

# Extending the Applicability of Stirling's Method

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**Abstract:** Stirling's method is considered as an alternative to Newton's method when the latter fails to converge to a solution of a nonlinear equation. Both methods converge quadratically under similar convergence criteria and require the same computational effort. However, Stirling's method has shortcomings too. In particular, contractive conditions are assumed to show convergence. However, these conditions limit its applicability. The novelty of our paper lies in the fact that our convergence criteria do not require contractive conditions. Hence, we extend its applicability of Stirling's method. Numerical examples illustrate our new findings.

**Keywords:** Stirling's method; Newton's method; convergence; Fréchet derivative; banach space

## 1. Introduction

In this work we deal with finding a fixed point  $x^*$  of the equation

$$x = F(x), \quad (1)$$

where  $F$  is a Fréchet-differentiable operator defined on a convex subset  $D$  of a Banach space  $X$  with values into itself. By  $I$  we denote the identity linear operator in  $L(X, X)$ . The symbol  $L(X, X)$  stands for the space of bounded linear operators from  $X$  into  $X$ .

Many applications from different areas, including education, reduce to dealing with Equation (1) utilizing mathematical modelling [1–24]. However, the solution  $x^*$  is found in closed form only in rare cases. This problem leads to the usage of methods that are iterative in nature.

We study Stirling's method given for all  $n = 0, 1, 2, \dots$  by

$$x_{n+1} = x_n - (I - F'(F(x_n)))^{-1}(x_n - F(x_n)), \quad (2)$$

where  $x_0 \in D$ . Further we will introduce an operator  $\Gamma(x) \in L(X, X)$  such that  $\Gamma(x) = (I - F'(F(x)))^{-1}$  with  $x \in D$ , and denote  $\Gamma_0 = \Gamma(x_0)$  for use in later Sections.

This method converges quadratically as Newton's method does, and also requires the same computational effort (see details in [1,22]). It is considered to be a useful alternative in cases where Newton's method fails to converge (see such examples in [22]). However, the usage of Stirling's method has a drawback, since the convergence criteria require contractions. We have detected some other problems listed in Remarks 3 and 4. These drawbacks limit the applicability of Stirling's method.

In order to extend its applicability, we do not use contractive conditions in our semi-local as well as the local convergence results.

The rest of the work is structured as follows. Section 2 includes the semi-local convergence analysis. Section 3 contains the local analysis. The numerical results are given in Section 4.

### 2. Semi-Local Convergence Analysis

Let  $L_0 > 0, L > 0$  and  $\gamma \geq 0$ . Consider a real sequence  $\{t_n\}$  as

$$t_0 = 0, t_1 = \gamma, t_{n+2} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2}{2(1 - L_0 t_{n+1})}. \tag{3}$$

Next, we study the convergence of sequence  $\{t_n\}$  by developing relevant lemmas and theorems.

**Lemma 1.** *Suppose that*

$$h = L_1 \gamma < \frac{1}{2}, \tag{4}$$

where

$$L_1 = \frac{1}{8}(L + 4L_0 + \sqrt{L^2 + 8L_0L}).$$

Then, sequence  $\{t_n\}$  generated for  $t_0 = 0$  by (4) is increasing, converges to its unique least upper bound  $t^*$ , so that

$$d_1 \leq t^* \leq d_2, \tag{5}$$

where

$$d_1 = \frac{1 - \exp[-\frac{L_0 \gamma}{(1-L_0 \gamma)(1-\delta L_0 \gamma)}]}{L_0}$$

$$d_2 = \frac{1 - \exp[\frac{2L_0 \gamma}{2-L_0 \gamma} + \frac{2\delta_1}{2-\delta_1}]}{L_0},$$

$$\delta = \frac{L}{2L_0(1 - L_0 \gamma)^2}$$

and

$$\delta_1 = \frac{L}{2L_0} \left(\frac{\delta}{1 - \delta}\right)^2.$$

**Proof.** It is convenient to first simplify sequence  $\{t_n\}$ . Define sequence  $\{\alpha_n\}$  by  $\alpha_n = 1 - L_0 t_n$ . Then, by (4) we can write  $\alpha_0 = 1, \alpha_1 = 1 - L_0 \gamma, \alpha_{n+1} = \alpha_n - \frac{L(\alpha_n - \alpha_{n-1})^2}{2L_0 \alpha_n}$ . Moreover, define sequence  $\{\theta_n\}$  by  $\theta_n = 1 - \frac{\alpha_n}{\alpha_{n-1}}$ . Then, we can write  $\theta_1 = L_0 \gamma, \theta_{n+1} = \frac{L}{2L_0} \left(\frac{\theta_n}{1 - \theta_n}\right)^2$ . We have by (4) that  $\delta \theta_1 < 1$  and  $0 < \theta_2 < \theta_1$ . Suppose that  $0 < \theta_k < \theta_{k-1}$  and  $\delta \theta_k < 1$ . Then, we get in turn that

$$\theta_{k+1} = \frac{L}{2L_0} \left(\frac{\theta_k}{1 - \theta_k}\right)^2 < \delta \theta_k^2 < \theta_k \tag{6}$$

and

$$\delta \theta_{k+1} < \delta \theta_k < 1. \tag{7}$$

Hence,  $\{\beta_n\}$  is a decreasing sequence, so  $\alpha_n = (1 - \beta_n)\alpha_{n-1}$  and  $t_n = \frac{1 - \alpha_n}{L_0}$  are also decreasing sequences. In particular,

$$\alpha_n = (1 - \beta_n)\alpha_{n-1} = \dots = (1 - \beta_n) \dots (1 - \beta_1)\alpha_0 = (1 - \beta_n) \dots (1 - \beta_0).$$

From  $0 < \beta_1 = L_0\gamma < 1$ , we get  $0 < \alpha_n < 1$ , so  $t_n = \frac{1-\alpha_n}{\gamma} < \frac{1}{\gamma}$ . That is sequence  $\{t_n\}$  is increasing, bounded from above by  $\frac{1}{L_0}$ , so it converges to  $t^*$ .

Next, we show (4). We can write

$$\alpha^* = \lim_{n \rightarrow \infty} \alpha_n = \prod_{n=1}^{\infty} (1 - \beta_n),$$

or

$$\log \frac{1}{\alpha^*} = \sum_{n=1}^{\infty} \log \frac{1}{1 - \beta_n}.$$

Using the estimate

$$2 \frac{t-1}{t+1} \leq \log t \leq \frac{t^2-1}{2t} \text{ for } t > 1,$$

we get first an upper bound for  $\log \frac{1}{\alpha^*}$  by (5) and (6) and the inequality  $2^n \geq n + 1$  for  $n = 0, 1, 2, \dots$ :

$$\begin{aligned} \log \frac{1}{\alpha^*} &\leq \sum_{n=1}^{\infty} \frac{\beta_n(2 - \beta_n)}{2(1 - \beta_n)} \leq \frac{1}{1 - \beta_1} \sum_{n=0}^{\infty} \beta_{n+1} \\ &\leq \frac{1}{\delta(1 - \beta_1)} \sum_{n=1}^{\infty} (\delta\theta_1)^{2^n} \leq \frac{1}{\delta(1 - \beta_1)} \sum_{n=1}^{\infty} (\delta\beta_1)^n \\ &= \frac{\beta_1}{(1 - \beta_1)(1 - \delta\beta_1)}, \end{aligned}$$

which together with  $t^* = \frac{1-\alpha^*}{L_0}$  imply  $t^* \leq d_2$ . The lower bound in (4) is obtained similarly from the estimate:

$$\log \frac{1}{\alpha^*} \geq 2 \sum_{n=1}^{\infty} \frac{\alpha_n}{2 - \alpha_n} > \frac{2\alpha_1}{2 - \alpha_1} + \frac{2\alpha_2}{2 - \alpha_2}.$$

□

**Lemma 2.** Suppose that

$$h = \frac{1}{2}. \tag{8}$$

Then, sequence  $\{t_n\}$  is increasingly converging to  $\frac{1}{L_0}$ .

**Proof.** We have  $\alpha_n = (1 - L_0R)^n, \beta_n = L_0\gamma$  and  $t_n = \frac{1-(1-L_0\gamma)^n}{L_0}$ . Then, by (8), we get  $0 \leq L_0\gamma < 1$ . □

In what follows the set denoted by  $U(x, a)$  is a ball with center  $x \in X$  and of radius  $a > 0$ .

To simplify, the notation, by  $\| \cdot \|$  in this work, we denote the operator norm or the norm on the Banach space. The semi-local convergence analysis is based on the conditions (C):

(C<sub>1</sub>)  $F : D \subset X \rightarrow X$  is a Fréchet differentiable operator and there exist  $x_0 \in D, c > 0, \gamma \geq 0$  such that  $\Gamma_0 = (I - F'(F(x_0)))^{-1} \in L(X, X)$  with

$$\|I - F'(F(x_0))\| \leq c$$

and

$$\|\Gamma_0(x_0 - F(x_0))\| \leq \gamma.$$

(C<sub>2</sub>) There exist  $a_0 \in [0, 1), b_0 > 0$  such that for each  $x \in D$

$$\|F(x) - F(x_0)\| \leq a_0\|x - x_0\|$$

and

$$\|\Gamma_0(F'(F(x)) - F'(F(x_0)))\| \leq b_0 \|F(x) - F(x_0)\|.$$

(C<sub>3</sub>) Let  $r_0 = \frac{1}{a_0 b_0}$  and  $D_0 = D \cap U(x_0, r_0)$ . There exist  $b > 0, b_1 > 0$  such that for each  $x, y \in D_0$

$$\|\Gamma_0(F'(x) - F'(y))\| \leq b \|x - y\|$$

and

$$\|F'(F(x)) - F'(F(x_0))\| \leq b_1 \|F(x) - F(x_0)\|.$$

(C<sub>4</sub>) Hypotheses of Lemmas 1 and 2 hold with

$$L = 2b(c + \frac{b_1}{b_0} + \frac{1}{2})$$

and

$$L_0 = a_0 b_0.$$

(C<sub>5</sub>)

$$\frac{\|F(x_0) - x_0\|}{1 - a_0} \leq t^*.$$

(C<sub>6</sub>) The ball  $\bar{U}(x_0, t^*)$  is constructed such that

$$\bar{U}(x_0, t^*) \subseteq D.$$

We suppose from now on that the conditions (C) hold.

Next, the semi-local convergence result is given for Stirling’s method (2).

**Theorem 1.** Under conditions (C), sequence  $\{x_n\}$  generated by Stirling’s method (2) is well defined, remains in  $U(x_0, t^*)$  for each  $n = 0, 1, 2, \dots$  and converges to  $x^* \in \bar{U}(x_0, t^*)$  which satisfies  $x^* = F(x^*)$  with Q-order of convergence 2. Moreover, the following estimates hold

$$\|x_n - x^*\| \leq t^* - t_n,$$

and  $x^*$  is the only fixed point of  $F$  in  $U(x_0, t^{**})$ , with

$$t^{**} = \frac{2}{b} - (2a_0 + 1)t^*.$$

**Proof.** Let  $x \in \bar{U}(x_0, t^*)$ . We get by (C<sub>2</sub>) and (C<sub>5</sub>) that

$$\|F(x) - x_0\| \leq \|F(x) - F(x_0)\| + \|F(x_0) - x_0\| \leq a_0 \|x - x_0\| + \|F(x_0) - x_0\| \leq a_0 t^* + \|F(x_0) - x_0\| \leq t^*,$$

so  $F(x) \in \bar{U}(x_0, t^*)$ . Using (C<sub>2</sub>) and the Lemmas 1 and 2, we have in turn that

$$\begin{aligned} \|\Gamma_0(\Gamma(x) - \Gamma_0)\| &= \|\Gamma_0((F'(F(x)) - F'(F(x_0))))\| \\ &\leq b_0 a_0 \|x - x_0\| = L_0 \|x - x_0\| \leq L_0 t^* < 1. \end{aligned} \tag{9}$$

By the Lemma of Banach on invertible operators [21] (Perturbation Lemma 2.3.2, p. 45)  $\Gamma(x)^{-1} \in L(\mathcal{X}, X)$ , and

$$\|\Gamma(x)(I - F'(F(x_0)))\| \leq \frac{1}{1 - L_0 \|x - x_0\|}. \tag{10}$$

Using Stirling’s method (2):

$$\begin{aligned}
 x_{k+1} - F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F(x_{k+1}) + F(x_k) \\
 &= F'(y_k)(x_{k+1} - x_k) - (F(x_{k+1}) - F(x_k)) \\
 &= \int_0^1 [F'(y_k) - F'(x_k + \theta(x_{k+1} - x_k))](x_{k+1} - x_k)d\theta.
 \end{aligned}
 \tag{11}$$

Then, in view of (C<sub>2</sub>), (C<sub>3</sub>) and Equation (11), we obtain in turn that

$$\begin{aligned}
 &||\Gamma_0(x_{k+1} - F(x_{k+1}))|| \\
 \leq &b \int_0^1 ||y_k - x_k - \theta(x_{k+1} - x_k)|| |x_{k+1} - x_k| d\theta \\
 \leq &b[||y_k - x_k|| + \frac{1}{2}||x_{k+1} - x_k||] |x_{k+1} - x_k| \\
 \leq &b[||I - F'(F(x_0))|| + ||F'(y_k) - F'(F(x_0))||] + ||x_{k+1} - x_k|| + \frac{1}{2}||x_{k+1} - x_k|| |x_{k+1} - x_k| \\
 \leq &b(c + \frac{b_1 a_0}{a_0 b_0} + \frac{1}{2}) ||x_{k+1} - x_k||^2 \\
 = &\frac{L}{2} ||x_{k+1} - x_k||^2.
 \end{aligned}
 \tag{12}$$

Next, we can connect the preceding estimates on sequence {x<sub>k</sub>} with {t<sub>k</sub>}. Indeed, we get by (C<sub>1</sub>) and Equation (3) that

$$||x_1 - x_0|| = ||\Gamma_0(x_0 - F(x_0))|| \leq \gamma = t_1 = t_1 - t_0.$$

By induction, Equations (3), (4), (10) and (12), we have in turn that

$$\begin{aligned}
 ||x_{k+1} - x_k|| &= ||\Gamma_k(x_k - F(x_k))|| \leq ||\Gamma_k(I - F'(F(x_0)))|| ||\Gamma_0(x_k - F(x_k))|| \\
 &\leq \frac{L(t_k - t_{k-1})^2}{2(1 - L_0 t_k)} = t_{k+1} - t_k.
 \end{aligned}
 \tag{13}$$

Hence, {t<sub>k</sub>} defined by Equation (3) is a majorizing sequence for {x<sub>k</sub>}. By Lemmas 1 and 2, sequence {t<sub>k</sub>} is complete as convergent to t\*. It then follows by Equation (13) that sequence {x<sub>k</sub>} is also complete so it converges to some x\* ∈ Ū(x<sub>0</sub>, t\*). By the estimate (see (12))

$$||\Gamma_0(x_{k+1} - F(x_{k+1}))|| \leq \frac{L}{2} ||x_{k+1} - x_k||^2 \leq \frac{L}{2} (t_{k+1} - t_k)^2, \tag{14}$$

we deduce that x\* = F(x\*) by letting k → ∞. Estimate ||x<sub>n</sub> - x\*|| ≤ t\* - t<sub>n</sub> follows from Equation (13) and for λ =  $\frac{L}{2(1-L_0 t^*)}$ , we get that

$$\begin{aligned}
 ||x_{k+1} - x_k|| &\leq \frac{L}{2(1 - L_0 t_k)} ||x_k - x_{k-1}||^2 \\
 &\leq \lambda ||x_k - x_{k-1}||^2,
 \end{aligned}$$

which implies that the  $Q$ -order convergence of Stirling’s method (2) is two. Furthermore, to show the uniqueness part, let  $y^* \in U(x_0, t^{**})$  with  $F(y^*) = y^*$ . Define the operator  $Q$  by  $Q = - \int_0^1 \Gamma_0 F'(x^* + \theta(y^* - x^*))d\theta$ . In view of (C<sub>2</sub>) and (C<sub>3</sub>), we obtain in turn that

$$\begin{aligned} \|I - (\Gamma_0 - Q)\| &= \left\| \int_0^1 \Gamma_0 [F'(x^* + \theta(y^* - x^*)) - F'(F(x_0))]d\theta \right\| & (15) \\ &\leq b \int_0^1 \|x^* + \theta(y^* - x^*) - F(x_0)\|d\theta \\ &\leq b[\|F(x^*) - F(x_0)\| + \frac{1}{2}\|x^* - x_0\| + \frac{1}{2}\|y^* - x_0\|] \\ &\leq b[(a_0 + \frac{1}{2})t^* + \frac{1}{2}t^{**}] < 1. \end{aligned}$$

Then, by (15)  $(\Gamma_0 - Q)^{-1} \in L(X, X)$ . Finally, we obtain  $y^* = x^*$  using the identity

$$0 = \Gamma_0(y^* - F(y^*) - x^* + F(x^*)) = (\Gamma_0 - Q)(y^* - x^*).$$

□

**Remark 1.**

(a) The Stirling’s method usual conditions corresponding to (C<sub>2</sub>) (first condition) are given by [22]:

$$(C_2)' \quad \|F'(x)\| \leq a \text{ for each } x \in D \text{ and } a \in [0, 1).$$

That is, operator  $F$  must be a contraction on  $D$ . Moreover, the convergence of Stirling’s method was shown in [22] under (C<sub>2</sub>),  $D_0 = D$  and  $a \in (0, \frac{1}{3}]$ . However, in the present study no such assumption is made. Hence, the applicability of Stirling’s method (2) is extended. Notice also that we can have  $a_0 \leq a, b_0 \leq b$  and  $c$  can be chosen as  $b = cb_1$ .

- (b) Estimate (4) is similar to the sufficient convergence Kantorovich-type criteria for the semi-local convergence of Newton’s method given by us in [4]. However, the constants  $\bar{b}_0$  and  $\bar{b}$  are the center-Lipschitz and Lipschitz constants for operator  $F$  (see also part (e)).
- (c) If set  $D_0$  is switched by  $D_1 = D \cap U(x_1, r_0 - \|x_0 - F(x_0)\|)$ , since  $D_1 \subseteq D$  and the iterates remain in  $D_1$  the results can be improved even further. The corresponding constants to  $b$  and  $b_1$  will be at least as small.
- (d) In view of the proof of Theorem 1, scalar sequence  $\{s_n\}$  defined by

$$s_0 = 0, s_1 = R, s_{n+1} = s_n + \frac{k_n(s_n - s_{n-1})^2}{1 - L_0 s_n},$$

is also a majorizing sequence for Stirling’s method (2), where

$$k_n = ab(c + b_1 a_0 s_n + \frac{1}{2}) < L$$

$$s_n \leq t_n,$$

$$s_{n+1} - s_n \leq t_{n+1} - t_n$$

and

$$s^* = \lim_{n \rightarrow \infty} s_n \leq t^*.$$

(e) Newton’s method for Equation (1) is given for all  $n = 0, 1, 2, \dots$  by

$$y_{n+1} = y_n - (I - F'(y_n))^{-1}(y_n - F(y_n)).$$

Consider, items  $\bar{c}, \bar{\gamma}, \bar{L}_0, \bar{L}, \bar{L}_1, \bar{\Gamma}_0, \bar{b}_0, \bar{b}, \bar{b}_1, \bar{r}_0, \bar{D}_0$  and  $\bar{h}$ , corresponding to  $c, \gamma, L_0, L, L_1, \Gamma_0, b_0, b, b_1, r_0, D_0$  and  $h$  respectively as

$$\begin{aligned} \|I - F'(x_0)\| &\leq \bar{c}, \\ \|\bar{\Gamma}_0(x_0 - F(x_0))\| &\leq \bar{\gamma}, \\ \|\bar{\Gamma}_0(F'(x) - F'(x_0))\| &\leq \bar{b}_0\|x - x_0\|, \\ \|\bar{\Gamma}_0(F'(x) - F'(y))\| &\leq \bar{b}\|x - y\|, \\ \bar{L} &= 2\bar{b}(\bar{c} + \frac{\bar{b}_1}{\bar{b}_0} + \frac{1}{2}), \bar{b}_1 = \bar{b}_0, \\ \bar{L}_0 &= \bar{a}_0\bar{b}_0, \\ \bar{r}_0 &= \frac{1}{\bar{a}_0\bar{b}_0}, \\ \bar{D}_0 &= D \cap U(x_0, \bar{r}_0), \end{aligned}$$

and

$$\bar{h} = \bar{L}_1, \bar{\gamma} \leq \frac{1}{2},$$

where

$$\bar{L}_1 = \frac{1}{8}(\bar{L} + 4\bar{L}_0 + \sqrt{\bar{L}^2 + 8\bar{L}_0\bar{L}}).$$

The scalar sequence  $\bar{t}_n$  is defined as

$$\bar{t}_0 = 0, \bar{t}_1 = \bar{\gamma}, \bar{t}_{n+2} = \bar{t}_{n+1} + \frac{\bar{L}(\bar{t}_{n+1} - \bar{t}_n)^2}{2(1 - \bar{L}_0\bar{t}_{n+1})}$$

Then, Stirling's method sufficient convergence criteria, error bounds and information on the uniqueness of the solution are better than Newton's method when the "bar" constants and sets are smaller than the non bar constants. Similar favorable comparison can be made in the local convergence case that follows.

### 3. Local Convergence

The conditions  $(\mathcal{H})$  are used in the local convergence analysis of Stirling's method (2):

$(\mathcal{H}_1)$   $F : D \subset X \rightarrow X$  is a Fréchet differentiable operator, and there exists  $x^* \in D$  such that  $\Gamma_* = (I - F'(x^*))^{-1} \in L(X, X)$  and  $F(x^*) = x^*$ .

$(\mathcal{H}_2)$  There exist  $\mu \in (0, 1), \zeta_0 > 0$  such that for each  $x \in D$

$$\|F(x) - F(x^*)\| \leq \mu\|x - x^*\|$$

and

$$\|\Gamma_*(F'(F(x)) - F'(F(x^*)))\| \leq \zeta_0\|F(x) - F(x^*)\|.$$

$(\mathcal{H}_3)$  Let  $D_0^* = D \cap U(x^*, R_0), R_0 = \frac{1}{\zeta_0\mu}$ . There exists  $\zeta > 0$  such that for each  $x, y \in D_0^*$

$$\|\Gamma_*(F'(x) - F'(y))\| \leq \zeta\|x - y\|.$$

$(\mathcal{H}_4)$  The ball  $\bar{U}(x^*, R)$  is constructed such that  $\bar{U}(x^*, R) \subseteq D$ , where

$$R = \frac{1}{(\mu + \frac{1}{2})\zeta + \mu\zeta_0}.$$

**Theorem 2.** Suppose that conditions  $(\mathcal{H})$  hold. Then, sequence  $\{x_n\}$  generated for  $x_0 \in U(x^*, R) - \{x^*\}$  by Stirling’s method (2) is well defined in  $U(x^*, R)$ , remains in  $U(x^*, R)$  for each  $n = 0, 1, 2, \dots$  and converges to  $x^* \in \bar{U}(x^*, R)$ . Moreover, the following inequality holds

$$\|x_{n+1} - x^*\| \leq \frac{\xi(\mu + \frac{1}{2})\|x_n - x^*\|^2}{1 - \mu\tilde{\xi}_0\|x_n - x^*\|}. \tag{16}$$

Furthermore, if  $R_1 = \frac{2}{\xi}$ ,  $x^*$  is the only fixed point of  $F$  on  $U(x^*, R_1)$ .

**Proof.** We shall show using mathematical induction that sequence  $\{x_n\}$  is well defined, remains in  $U(x^*, R)$  and converges to  $x^*$  so that (16) is satisfied. We have by  $(\mathcal{H}_1)$  and  $(\mathcal{H}_2)$  for  $x \in U(x^*, R)$  that

$$\|F(x) - x^*\| = \|F(x) - F(x^*)\| \leq \mu\|x - x^*\| \leq R,$$

so  $F(x) \in U(x^*, R)$ . Then by  $(\mathcal{H}_2)$

$$\begin{aligned} \|\Gamma_*(I - F'(F(x))) - \Gamma_*\| &= \|\Gamma_*(F'(F(x)) - F'(F(x^*)))\| \\ &\leq \tilde{\xi}_0\|F(x) - F(x^*)\| \leq \xi_0\mu\|x - x^*\| \leq \xi_0\mu R < 1. \end{aligned} \tag{17}$$

Hence,  $\Gamma(x) \in L(X, X)$  and

$$\|\Gamma(x)(I - F(F(x^*)))\| \leq \frac{1}{1 - \mu_0\tilde{\xi}_0\|x - x^*\|}. \tag{18}$$

In particular, (18) holds for  $x = x_0$ , which shows that  $x_1$  is well defined by Stirling’s method for  $n = 0$ . We can write by  $(\mathcal{H}_1)$  that

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - (I - F'(F(x_0)))^{-1}(x_0 - F(x_0)) \\ &= (I - F'(F(x_0)))^{-1}[F(x_0) - F(x^*) - F'(F(x_0))(x_0 - x^*)] \\ &= (I - F'(F(x_0)))^{-1}\left[\int_0^1 (F'(x^* + \theta(x_0 - x^*)) - F'(F(x_0)))(x_0 - x^*)d\theta\right]. \end{aligned} \tag{19}$$

We get in turn by  $(\mathcal{H}_2)$  and  $(\mathcal{H}_3)$

$$\begin{aligned} &\|\Gamma_* \int_0^1 (F'(x^* + \theta(x_0 - x^*)) - F'(F(x_0)))(x_0 - x^*)d\theta\| \\ &\leq \xi \int_0^1 \|x^* + \theta(x_0 - x^*) - F(x_0)\|\|x_0 - x^*\|d\theta \\ &\leq \xi[\|F(x^*) - F(x_0)\| + \theta\|x_0 - x^*\|]d\theta \\ &\leq \xi(\mu + \frac{1}{2})\|x_0 - x^*\| \end{aligned} \tag{20}$$

Then, by (18)–(20), we get that also

$$\begin{aligned} \|x_1 - x^*\| &\leq \|(I - F'(F(x_0)))^{-1}\Gamma_*\| \\ &\leq \frac{\xi(\mu + \frac{1}{2})\|x_0 - x^*\|^2}{1 - \mu\tilde{\xi}_0\|x_0 - x^*\|} \leq \|x_0 - x^*\| < R, \end{aligned}$$

so (16) holds for  $n = 0$  and  $x_1 \in U(x^*, R)$ . Switch  $x_0$  by  $x_k$  in the preceding estimates, we arrive at (16). In view of the estimate  $\|x_{k+1} - x^*\| < \|x_k - x^*\| < R$ , we conclude that  $\lim_{k \rightarrow \infty} x_k = x^*$  and  $x_{k+1} \in U(x^*, R)$ . Let  $x_0 = x^*$  in (15) to show the uniqueness part.  $\square$

**Remark 2.** The local results in the literature use  $(C_2)'$  and  $D_0^* = D$ . But  $(H_2)$  is weaker than  $(C_2)'$ . Hence, we extend the applicability of Stirling’s method (2) in the local case too.

#### 4. Numerical Example with Concluding Remarks

In the next example, we compare Stirling’s method with Newton’s method.

**Example 1.** Let  $D = X = \mathbb{R}$ . Consider function  $F$  on  $D$  as

$$F(x) = \begin{cases} -\frac{1}{3}x, & x \leq 3 \\ \frac{1}{3}x^2 - \frac{7}{3}x + 3, & 3 \leq x \leq 4 \\ \frac{1}{3}(x - 7), & x \geq 4. \end{cases}$$

Clearly, the quadratic polynomial joins smoothly with the linear parts.

- (I) Semilocal case (i). If we choose  $x_0 = 3$ , we see that  $x_1 = y_1 = x^* = 0$ . Moreover, the semi-local convergence criteria of Theorem 1 are satisfied (with  $\gamma = 0, a_0 = \frac{1}{3}$  and  $c = \frac{4}{3}$ ).
- (II) Local convergence criteria of Theorem 2 (with  $\mu = \frac{1}{3}$ , since the derivative of the quadratic polynomial satisfies  $\frac{1}{3}|2x - 7| \leq \frac{1}{3}$  for all  $x \in [3, 4]$ ).
- (III) In Tables 1 and 2 we present some cases in which Stirling’s method stands better than Newton’s one.

**Table 1.** Iteration of Newton’s and Stirling’s method with different starting points.

Iteration	Newton’s Method	Stirling’s Method
0	3.4975	3.4975
1	−646.501	0.0618766
2	$-1.13687 \times 10^{-13}$	0

**Table 2.** Iteration of Newton’s and Stirling’s method with different starting points.

Iteration	Newton’s Method	Stirling’s Method
0	3.5	3.5
1	$\infty$	0.0625
2	−	0

In the current study, we have successfully demonstrated our claims on Stirling’s method by focusing on very classic problems, but in the future we will consider studying other complex problems such as solving symmetric ordinary differential equations with a more favorable theory.

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