) = 0, \quad g_\theta(P,P) = \varepsilon(= \pm 1),$$

for any $X, Y \in \mathcal{H}(M)$. g_θ is a pseudo-Riemannian metric on M. Hence,

$$\varphi,\ \xi = -P,\ \eta = -\theta,\ g = g_\theta$$

is a contact pseudo-metric structure on M. Conversely, a contact pseudo-metric structure (φ, ξ, η, g) defines a non-degenerate almost CR structure on M given by $(\mathcal{H}(M), J, \theta)$, where $\mathcal{H}(M) = ker\eta$, $\theta = -\eta$ and $J = \varphi\,|_{\mathcal{H}(M)}$. Then, we have

Proposition 4 ([5])**.** *The notion of a non-degenerate almost CR structure is equivalent to the notion of a contact pseudo-metric structure.*

Tanaka ([16]) defined the canonical affine connection, called the Tanaka–Webster connection, on a non-degenerate CR manifold. D. Perrone defined the *generalized Tanaka–Webster connection* [5] on a contact pseudo-metric manifold $M = (M^{2n+1}, \eta, \xi, \varphi, g)$.

In this section, we consider the *generalized Tanaka–Webster connection* on a contact Lorentzian manifold M.

The generalized Tanaka–Webster connection $\hat{\nabla}$ is defined by (cf. [3], [16])

$$\hat{\nabla}_X Y = \nabla_X Y - \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi,$$

for all vector fields X, Y on M. $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + A(X, Y).$$

Then, using (5) and (8), we have

$$A(X, Y) = -\eta(X)\varphi Y - \eta(Y)(\varphi X - \varphi h X) - g(\varphi X - \varphi h X, Y)\xi. \tag{12}$$

Next, if we define the torsion $\hat{T}(X, Y) = [X, Y] - \hat{\nabla}_X Y + \hat{\nabla}_Y X$ for the Tanaka–Webster connection $\hat{\nabla}$ in M ([17]), then we get

$$\hat{T}(X, Y) = 2g(\varphi X, Y)\xi + \eta(X)\varphi h Y - \eta(Y)\varphi h X. \tag{13}$$

In particular, for a K-contact manifold, (12) and the above equation reduce as follows:

$$\begin{aligned} A(X, Y) &= -\eta(X)\varphi Y - \eta(Y)\varphi X - g(\varphi X, Y)\xi, \\ \hat{T}(X, Y) &= 2g(\varphi X, Y)\xi. \end{aligned}$$

Using (2)–(9), we have

Theorem 1. *The generalized Tanaka–Webster connection $\hat{\nabla}$ on a contact Lorentzian manifold $M = (M^{2n+1}; \eta, \xi, \varphi, g)$ is the unique linear connection satisfying the following conditions*:

(a) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$,
(b) $\hat{\nabla} g = 0$,
(c) $\hat{T}(X, Y) = 2g(\varphi X, Y)\xi$, $X,\ Y \in \mathcal{D}$,
(d) $\hat{T}(\xi, \varphi Y) = -\varphi\hat{T}(\xi, Y)$, $Y \in \mathcal{D}$,
(e) $(\hat{\nabla}_X \varphi)Y = Q(X, Y)$, $X,\ Y \in TM$.

The Tanaka–Webster connection on a non-degenerate (integrable) CR manifold is defined as the unique linear connection satisfying (a), (b), (c), (d), and $Q = 0$ (CR integrability), where Q is a $(1, 2)$-tensor field on M defined by $Q(X, Y) = (\nabla_X \varphi)Y - g(X - hX, Y)\xi - \eta(Y)(X - hX)$.

Thus, in [5] (page 217), we find:

Corollary 1. *Let $(M^{2n+1}, \eta, \xi, \varphi, g)$ be a contact Lorentzian manifold. Then, the (M^{2n+1}, D) is a (strongly pseudoconvex) CR manifold if and only if*

$$(\nabla_X \varphi)Y = g(X - hX, Y)\xi + \eta(Y)(X - hX).$$

In particular, if M^{2n+1} is a Lorentzian Sasakian manifold, then it satisfies (7). In fact, every three-dimensional contact Lorentzian manifold is a (strongly pseudoconvex) CR manifold. Thus, a three-dimensional K-contact manifold is Sasakian.

4. Slant Curves in Non-Degenerate CR Manifolds

Let $\gamma : I \to M^3$ be a unit speed curve in Lorentzian three-manifolds M^3 such that γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$. The constant ε_1 is called the *causal character* of γ. A unit speed curve γ is said to be spacelike or timelike if its causal character is 1 or -1, respectively. A unit speed curve γ is said to be a *Frenet curve* if $g(\gamma'', \gamma'') \neq 0$. A Frenet curve γ admits an orthonormal frame field $\{T = \gamma', N, B\}$ along γ. Then, the *Frenet–Serret* equations, following [14] and [18], are:

$$\begin{cases} \hat{\nabla}_{\gamma'} T = \phantom{-\varepsilon_1 \hat{\kappa} T} \varepsilon_2 \hat{\kappa} N, \\ \hat{\nabla}_{\gamma'} N = -\varepsilon_1 \hat{\kappa} T \quad - \varepsilon_3 \hat{\tau} B, \\ \hat{\nabla}_{\gamma'} B = \phantom{-\varepsilon_1 \hat{\kappa} T} \varepsilon_2 \hat{\tau} N, \end{cases} \tag{14}$$

where $\hat{\kappa} = |\hat{\nabla}_{\gamma'} \gamma'|$ is the *geodesic curvature* of γ and $\hat{\tau}$ is its *geodesic torsion* for the Tanaka–Webster connection $\hat{\nabla}$. The vector fields T, N, and B are called the tangent vector field, principal normal vector field, and binormal vector field of γ, respectively.

The constants ε_2 and ε_3 are defined by $g(N, N) = \varepsilon_2$ and $g(B, B) = \varepsilon_3$, and are called the *second causal character* and *third causal character* of γ, respectively. Thus, this satisfies $\varepsilon_1 \varepsilon_2 = -\varepsilon_3$.

A Frenet curve γ is a *pseudo-Hermitian geodesic* if and only if $\hat{\kappa} = 0$. A Frenet curve γ with constant geodesic curvature and zero geodesic torsion is called a *pseudo-Hermitian pseudo-circle*. A *pseudo-Hermitian pseudo-helix* is a Frenet curve γ whose geodesic curvature and torsion are constant.

4.1. Slant Curves

A one-dimensional integral submanifold of D in a three-dimensional contact manifold is called a *Legendre curve*, especially to avoid confusion with an integral curve of the vector field ξ. As a generalization of the Legendre curve, the notion of slant curves was introduced in [1] for a contact Riemannian three-manifold, that is, a curve in a contact three-manifold is said to be *slant* if its tangent vector field has a constant angle with the Reeb vector field.

Similarly to in the contact Riemannian three-manifolds, a curve in a contact Lorentzian three-manifold is said to be *slant* if its tangent vector field has a constant angle with the Reeb vector field (i.e., $g(\gamma', \xi)$ is a constant). In particular, if $g(\gamma', \xi) = 0$ then γ is a Legendre curve.

Let γ be a Frenet curve in a Sasakian Lorentzian three-manifold M^3. Then, we get

$$\hat{\nabla}_{\gamma'} \gamma' = \nabla_{\gamma'} \gamma' - 2\eta(\gamma')\varphi\gamma'.$$

If γ is a slant curve, then since $\eta(\gamma') = a$, a is a constant, $\hat{\nabla}_{\gamma'} \gamma' = 0$ if and only if $\nabla_{\gamma'} \gamma' = 2a\varphi\gamma'$. Hence, we have:

Proposition 5. *A Frenet curve γ in a Sasakian Lorentzian three-manifold M^3 is a slant curve. Then, γ is a geodesic for $\hat{\nabla}$ if and only if it is a magnetic curve (for ∇).*

Recently, we studied slant curves and magnetic curves in Sasakian Lorentzian three-manifolds (see [19]). If a curve γ satisfies $\nabla_{\gamma'} \gamma' = q\varphi\gamma'$, then we call it a contact magnetic curve in a contact Riemannian and Lorentzian manifold; we proved that γ is a slant curve if and only if M is Sasakian.

Now, we assume that $\eta(T) = a$, where a is a function. Using (5) and differentiating $g(T, \xi) = -a$ along γ for a Tanaka–Webster connection $\hat{\nabla}$, then

$$-a' = g(\varepsilon_2 \hat{\kappa} N, \xi) + g(T, \hat{\nabla}_T \xi) = -\varepsilon_2 \hat{\kappa} \eta(N).$$

This equation implies:

Proposition 6. *A non-geodesic Frenet curve γ for $\hat{\nabla}$ in a Sasakian Lorentzian three-manifold M^3 is a slant curve if and only if $\eta(N) = 0$.*

Moreover, we have:

Lemma 1. *Let γ be a non-geodesic slant curve in the three-dimensional almost contact Lorentzian manifold M. Then, we find an orthonormal frame field in M as follows:*

$$T = \gamma', \quad N = \frac{\varphi T}{\sqrt{\varepsilon_1 + a^2}}, \quad B = \frac{\xi + \varepsilon_1 a T}{\sqrt{\varepsilon_1 + a^2}},$$

and $\xi = -\varepsilon_1 a T + \sqrt{\varepsilon_1 + a^2} B$.
Thus, γ is a spacelike curve with a spacelike normal vector field or timelike curve.

Differentiating $\xi = -\varepsilon_1 a T + \sqrt{\varepsilon_1 + a^2} B$ along γ for $\hat{\nabla}$ and using (14), we have:

Proposition 7. *A non-geodesic Frenet curve γ in a Sasakian Lorentzian three-manifold M^3 is a slant curve. Then, the ratio of $\hat{\kappa}$ and $\hat{\tau}$ is constant.*

4.2. $\hat{\nabla}$-Jacobi Equations

We find that the non-vanishing Tanaka–Webster connections $\hat{\nabla}$ of the Bianchi–Cartan–Vranceanu model space are

$$\hat{\nabla}_{u_1} u_1 = c\, y u_2, \ \ \hat{\nabla}_{u_1} u_2 = -c\, y u_1, \ \ \hat{\nabla}_{u_2} u_1 = -c\, x u_2, \ \ \hat{\nabla}_{u_2} u_2 = c\, x u_1.$$

By using the above data, we calculate the Tanaka–Webster curvature tensor $\hat{R}(X, Y)Z = \hat{\nabla}_{[X,Y]} Z - \hat{\nabla}_X(\hat{\nabla}_Y Z) + \hat{\nabla}_Y(\hat{\nabla}_X Z)$. Then, we find that

$$\hat{R}(u_1, u_2)u_2 = -2c u_1, \ \ \hat{R}(u_1, u_2)u_1 = 2c u_2, \tag{15}$$

and all others are zero.

As $\hat{H} = H - 3$, we find that constant holomorphic sectional curvature $\hat{H} = 2c$ for the Tanaka–Webster connection $\hat{\nabla}$. Hereafter, we denote the Lorentzian Bianchi–Cartan–Vranceanu model space for $\hat{\nabla}$ by $\mathcal{M}^3_1(\hat{H})$.

Using (14), we get

$$\hat{\nabla}^3_T T = 3\varepsilon_3 \hat{\kappa}\hat{\kappa}' T + \varepsilon_2 (\hat{\kappa}'' - \varepsilon_2 \hat{\kappa}(\varepsilon_1 \hat{\kappa}^2 + \varepsilon_3 \hat{\tau}^2)) N + \varepsilon_1 (2\hat{\kappa}' \hat{\tau} + \hat{\kappa}\hat{\tau}') B.$$

From the curvature tensor (15) and Proposition 3, we have

$$\begin{aligned}
&\hat{R}(\hat{\kappa} N, T)T \\
=&\hat{\kappa}\hat{R}(N_1 e_1 + N_2 e_2 + N_3 e_3, T_1 e_1 + T_2 e_2 + T_3 e_3)(T_1 e_1 + T_2 e_2 + T_3 e_3) \\
=& -2c\varepsilon_2 \hat{\kappa} \left[B_3^2 N - N_3 B_3 B \right]
\end{aligned}$$

and

$$\hat{\nabla}^3_T T - \varepsilon_2 \hat{R}(\hat{\kappa}N, T)T = 3\varepsilon_3 \hat{\kappa}\hat{\kappa}' T + \left[\varepsilon_2 \hat{\kappa}'' - \hat{\kappa}(\varepsilon_1 \hat{\kappa}^2 + \varepsilon_3 \hat{\tau}^2 - 2cB_3^2)\right] N + \left[\varepsilon_1 (2\hat{\kappa}'\hat{\tau} + \hat{\kappa}\hat{\tau}') - 2c\hat{\kappa}N_3 B_3\right] B.$$

Hence, we have:

Proposition 8. *Let $\gamma : I \to M$ be a non-geodesic slant Frenet curve in the Lorentzian Sasakian space forms $\mathcal{M}^3_1(\hat{H})$ for the Tanaka–Webster connection $\hat{\nabla}$. Then, γ satisfies $\hat{\nabla}^3_T T - \hat{R}(\hat{\nabla}_T T, T)T = 0$ if and only if γ is a pseudo-Hermitian pseudo-helix with $\hat{\kappa}^2 - \hat{\tau}^2 = 2c\varepsilon_1 \eta(B)^2$.*

Using (14), we calculate

$$\hat{\nabla}^2_T T = \varepsilon_3 \hat{\kappa}^2 T + \varepsilon_2 \hat{\kappa} N - \varepsilon_1 \hat{\kappa}\hat{\tau} B,$$

and we get

$$g(\hat{\nabla}^2_T T, \varphi\gamma') = \varepsilon_2 \hat{\kappa}' \sqrt{\varepsilon_1 + a^2}.$$

Thus, we have:

Proposition 9. *A non-geodesic slant Frenet curve γ in a three-dimensional Sasakian Lorentzian manifold $\mathcal{M}^3_1(\hat{H})$ satisfies $g(\hat{\nabla}^2_T T, \varphi\gamma') = 0$ if and only if $\hat{\kappa}$ is a non-zero constant.*

Hence, we obtain:

Theorem 2. *Let $\gamma : I \to M$ be a non-geodesic slant Frenet curve in the Lorentzian Sasakian space forms $\mathcal{M}^3_1(\hat{H})$ for the Tanaka–Webster connection $\hat{\nabla}$. Then, γ satisfies the $\hat{\nabla}$-Jacobi equation for a $\hat{\nabla}$-geodesic vector field if and only if it is a pseudo-Hermitian pseudo-helix with $\hat{\kappa}^2 - \hat{\tau}^2 = 2c\varepsilon_1 \eta(B)^2$.*

Let γ be a slant Frenet curve in Lorentzian Sasakian space forms $\mathcal{M}^3_1(\hat{H})$ parametrized by arc-length. Then, the tangent vector field T has the form

$$T = \gamma' = \sqrt{\varepsilon_1 + a^2}\cos\beta u_1 + \sqrt{\varepsilon_1 + a^2}\sin\beta u_2 + a u_3, \tag{16}$$

where $a = constant$, $\beta = \beta(s)$. Using (10), since γ is a non-geodesic, we may assume that $\hat{\kappa} = \sqrt{\varepsilon_1 + a^2}(\beta' + cy\sqrt{\varepsilon_1 + a^2}\cos\beta - cx\sqrt{\varepsilon_1 + a^2}\sin\beta) > 0$ without loss of generality. Then, we get the normal vector field

$$N = -\sin\beta u_1 + \cos\beta u_2.$$

The binormal vector field $\varepsilon_3 B = T \wedge_L N = -a\cos\beta u_1 - a\sin\beta u_2 - \sqrt{\varepsilon_1 + a^2} u_3$. From the Lemma 1, we see that $\varepsilon_2 = 1$, so we have $\varepsilon_3 = -\varepsilon_1$. Hence, we have the binormal vector field

$$B = \varepsilon_1 (a\cos\beta u_1 + a\sin\beta u_2 + \sqrt{\varepsilon_1 + a^2} u_3).$$

Using the Frenet–Serret Equation (14), we have:

Lemma 2. *Let γ be a slant Frenet curve in Lorentzian Sasakian space forms $\mathcal{M}^3_1(\hat{H})$ parametrized by arc-length. Then, γ admits an orthonormal frame field $\{T, N, B\}$ along γ and*

$$\begin{aligned} \hat{\kappa} &= \sqrt{\varepsilon_1 + a^2}\{\beta' + c\sqrt{\varepsilon_1 + a^2}(y\cos\beta - x\sin\beta)\}, \\ \hat{\tau} &= -\varepsilon_1 a\{\beta' + c\sqrt{\varepsilon_1 + a^2}(y\cos\beta - x\sin\beta)\}. \end{aligned} \tag{17}$$

From this, we find that $\hat{\kappa}^2 - \hat{\tau}^2 = 2c\varepsilon_1 \eta(B)^2$ if and only if $\{\beta' + c\sqrt{\varepsilon_1 + a^2}(y\cos\beta - x\sin\beta)\}^2 = 2c(\varepsilon_1 + a^2)$. Hence, we have:

Corollary 2. *Let $\mathcal{M}_1^3(\hat{H})$ be a Lorentzian Sasakian space form with $c \leq 0$. Then, there does not exist a non-geodesic slant Frenet curve satisfying the $\hat{\nabla}$-Jacobi equations for $\hat{\nabla}$-geodesic vector fields.*

Since for $c > 0$, we get $\hat{H} = H - 3 = 2c > 0$, we now construct a non-geodesic slant Frenet curve γ satisfying (1) in Lorentzian space forms $M_1^3(\hat{H})$ for $\hat{H} = 2c > 0$.

Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in Lorentzian space forms $M_1^3(\hat{H})$ for $\hat{H} = 2c > 0$. Then, the tangent vector field T of γ is

$$T = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z},$$

using the relations:

$$\frac{\partial}{\partial x} = \frac{1}{\{1 + \frac{c}{2}(x(s)^2 + y(s)^2)\}}(u_1 + yu_3), \ \frac{\partial}{\partial y} = \frac{1}{\{1 + \frac{c}{2}(x(s)^2 + y(s)^2)\}}(u_2 - xu_3), \ \frac{\partial}{\partial z} = u_3.$$

If γ is a slant Frenet curve in Lorentzian space forms $M_1^3(\hat{H})$ for $\hat{H} = 2c > 0$, then from (16), the system of differential equations for γ is given by

$$\frac{dx}{ds}(s) = \sqrt{\varepsilon_1 + a^2}\cos\beta(s)\{1 + \frac{c}{2}(x(s)^2 + y(s)^2)\}, \tag{18}$$

$$\frac{dy}{ds}(s) = \sqrt{\varepsilon_1 + a^2}\sin\beta(s)\{1 + \frac{c}{2}(x(s)^2 + y(s)^2)\}, \tag{19}$$

$$\frac{dz}{ds}(s) = a + \sqrt{\varepsilon_1 + a^2}(x(s)\sin\beta(s) - y(s)\cos\beta(s)). \tag{20}$$

From the Theorem 2 and (17), we have:

Corollary 3. *Let $\gamma : I \to M_1^3(\hat{H})$ be a non-geodesic slant Frenet curve satisfying the $\hat{\nabla}$-Jacobi equations for the $\hat{\nabla}$-geodesic in Lorentzian space forms $M_1^3(\hat{H})$ for $\hat{H} = 2c > 0$. Then*

$$\beta' + c\sqrt{\varepsilon_1 + a^2}(y\cos\beta - x\sin\beta) = \pm\sqrt{2c(\varepsilon_1 + a^2)}. \tag{21}$$

Together with (21), we see that the Equation (20) becomes

$$\frac{dz}{ds} = \frac{1}{c}(\beta' \pm \sqrt{2c(\varepsilon_1 + a^2)}) + a.$$

Thus, we have

$$z(s) = \frac{1}{c}\beta(s) + \{a \pm \frac{1}{c}\sqrt{2c(\varepsilon_1 + a^2)}\}s + z_0,$$

where z_0 is a constant. We now compute the x and y coordinates. We put $h(s) := 1 + \frac{c}{2}(x(s)^2 + y(s)^2)$. Then, (18) and (19) become

$$\frac{dx}{ds} = \sqrt{\varepsilon_1 + a^2}\cos\beta(s)h(s), \ \frac{dy}{ds} = \sqrt{\varepsilon_1 + a^2}\sin\beta(s)h(s),$$

respectively. We note that the function $h(s)$ satisfies the following Ordinary Differential Equation:

$$\frac{d}{ds}\log|h(s)| = c\sqrt{\varepsilon_1 + a^2}(\cos\beta(s)x(s) + \sin\beta(s)y(s)).$$

Differentiating (21), we have

$$\frac{d^2}{ds^2}\beta(s) = \frac{d\beta}{ds}(s)\frac{d}{ds}\log|h(s)|.$$

First, if $d\beta/ds = 0$ for all s, then $(x(s), y(s))$ is a line in the orbit space. Hence, we have the following parametrization:

$$\begin{cases} x(s) = \sqrt{\varepsilon_1 + a^2}\cos\beta_0 \int h(s)ds, \\ y(s) = \sqrt{\varepsilon_1 + a^2}\sin\beta_0 \int h(s)ds, \\ z(s) = \{a \pm \frac{1}{c}\sqrt{2c(\varepsilon_1 + a^2)}\}s + z_0, \end{cases}$$

where $\int h(s)ds = \sqrt{-\frac{2}{c(\varepsilon_1+a^2)}} + \{p\exp(-\sqrt{-2c(\varepsilon_1 + a^2)}s) - \sqrt{-\frac{c(\varepsilon_1+a^2)}{8}}\}^{-1}, \quad p \in \mathbb{R}$, and $c < 0$. So, we conclude that β is not constant along γ.

Next, we assume that $\frac{d\beta}{ds}\mid_{s=s_0} \neq 0$ for some $s = s_0$. Then, we get $h(s) = r\frac{d\beta}{ds}s$, $r \in R$. Thus, we have

$$\begin{cases} x(s) = r\sqrt{\varepsilon_1 + a^2}\sin\beta(s) + x_0, \\ y(s) = -r\sqrt{\varepsilon_1 + a^2}\cos\beta(s) + y_0. \end{cases}$$

Since $c > 0$, the orbit space is the whole plane $\mathbb{R}^2(x, y)$. The projected curve $\bar{\gamma}(s)$ is a circle $(x - x_0)^2 + (y - y_0)^2 = r^2(\varepsilon_1 + a^2)$. We may assume that $\bar{\gamma}$ is a circle centered at $(0, 0)$. Then, the angle function β is given by

$$\beta(s) = \frac{1}{r}\left(\frac{c}{2}r^2(\varepsilon_1 + a^2) + 1\right)s + \beta_0.$$

Therefore, we obtain:

Theorem 3. *Let $\gamma : I \to M_1^3(\hat{H})$ be a non-geodesic slant Frenet curve satisfying the $\hat{\nabla}$-Jacobi equations for the $\hat{\nabla}$-geodesic in Lorentzian space forms $M_1^3(\hat{H})$ for $\hat{H} = 2c > 0$. Then, its parametric equations are given by*

$$\begin{cases} x(s) = r\sqrt{\varepsilon_1 + a^2}\sin(\frac{1}{r}\left(\frac{c}{2}r^2(\varepsilon_1 + a^2) + 1\right)s + \beta_0) + x_0, \\ y(s) = -r\sqrt{\varepsilon_1 + a^2}\cos(\frac{1}{r}\left(\frac{c}{2}r^2(\varepsilon_1 + a^2) + 1\right)s + \beta_0) + y_0, \\ z(s) = [a + \frac{1}{c}\{\frac{1}{r}\left(\frac{c}{2}r^2(\varepsilon_1 + a^2) + 1\right) \pm \sqrt{2c(\varepsilon_1 + a^2)}\}]s + z_0, \end{cases}$$

where $r \in R$ and β_0, x_0, y_0, z_0 are constants.

If γ is a timelike curve, then $\varepsilon = -1$ and $a = \cosh\alpha_0$. If γ is a spacelike curve, then $\varepsilon = 1$ and $a = \sinh\alpha_0$. In particular, if $\varepsilon = 1$ and $\eta(\gamma') = a = 0$, then we have:

Example 1 (Legendre curves). *Let $\gamma : I \to M_1^3(\hat{H})$ be a non-geodesic Legendre Frenet curve satisfying the $\hat{\nabla}$-Jacobi equations for the $\hat{\nabla}$-geodesic in Lorentzian space forms $M_1^3(\hat{H})$ for $\hat{H} = 2c > 0$. Then, its parametric equations are given by*

$$\begin{cases} x(s) = r\sin(\frac{1}{r}\left(\frac{c}{2}r^2 + 1\right)s + \beta_0) + x_0, \\ y(s) = -r\cos(\frac{1}{r}\left(\frac{c}{2}r^2 + 1\right)s + \beta_0) + y_0, \\ z(s) = \frac{1}{c}\{\frac{1}{r}\left(\frac{c}{2}r^2 + 1\right) \pm \sqrt{2c}\}s + z_0, \end{cases}$$

where $r \in R$ and β_0, x_0, y_0, z_0 are constants.

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