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Twisted Weyl Groups of Compact Lie Groups and Nonabelian Cohomology

Ming Liu [†] and Xia Zhang ^{*,†}

School of Mathematical Sciences, Tiangong University, Tianjin 300387, China; liuming@tiangong.edu.cn

* Correspondence: zhangxia@tiangong.edu.cn

† These authors contributed equally to this work.

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Abstract: Given a compact connected Lie group G with an S^1 -module structure and a maximal compact torus T of G^{S^1} , we define twisted Weyl group $W(G, S^1, T)$ of G associated to S^1 -module and show that two elements of T are δ -conjugate if and only if they are in one $W(G, S^1, T)$ -orbit. Based on this, we prove that the natural map $W(G, S^1, T) \setminus H^1(S^1, T) \rightarrow H^1(S^1, G)$ is bijective, which reduces the calculation for the nonabelian cohomology $H^1(S^1, G)$.

Keywords: twisted Weyl groups; compact Lie groups; δ -conjugate; nonabelian cohomology

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1. Introduction

Let G be a compact connected Lie group and σ be an automorphism on G [1]. The twisted conjugate action τ^σ [2] of G on itself associated to σ is defined as

$$\tau^\sigma : G \times G \rightarrow G, (g, h) \mapsto gh\sigma(g)^{-1} \triangleq \tau_g^\sigma(h).$$

Two elements g_1 and g_2 of G are called σ -conjugate if they are in one twisted orbit of G associated to σ , i.e., there exists an element $g_0 \in G$ such that

$$g_2 = \tau_{g_0}^\sigma(g_1).$$

Based on this, given a compact connected Lie group G with an additional $\mathbb{Z}_n$ -module structure and T a maximal compact torus of $G_0^{\mathbb{Z}_n}$, An J. [3] defined the twisted Weyl group $W(G, \mathbb{Z}_n, T)$ of G associated to $\mathbb{Z}_n$ -module and reduced the calculation of $H^1(\mathbb{Z}_n, G)$ to the action of $W(G, \mathbb{Z}_n, T)$ on T , where $H^1(\mathbb{Z}_n, G)$ is the first nonabelian cohomology of $\mathbb{Z}_n$ with coefficients in G [2].

Motivated by the underlying work, in this paper we consider the case of G with an nonabelian S^1 -module structure, where S^1 is the one-dimensional torus. Picking a topological generator δ of S^1 , which can be regarded as an automorphism of G [4], we first define the twisted Weyl group $W(G, \delta, T)$ of G associated to δ and then define the twisted Weyl group $W(G, S^1, T)$ associated to S^1 -module, where T is a maximal compact torus in G^{S^1} . Closed by the definition, we study the action of $W(G, S^1, T)$ on T and show that two elements in T are δ -conjugate if and only if they are in one $W(G, S^1, T)$ -orbit. Furthermore, we prove that $W(G, S^1, T)$ is a finite group, which is the same as the case of classical Weyl groups.

Based on the underlying properties of $W(G, S^1, T)$, we study the action of $W(G, S^1, T)$ on the first cohomology $H^1(S^1, T)$ of the compact Lie group S^1 with coefficients in T [5], and prove that the natural map

$$W(G, S^1, T) \setminus H^1(S^1, T) \rightarrow H^1(S^1, G)$$

induced by the natural embedding $T \hookrightarrow G$ is a bijection. Using the result, one can reduce the calculation for $H^1(S^1, G)$ to the calculation for the orbit space $W(G, S^1, T) \setminus H^1(S^1, T)$. Indeed, using this formula, one can simplify some calculations in dynamical systems theory, especially in fractional dynamics and fractional-wavelet analysis of some positive definite distributions [6–8].

In Section 2, we exhibit the definition of the twisted Weyl group $W(G, S^1, T)$ of G associated to S^1 -module together with its some properties. In Section 3, we construct a one-to-one correspondence between the orbit space of the action of $W(G, S^1, T)$ on $H^1(S^1, T)$ and $H^1(S^1, G)$. In Section 4, we discuss some new developments in the field as well as its relations with amenability of groups [9–12]. For basic knowledge on compact Lie groups and twisted conjugate actions, one can refer [2,13,14]; for the nonabelian cohomology of Lie groups, one can refer [5,15–17].

2. Twisted Weyl Groups of Compact Lie Groups

Let G be a compact connected Lie group with an S^1 -module structure and δ be a topological generator of S^1 . From the definition of S^1 -module structure, δ can be regarded as an automorphism on G . Denote by τ^δ the twisted conjugate action of G associated to δ .

First of all, we define the twisted Weyl group of G associated to δ .

Definition 1. Define

$$N_{\delta,G}(T) := \{g \in G \mid \tau_g^\delta(T) = T\},$$

$$Z_{\delta,G}(T) := \{g \in G \mid \tau_g^\delta(t) = t, \forall t \in T\},$$

where T is a maximal compact torus of $G^\delta := \{g \in G \mid \delta(g) = g\}$.

It is easy to know that $N_{\delta,G}(T)$ and $Z_{\delta,G}(T)$ are both closed subgroups of G and that $Z_{\delta,G}(T)$ is a normal subgroup of $N_{\delta,G}(T)$.

Definition 2. Define

$$W(G, \delta, T) := N_{\delta,G}(T) / Z_{\delta,G}(T).$$

$W(G, \delta, T)$ is called the twisted Weyl group of G associated to δ .

As an abstract group, the group operations on $W(G, \delta, T)$ are defined as following. For all $w_1, w_2 \in W(G, \delta, T)$,

$$w_1 \cdot w_2 := [g_1 g_2],$$

where $g_1 \in w_1, g_2 \in w_2, [g_1 g_2]$ represents the equivalence class of $g_1 g_2$ in $N_{\delta,G}(T) / Z_{\delta,G}(T)$; for all $w \in W(G, \delta, T)$,

$$w^{-1} := [g^{-1}],$$

where $g \in w$.

Next, we exhibit the definition of the twisted Weyl group of G associated to S^1 -module.

Definition 3. Define,

$$N_{S^1}(T) := \{g \in G \mid \tau_g^\delta(T) = T, \forall \delta \in S^1\},$$

$$Z_{S^1}(T) := \{g \in G \mid \tau_g^\delta(t) = t, \forall t \in T, \forall \delta \in S^1\},$$

where T is a maximal compact torus of $G^{S^1} := \{g \in G \mid a(g) = g, \forall a \in S^1\}$.

It is clear that they are both closed subgroups of G and that $Z_{S^1}(T)$ is a normal subgroup of $N_{S^1}(T)$.

Proposition 1. Let δ be a topological generator of S^1 . Then,

$$G^{S^1} = G^\delta.$$

Proof. It suffices to prove $G^{S^1} \supset G^\delta$. Let $g \in G^\delta$ and set

$$S_g^1 := \{\delta \in S^1 \mid \delta(g) = g\}.$$

For the underlying $g \in G^\delta$, it is obvious that $\delta(g) = g$. Thus, for all $n \in \mathbb{Z}$,

$$\delta^n(g) = \delta^{n-1}(\delta(g)) = \delta^{n-1}(g) = \dots = g.$$

Since δ is a topological generator of S^1 , S_g^1 is dense in S^1 . Again S_g^1 is a closed subgroup of S^1 ,

$$S_g^1 = S^1.$$

Then, $a(g) = g, \forall a \in S^1, g \in G^{S^1}$. So,

$$G^{S^1} \supset G^\delta.$$

□

By Proposition 1, G^{S^1} is connected. For the underlying two subgroups $N_{S^1}(T)$ and $Z_{S^1}(T)$ of G , we claim that:

Proposition 2. For a given topological generator δ_0 of S^1 and a maximal compact torus T of G^{S^1} ,

- (i) $N_{S^1}(T) = N_{\delta_0, G}(T)$;
- (ii) $Z_{S^1}(T) = Z_{\delta_0, G}(T)$.

Proof. (i) It suffices to show that $N_{S^1}(T) \supset N_{\delta_0, G}(T)$.

Suppose that $g_0 \in N_{\delta_0, G}(T)$. Then

$$g_0 T \delta_0(g_0)^{-1} = T.$$

Define

$$S_{g_0, T}^1 := \{\delta \in S^1 \mid g_0 T \delta(g_0)^{-1} = T\}.$$

Then, $S_{g_0, T}^1$ is a dense subset of S^1 . In fact, for all $n \in \mathbb{Z}$,

$$\begin{aligned} & g_0 T \delta_0^n(g_0)^{-1} \\ &= g_0 T \delta_0^n(g_0^{-1}) \\ &= (g_0 T \delta_0(g_0^{-1}))(\delta_0(g_0) T \delta_0^2(g_0^{-1})) \dots (\delta_0^{n-1}(g_0) T \delta_0^n(g_0^{-1})) \\ &= (g_0 T \delta_0(g_0^{-1})) \delta(g_0 T \delta_0(g_0^{-1})) \dots \delta^{n-1}(g_0 T \delta_0(g_0^{-1})) \\ &= T^n \\ &= T. \end{aligned}$$

Since δ_0 is a topological generator of S^1 , the set which is generated by δ_0 is a dense subset of S^1 . Again for all $n \in \mathbb{Z}$,

$$g_0 T \delta_0^n(g_0)^{-1} = T,$$

$S_{g_0, T}^1$ is dense in S^1 . It is obvious that $S_{g_0, T}^1$ is also a closed subgroup of S^1 . Hence,

$$S_{g_0, T}^1 = S^1,$$

which shows that

$$\tau_{g_0}^\delta(T) = T,$$

for all $\delta \in S^1$. So, $g_0 \in N_{S^1}(T)$, i.e.,

$$N_{S^1}(T) \supset N_{\delta_0, G}(T).$$

(ii) Similarly, it suffices to show that $Z_{S^1}(T) \supset Z_{\delta_0, G}(T)$.

Suppose that $g_0 \in Z_{\delta_0, G}(T)$. Then we have

$$g_0 t \delta_0(g_0)^{-1} = t,$$

for all $t \in T$.

For $g_0 \in Z_{\delta_0, G}(T)$, for any $t \in T$, define

$$S_{g_0, t}^1 := \{\delta \in S^1 \mid g_0 t \delta(g_0)^{-1} = t\}.$$

Analogously,

$$S_{g_0, t}^1 = S^1.$$

So,

$$Z_{S^1}(T) \supset Z_{\delta_0, G}(T).$$

□

From Proposition 2, one can get that the subgroups $N_{S^1}(T)$ and $Z_{S^1}(T)$ of G are independent with the choice of the topological generator δ of S^1 .

Definition 4. Let δ be a topological generator of S^1 and $W(G, \delta, T)$ be defined as above. Define,

$$W(G, S^1, T) := W(G, \delta, T).$$

$W(G, S^1, T)$ is called the twisted Weyl group of G associated to S^1 -module.

Following, we present some properties of $W(G, S^1, T)$.

Proposition 3. $W(G, S^1, T)$ is independent with the choice of T .

Proof. Let T' is another maximal compact torus of G^δ . Then by ([3], Proposition 2.11), there exists an element $g_0 \in G^\delta$ such that $T' = g_0 T g_0^{-1}$. Thus,

$$\begin{aligned} \tau_{g_0 g g_0^{-1}}^\delta(T') &= g_0 g g_0^{-1} T' \delta(g_0 g g_0^{-1})^{-1} \\ &= g_0 g g_0^{-1} T' g_0 \delta(g)^{-1} g_0^{-1} \\ &= g_0 g T \delta(g)^{-1} g_0^{-1} \\ &= g_0 T g_0^{-1} \\ &= T', \end{aligned}$$

for all $g \in N_{\delta, G}(T)$. Hence,

$$g_0 N_{\delta, G}(T) g_0^{-1} \subset N_{\delta, G}(T').$$

Similarly, we have

$$g_0 N_{\delta, G}(T) g_0^{-1} \supset N_{\delta, G}(T').$$

Therefore,

$$N_{\delta, G}(T') = g_0 N_{\delta, G}(T) g_0^{-1}.$$

Analogously,

$$Z_{\delta,G}(T') = g_0 Z_{\delta,G}(T) g_0^{-1}.$$

Then,

$$\begin{aligned} W_{\delta}(T') &= N_{\delta,G}(T') / Z_{\delta,G}(T') \\ &= g_0 N_{\delta,G}(T) g_0^{-1} / g_0 Z_{\delta,G}(T) g_0^{-1} \\ &\cong W_{\delta}(T), \end{aligned}$$

which shows that $W(G, S^1, T)$ is independent with the choice of T . $\square$

For the reason, we write $W(G, S^1, T)$ as $W(S^1)$ and write $W(G, \delta, T)$ as $W(\delta)$ in simplified forms, respectively.

Lemma 1. Denote by $\mathfrak{L}$ Lie functor. Then,

$$\mathfrak{L}(Z_{\delta,G}(T)) = \mathfrak{L}(N_{\delta,G}(T)) = \mathfrak{L}(T).$$

Proof. Above all, we show the first equality. It suffices to show the reverse inclusion for the clear fact that $\mathfrak{L}(Z_{\delta,G}(T)) \subset \mathfrak{L}(N_{\delta,G}(T))$. For all $X \in \mathfrak{L}(N_{\delta,G}(T))$, $s \in \mathbb{R}$, $t \in T$, we have

$$\begin{aligned} &\tau_{e^{sX}}(t) \\ &= e^{sX} t \delta(e^{sX})^{-1} \\ &= e^{sX} t e^{-s\delta(X)} \\ &= e^{sX} e^{-s\text{Ad}(t)\delta(X)} t \in T. \end{aligned}$$

Then, $(1 - \text{Ad}(t)\delta)(X) \in \mathfrak{L}(T)$. Now, we show

$$(1 - \text{Ad}(t)\delta)(X) = 0,$$

for all $t \in T$. In fact, $\mathfrak{L}(T) \subset \ker(1 - \text{Ad}(t)\delta)$ implies that $(1 - \text{Ad}(t)\delta)^2(X) = 0$. Since $d\delta$ and $\text{Ad}(t)$ are both 1-semisimple (see in [3]), $\text{Ad}(t) \circ d\delta$ is 1-semisimple. Hence, $\ker(1 - \text{Ad}(t)\delta) = \ker(1 - \text{Ad}(t)\delta)^2$. Then, we have $(1 - \text{Ad}(t)\delta)(X) = 0$ for all $t \in T$.

Next, we show the equality $\mathfrak{L}(Z_{\delta,G}(T)) = \mathfrak{L}(T)$. For all $g \in Z_{\delta,G}(T)$, $t \in T$, $\tau_g^\delta(t) = t$. Take e for the unit of T . Then $\tau_g(e) = e$, i.e., $g\delta(g)^{-1} = e$. Thus, $\delta(g) = g$ and hence $g \in G^\delta$. So, $Z_{\delta,G}(T) \subset G^\delta$. Moreover,

$$Z_{\delta,G}(T) = Z_{\delta,G}(T) \cap G^\delta = Z_{\delta,G^\delta}(T) = T,$$

which shows $Z_{\delta,G}(T) = T$. Then,

$$\mathfrak{L}(Z_{\delta,G}(T)) = \mathfrak{L}(T).$$

Therefore, $\mathfrak{L}(Z_{\delta,G}(T)) = \mathfrak{L}(N_{\delta,G}(T)) = \mathfrak{L}(T)$. $\square$

Proposition 4. As an abstract group, $W(G, S^1, T)$ is a finite group.

Proof. By Lemma 1, it is clear. $\square$

Remark 1. In the case of classical Weyl groups, let G be a compact connected Lie group, T be a maximal torus of G , and $\mathfrak{g}_\mathbb{O}$ and $\mathfrak{t}_\mathbb{O}$ be the Lie algebras of G and T respectively. Denote by $\mathfrak{g}$ and $\mathfrak{t}$ the complexifications of $\mathfrak{g}_\mathbb{O}$ and $\mathfrak{t}_\mathbb{O}$, and denote $\Delta(\mathfrak{g}, \mathfrak{t})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. In analytical level, Weyl group $W(G, T)$ is defined as the quotient of normalizer by centralizer $W(G, T) = N_G(T) / Z_G(T)$. In algebraical level, Weyl group $W(\Delta(\mathfrak{g}, \mathfrak{t}))$ is defined as the subgroup of the orthogonal group on $\mathfrak{g}$ generated by the root reflections s_α for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$. When $W(G, T)$ is considered as acting on $(\mathfrak{t}_\mathbb{O})^*$, $W(G, T)$ coincides with $W(\Delta(\mathfrak{g}, \mathfrak{t}))$ [14].

In the case of twisted Weyl groups, let G be a compact connected Lie group with an $\mathbb{Z}_n$ -module structure, T be a maximal torus of $G^{\mathbb{Z}_n}$, and $\mathfrak{g}, \mathfrak{t}, \mathfrak{g}_\delta$ and $\mathfrak{t}_\delta$ described as above. In [3], An J. defined the twisted Weyl group

$$W(G, \mathbb{Z}_n, T) := W(G, \delta, T) = N_\delta(T) / Z_\delta(T),$$

where δ is a generator of $\mathbb{Z}_n$. Motivated by the underlying equality between $W(G, T)$ and $W(\Delta(\mathfrak{g}, \mathfrak{t}))$, one can also consider the algebraic twisted Weyl group. Since δ can be regarded as an automorphism on G , its differential $d\delta$ can be thought as an automorphism on $\mathfrak{g}$. Define

$$\mathfrak{g}_\alpha^\delta := \{X \in \mathfrak{g} \mid \text{ad}(d\delta)(H)X = \alpha(H)X, \forall H \in \mathfrak{t}\}.$$

If $\mathfrak{g}_\alpha^\delta \neq 0$, α is called a twisted root of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Denote by $\Delta(\mathfrak{g}, \delta, \mathfrak{t})$ the set of twisted roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. For $\alpha \in \Delta(\mathfrak{g}, \delta, \mathfrak{t})$, the twisted root reflection is defined as

$$s_\alpha^\delta(\gamma) := \gamma - \frac{2 \langle d\delta(\gamma), \alpha \rangle}{|\alpha|^2} \alpha.$$

Define $W(\Delta(\mathfrak{g}, \delta, \mathfrak{t}))$ as the group generated by the twisted root reflections s_α^δ for $\alpha \in \Delta(\mathfrak{g}, \delta, \mathfrak{t})$. Similar as the proof in ([14], Theorem 4.54), one can obtain that

$$W(G, \mathbb{Z}_n, T) = W(\Delta(\mathfrak{g}, \delta, \mathfrak{t})),$$

when $W(G, \mathbb{Z}_n, T)$ is considered as acting on $(\mathfrak{t}_\delta)^*$. For analytical twisted Weyl group $W(G, S^1, T)$, we have an analogous algebraical counterpart described as above.

The twisted conjugate action τ^δ of G associated to δ naturally induces the action of twisted Weyl group $W(S^1)$ on T :

$$W(S^1) \times T \rightarrow T, w \cdot t := \tau_g^\delta(t),$$

where δ is a topological generator of S^1 , $w \in W(S^1)$, $t \in T$, $g \in w$. Now, we show the following property of the action of $W(S^1)$ on T .

Proposition 5. Let G be a compact connected Lie group associated to an S^1 -module, δ be a topological generator of S^1 , T be a maximal compact torus of G^{S^1} and $W(S^1)$ be the twisted Weyl group of G associated to S^1 -module. Then two elements of T are δ -conjugate if and only if they are in one $W(S^1)$ -orbit.

Proof. Denote by O_t^W the $W(S^1)$ -orbit over $t \in T$, i.e.,

$$O_t^W := \{\tau_g^\delta(t) \mid w \in W(S^1), g \in w\}.$$

$\Rightarrow$) If $t_1, t_2 \in O_t^W$, then there exist $g_1 \in w_1 \in W(S^1)$, $g_2 \in w_2 \in W(S^1)$ such that $t_1 = \tau_{g_1}^\delta(t)$, $t_2 = \tau_{g_2}^\delta(t)$. Hence, $t = g_1^{-1}t_1\delta(g_1)$. So,

$$\begin{aligned} t_2 &= g_2 t \delta(g_2)^{-1} \\ &= g_2 g_1^{-1} t_1 \delta(g_1) \delta(g_2)^{-1} \\ &= (g_2 g_1^{-1}) t_1 \delta(g_2 g_1^{-1})^{-1}, \end{aligned}$$

which shows that $t_2 = \tau_{g_2 g_1^{-1}}^\delta(t_1)$, $g_2 g_1^{-1} \in G$. So, t_1, t_2 are δ -conjugate.

$\Leftarrow$) If t_1, t_2 are δ -conjugate. i.e., there exists an element $g \in G$ such that $t_2 = \tau_g^\delta(t_1)$. It needs to show that t_1, t_2 are in one $W(S^1)$ -orbit, i.e., to find an element $w_0 \in W(S^1)$ such that $t_2 = w_0 \cdot t_1$ or to find an element $g_0 \in N_{\delta, G}(T)$ such that

$$t_2 = \tau_{g_0}^\delta(t_1).$$

Define

$$G_{t_1} := \{h \in G \mid \tau_h^\delta(t_1) = t_1\}.$$

Then $T \subset G_{t_1}$, which holds for the fact T is connected and $\tau_t^\delta(t_1) = t_1$ for all $t \in T$. Again,

$$\begin{aligned} h \in G_{t_1} &\Leftrightarrow \tau_h^\delta(t_1) = ht_1\delta(h)^{-1} = t_1 \\ &\Leftrightarrow h = t_1\delta(h)t_1^{-1} \\ &\Leftrightarrow h \in G^{t_1 \circ \delta}, \end{aligned}$$

where t_1 represents the automorphism induced by t_1 . Thus,

$$G_{t_1} = G^{t_1 \circ \delta}.$$

Then by ([2], Theorem 2.1),

$$\text{rank}(G^\delta) = \text{rank}(G^{t_1 \circ \delta}) = \text{rank}(G_{t_1}),$$

where $\text{rank}(G^\delta)$ represents the dimension of the maximal compact torus of G^δ . Hence, T is maximal compact torus of G_{t_1} . In fact, for the underlying $g \in G$, $g^{-1}Tg$ is also a maximal compact torus of G_{t_1} : for any $t \in T$, we have

$$\begin{aligned} \tau_{g^{-1}tg}^\delta(t_1) &= g^{-1}tg t_1 \delta(g^{-1}tg)^{-1} \\ &= g^{-1}tg t_1 \delta(g)^{-1} \delta(t)^{-1} \delta(g^{-1})^{-1} \\ &= g^{-1}t t_2 t^{-1} \delta(g^{-1})^{-1} \\ &= g^{-1}t_2 \delta(g^{-1})^{-1} \\ &= t_1, \end{aligned}$$

which shows that $g^{-1}Tg \subset G_{t_1}$. Thus, T and $g^{-1}Tg$ are both torus of G_{t_1} . Then there exists an element $x \in G_{t_1}$ such that $xTx^{-1} = g^{-1}Tg$. Pick $g_0 = gx$, then $g_0Tg_0^{-1} = T$.

Now we show $g_0 \in N_{\delta,G}(T)$ and $\tau_{g_0}^\delta(t_1) = t_2$. By

$$\begin{aligned} \tau_{g_0}^\delta(T) &= g_0T\delta(g_0)^{-1} \\ &= g_0Tg_0^{-1}g_0t_1\delta(g_0)^{-1} \\ &= Tgx t_1 \delta(x)^{-1} \delta(g)^{-1} \\ &= Tg t_1 \delta(g)^{-1} \\ &= Tt_2 \\ &= T, \end{aligned}$$

we get $g_0 \in N_{\delta,G}(T)$; by

$$\begin{aligned} \tau_{g_0}^\delta(t_1) &= g_0t_1\delta(g_0)^{-1} \\ &= gxt_1\delta(gx)^{-1} \\ &= gxt_1\delta(x)^{-1}\delta(g)^{-1} \\ &= gt_1\delta(g)^{-1} \\ &= t_2, \end{aligned}$$

we get $\tau_{g_0}^\delta(t_1) = t_2$. $\square$

3. Twisted Weyl Groups of Compact Lie Groups and Nonabelian Cohomology

In this section, we discuss the relationship between twisted Weyl group $W(S^1)$ and the first nonabelian cohomology $H^1(S^1, G)$ of S^1 with coefficients in G .

Let T be a maximal compact torus of G^{S^1} . As T is abelian and S^1 acts trivially on T , $H^1(S^1, T)$ coincides with $Z^1(S^1, T)$ and $\gamma \in Z^1(S^1, T)$ is a group homomorphism $\gamma : S^1 \rightarrow T$. Thus, $W(S^1)$ naturally acts on $H^1(S^1, T) = Z^1(S^1, T)$, i.e.,

$$W(S^1) \times H^1(S^1, T) \rightarrow H^1(S^1, T),$$

$$(w \cdot \gamma)(a) := g\gamma(a)a(g)^{-1},$$

where $w \in W(S^1)$, $\gamma \in H^1(S^1, T)$, $g \in w$, $a \in S^1$. Now, we show that the definition is well-defined. It suffices to show that $w \cdot \gamma \in Z^1(S^1, T)$.

In fact, for any $a_1, a_2 \in S^1$, we have

$$\begin{aligned} (w \cdot \gamma)(a_1)a_1((w \cdot \gamma)(a_2)) &= g\gamma(a_1)a_1(g)^{-1}a_1(g\gamma(a_2)a_2(g)^{-1}) \\ &= g\gamma(a_1)a_1(g)^{-1}a_1(g)a_1(\gamma(a_2))a_1(a_2(g)^{-1}) \\ &= g\gamma(a_1)\gamma(a_2)(a_1a_2)(g)^{-1} \\ &= g\gamma(a_1a_2)(a_1a_2)(g)^{-1} \\ &= (w \cdot \gamma)(a_1a_2), \end{aligned}$$

which shows that the underlying definition is well-defined. Thus, we have

$$(w \cdot \gamma)(a_1a_2) = (w \cdot \gamma)(a_1)(w \cdot \gamma)(a_2).$$

Hence, $w \cdot \gamma$ is a group homomorphism.

The natural embedding

$$i : T \hookrightarrow G$$

induces the natural map

$$i_1 : H^1(S^1, T) \rightarrow H^1(S^1, G), \quad [\gamma] \mapsto [i \circ \gamma],$$

and i_1 can be reduced to

$$i_1^{W(S^1)} : W(S^1) \setminus H^1(S^1, T) \rightarrow H^1(S^1, G), \quad O_\gamma^W \mapsto [i \circ \gamma],$$

where $O_\gamma^W := \{w \cdot \gamma \mid w \in W\}$ is the $W(S^1)$ -orbit over γ in $H^1(S^1, T)$. Thus, we construct a correspondence between $W(S^1) \setminus H^1(S^1, T)$ and $H^1(S^1, G)$ by $i_1^{W(S^1)}$.

Theorem 1. Let G be a compact connected Lie group associated to an S^1 -module, δ be a topological generator of S^1 , T be a maximal compact torus of G^{S^1} and $W(S^1)$ be the twisted Weyl group of G associated to S^1 -module. Then the map

$$i_1^{W(S^1)} : W(S^1) \setminus H^1(S^1, T) \rightarrow H^1(S^1, G)$$

is a bijection.

Proof. By ([17], Theorem 2.5), the natural map i_1 is a surjection and hence $i_1^{W(S^1)}$ is also a surjection.

Now we show $i_1^{W(S^1)}$ is an injection. Suppose that $\gamma_1, \gamma_2 \in H^1(S^1, T) = Z^1(S^1, T)$ have the same image under i_1 . Then, there exists an element $g \in G$ such that

$$\gamma_2(a) = g\gamma_1(a)a(g)^{-1},$$

for all $a \in A$. Picking a topological generator δ_0 of S^1 . Thus, we have

$$\gamma_2(\delta_0) = g\gamma_1(\delta_0)\delta_0(g)^{-1},$$

which shows that $\gamma_1(\delta_0)$ and $\gamma_2(\delta_0)$ are δ_0 -conjugate. By Proposition 5, $\gamma_1(\delta_0)$ and $\gamma_2(\delta_0)$ are in one $W(S^1) = W(\delta_0)$ -orbit. In other words, there exists an element $g_0 \in N_{\delta_0, G}(T)$ such that

$$\gamma_2(\delta_0) = g_0 \gamma_1(\delta_0) \delta_0 (g_0)^{-1},$$

which also means that there exists $w_0 \in W$ such that $g_0 \in w_0$ and

$$\gamma_2(\delta_0) = (w_0 \cdot \gamma_1)(\delta_0).$$

Since δ_0 is a topological generator of S^1 , one can obtain that

$$\gamma_2(a) = (w_0 \cdot \gamma_1)(a),$$

for all $a \in S^1$. Thus,

$$\gamma_2 = w_0 \cdot \gamma_1,$$

which shows that γ_1 and γ_2 are in one $W(S^1)$ -orbit. So,

$$i_1^{W(S^1)}(O_{\gamma_1}^W) = i_1^{W(S^1)}(O_{\gamma_2}^W),$$

and thus,

$$O_{\gamma_1}^{S^1} = O_{\gamma_2}^{S^1},$$

which shows that $i_1^{W(S^1)}$ is an injection. $\square$

Remark 2. Let A be a general compact group [4] and G be a compact connected Lie group with an A -module structure. Our motivation for this paper is to define the twisted Weyl group of G associated A -module. For this aim, we have to deal with the existence of the maximal compact torus in G_0^A , where G_0^A is the identity connected component of

$$G^A := \{g \in G \mid a(g) = g, \forall a \in A\}.$$

However, for a general compact group A , maximal compact torus in G_0^A may not exist [1]. Even for the existence of A -invariant maximal compact torus in G , it is not certain too [17]. From the discussions for the existence of maximal compact torus in G_0^A and G^A , we find the following two open problems.

Problem 1. Under what conditions there exists a nontrivial maximal compact torus in G_0^A ?

Problem 2. If T is a nontrivial maximal compact torus of G_0^A . Can the maximal compact torus T generalize to an A -invariant maximal compact torus T' of G ?

4. Discussions

In [9], Bartholdi studied the amenability of Γ -set, here Γ is a group, which was induced by John Von Neumann in 1929. Fundamentally, the notion exhibited the following property of a group acting on a Γ -set X : the Γ -set X right is called amenable if there exists a Γ -invariant mean m on the power set 2^X of X , namely a function

$$m : 2^X \rightarrow [0, 1],$$

satisfying $m(A \sqcup B) = m(A) + m(B)$, $m(X) = 1$ and

$$m(Ag) = m(A),$$

for all $A \subset X$ and $g \in \Gamma$.

In [9], Bartholdi presented some criterions to show a Γ -set X amenable. By ([9], Proposition 2.3), one can get that the underlying twisted Weyl group $W(S^1)$ is indeed amenable. For the amenability of twisted Weyl groups, we will study it in a sole paper.

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