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A Characterization of Quasi-Metric Completeness in Terms of α–ψ-Contractive Mappings Having Fixed Points

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Abstract: We obtain a characterization of Hausdorff left K-complete quasi-metric spaces by means of α–ψ-contractive mappings, from which we deduce the somewhat surprising fact that one the main fixed point theorems of Samet, Vetro, and Vetro (see "Fixed point theorems for α–ψ-contractive type mappings", *Nonlinear Anal.* **2012**, *75*, 2154–2165), characterizes the metric completeness.

Keywords: fixed point; quasi-metric space; left K-complete; α–ψ-contractive mapping

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1. Introduction and Preliminaries

In their interesting and germinal paper [1], Samet, Vetro, and Vetro obtained various fixed point theorems in terms of α–ψ contractions which allowed them to deduce, in an elegant and direct way, several important and well-known fixed point results from [2–5]. Many authors have continued the research of this type of contractions and their generalizations in different contexts (see e.g., [6–12]). Recently, Fulsa and Taş [13] have presented a careful and extensive study for several generalized α–ψ contractions in the realm of quasi-metric spaces.

In this note we obtain a characterization of Hausdorff left K-complete quasi-metric spaces by means of α–ψ-contractive mappings from which we deduce the somewhat surprising fact that one the main fixed point theorems of Samet, Vetro, and Vetro [1] (Theorem 2.2) characterizes the metric completeness (see Corollary 1 at the end of the paper).

Let us recall that the problem of characterizing the metric completeness in term of fixed point theorems has been studied and solved by several authors with different approaches (see e.g., [14–17]) and that this study has been extended in recent years to some types of generalized metric spaces as partial metric spaces [18,19] and quasi-metric spaces [20,21].

In order to help the reader, we recall some notions and properties of quasi-metric spaces which will be used in this paper. Our basic reference is [22].

A quasi-metric space is a pair $(\mathcal{X}, \rho)$ such that $\mathcal{X}$ is a set and ρ is a quasi-metric on $\mathcal{X}$, i.e., ρ is a function from $\mathcal{X} \times \mathcal{X}$ to $[0, \infty)$ such that for all $\zeta, \eta, \theta \in \mathcal{X}$:

(i) $\zeta = \eta$ if and only if $\rho(\zeta, \eta) = \rho(\eta, \zeta) = 0$, and

(ii) $\rho(\zeta, \theta) \leq \rho(\zeta, \eta) + \rho(\eta, \theta)$.

Given a quasi-metric ρ on $\mathcal{X}$ the family $\{B_\rho(\zeta, \varepsilon) : \zeta \in \mathcal{X}, \varepsilon > 0\}$, where $B_\rho(\zeta, \varepsilon) = \{\eta \in \mathcal{X} : \rho(\zeta, \eta) < \varepsilon\}$ for all $\zeta \in \mathcal{X}$ and $\varepsilon > 0$, is a base for a $\mathcal{T}_0$ topology τ_ρ on $\mathcal{X}$.

$(\mathcal{X}, \rho)$ is called a $\mathcal{T}_1$ quasi-metric space if τ_ρ is a $\mathcal{T}_1$ topology, and it is called a Hausdorff quasi-metric space if τ_ρ is a $\mathcal{T}_2$ topology.

A quasi-metric space $(\mathcal{X}, \rho)$ is said to be left K-complete if every left K-Cauchy sequence converges with respect to τ_ρ, where, by a left K-Cauchy sequence we mean a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $(\mathcal{X}, \rho)$ such that for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ satisfying $\rho(\zeta_n, \zeta_m) < \varepsilon$ whenever $n_\varepsilon \leq n \leq m$.

2. Results

We start this section by recalling some known concepts.

As usual, we denote by Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \geq 0$.

Let $\mathcal{X}$ be a set, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$. Following [1] (Definition 2.2), we say that $\mathcal{T}$ is α-admissible if $\alpha(\zeta, \eta) \geq 1$ implies $\alpha(\mathcal{T}\zeta, \mathcal{T}\eta) \geq 1$; $\zeta, \eta \in \mathcal{X}$.

As in the metric case [1] (Definition 2.1), given a quasi-metric space $(\mathcal{X}, \rho)$ we say that a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is an α–ψ-contractive mapping if there exist two functions $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $\psi \in \Psi$ such that $\alpha(\zeta, \eta)\rho(\mathcal{T}\zeta, \mathcal{T}\eta) \leq \psi(\rho(\zeta, \eta))$ for all $\zeta, \eta \in \mathcal{X}$.

The following slight modification of condition (iii) in Theorem 2.2 of [1] constitutes a crucial ingredient in obtaining our main result:

Let $(\mathcal{X}, \rho)$ be a quasi-metric space and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$. We say that $(\mathcal{X}, \rho)$ has property (A) (with respect to α) if for any sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ satisfying $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and such that $\rho(\zeta, \zeta_n) \to 0$ as $n \to \infty$ for some $\zeta \in \mathcal{X}$, it follows that $\alpha(\zeta, \zeta_n) \geq 1$ for all $n \in \mathbb{N}$.

Definition 1. *Given a quasi-metric space $(\mathcal{X}, \rho)$, an $\alpha - \psi$-contractive mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ will be called an α–ψ-SVV contractive mapping if: (i) $\mathcal{T}$ is α-admissible; (ii) there exists $\zeta_0 \in \mathcal{X}$ such that $\alpha(\zeta_0, \mathcal{T}\zeta_0) \geq 1$; (iii) $(\mathcal{X}, \rho)$ has property (A) (with respect to α).*

By using the preceding definition, Theorem 2.2 of [1] can be reformulated as follows: *Every α–ψ-SVV contractive mapping on a complete metric space has a fixed point.*

Our first result provides a quasi-metric extension of Theorem 2.2 of [1] (its proof is only an adaptation of the original proof of Samet, Vetro, and Vetro).

Theorem 1. *Every α–ψ-SVV contractive mapping on a left K-complete quasi-metric space has a fixed point.*

Proof of Theorem 1. Let $\mathcal{T}$ be an α–ψ-*SVV* contractive mapping on a Hausdorff left K-complete quasi-metric space $(\mathcal{X}, \rho)$. Then, there exists an α-admissible function such that $\mathcal{T}$ is α–ψ-contractive, $(\mathcal{X}, \rho)$ has property (A), and $\alpha(\zeta_0, \mathcal{T}\zeta_0) \geq 1$ for some $\zeta_0 \in \mathcal{X}$.

For each $n \in \mathbb{N}$ let $\zeta_n := \mathcal{T}^n\zeta_0$. If there exists $m \in \mathbb{N}$ such that $\zeta_{m-1} = \zeta_m$, then ζ_m is a fixed point of $\mathcal{T}$. Assume then that $\zeta_n \neq \zeta_m$ for all $n, m \in \mathbb{N}\cup\{0\}$. Since $\alpha(\zeta_0, \zeta_1) \geq 1$ and $\mathcal{T}$ is α-admissible we deduce that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. As in the proof of Theorem 2.1 of [1] we obtain $\rho(\zeta_n, \zeta_{n+1}) \leq \psi^n(\rho(\zeta_0, \zeta_1))$ and deduce that $(\zeta_n)_{n\in\mathbb{N}}$ is a left K-Cauchy sequence in $(\mathcal{X}, \rho)$ (see [1] (p. 2156)). Since $(\mathcal{X}, \rho)$ is left K-complete there exists $\theta \in \mathcal{X}$ such that $\rho(\theta, \zeta_n) \to 0$ as $n \to \infty$. From property (A) it follows that $\alpha(\theta, \zeta_n) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. We shall show that θ is a fixed point of $\mathcal{T}$. Indeed, for each $n \in \mathbb{N}\cup\{0\}$ we have: $\rho(\mathcal{T}\theta, \zeta_{n+1}) = \rho(\mathcal{T}\theta, \mathcal{T}\zeta_n) \leq \alpha(\theta, \zeta_n)\rho(\mathcal{T}\theta, \mathcal{T}\zeta_n) \leq \psi(\rho(\theta, \zeta_n))$.

Since $\rho(\theta, \zeta_n) > 0$, we deduce that $\psi(\rho(\theta, \zeta_n)) < \rho(\theta, \zeta_n)$ (see e.g., [1] (Lemma 2.1)), and, hence, $\rho(\mathcal{T}\theta, \zeta_n) \to 0$ as $n \to \infty$. Since $(\mathcal{X}, \rho)$ is Hausdorff we conclude that $\theta = \mathcal{T}\theta$. □

As for metric spaces [1] (Theorem 2.1), a slight modification of the proof of Theorem 1 shows the following result where the property (A) is replaced by continuity of $\mathcal{T}$. More precisely we have

Theorem 2. *Let $(\mathcal{X}, \rho)$ be a Hausdorff left K-complete quasi-metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an α–ψ-contractive mapping such that*

(i) $\mathcal{T}$ is α-admissible;
(ii) there exists $\zeta_0 \in \mathcal{X}$ such that $\alpha(\zeta_0, \mathcal{T}\zeta_0) \geq 1$;
(iii) $\mathcal{T}$ is continuous.

Then $\mathcal{T}$ has a fixed point.

Theorems 1 and 2 can not be generalized to $\mathcal{T}_1$ left K-complete quasi-metric spaces (see e.g., [23] (Example 5)).

Let us recall that if ρ is a quasi-metric on a set $\mathcal{X}$, then the function ρ^s defined on $\mathcal{X} \times \mathcal{X}$ by $\rho^s(\zeta, \eta) = \max\{\rho(\zeta, \eta), \rho(\eta, \zeta)\}$ is a metric on $\mathcal{X}$. We give an example for a quasi-metric space $(\mathcal{X}, \rho)$ where we can apply both Theorem 1 and Theorem 2 but not [1] (Theorem 2.2) because the metric space $(\mathcal{X}, \rho^s)$ is not complete.

Example 1. *Let $\mathcal{X} := \{0\} \cup \{1/n : n \in \mathbb{N}\} \cup \{n : n \in \mathbb{N} \backslash \{1\}\}$. It is routine to check that $(\mathcal{X}, \rho)$ is a Hausdorff quasi-metric space where (the quasi-metric) ρ is defined as follows:*

$\rho(\zeta, \zeta) = 0$ *for all* $\zeta \in \mathcal{X}$.
$\rho(0, 1/n) = 1/n$ *for all* $n \in \mathbb{N}$.
$\rho(1/n, 1/m) = 1/n$ *whenever* $n < m$.
$\rho(0, n) = 2^{-n}$ *for all* $n \in \mathbb{N} \backslash \{1\}$.
$\rho(n, m) = |2^{-n} - 2^{-m}|$ *for all* $n, m \in \mathbb{N} \backslash \{1\}$, *and*
$\rho(\zeta, \eta) = 1$ *otherwise.*

Observe that $(\mathcal{X}, \rho)$ is left K-complete: The sequence $(1/n)_{n \in \mathbb{N}}$ is left K-Cauchy and converges to 0, whereas the sequence $(n)_{n \in \mathbb{N}}$ is Cauchy in the metric space $(\mathcal{X}, \rho^s)$, and hence left K-Cauchy in $(\mathcal{X}, \rho)$, and also converges to 0. However, we have $\rho(n, 0) = 1$ for all $n \in \mathbb{N}$, and thus the metric space $(\mathcal{X}, \rho^s)$ is not complete.

Now define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ as $\mathcal{T}0 = 0$, $\mathcal{T}n = n + 1$ for all $n \in \mathbb{N}$, and $\mathcal{T}(1/n) = n$ for all $n \in \mathbb{N} \backslash \{1\}$.

We show that $\mathcal{T}$ is an $\alpha - \psi$-SVV contractive mapping for α given by $\alpha(0, n) = \alpha(n, n+1) = 1$ for all $n \in \mathbb{N}$, and $\alpha(\zeta, \eta) = 0$ otherwise; and $\psi \in \Psi$ given by $\psi(t) = t/2$ for all $t \geq 0$.

Indeed, since $\alpha(1, T1) = \alpha(1, 2) = 1$, we deduce by the definition of $\mathcal{T}$ and the construction of α that $\mathcal{T}$ is α-contractive. Also, the property (A) is clearly satisfied since $\rho(0, n) \to 0$ as $n \to \infty$, and $\alpha(0, n) = 1$ for all $n \in \mathbb{N}$. It remains to check that $\mathcal{T}$ is an α–ψ-contractive mapping. To this end, it suffices to consider the following two cases:

Case 1. $\zeta = 0$, $\eta = n$, $n \in \mathbb{N}$. Thus, we obtain

$$\alpha(\zeta, \eta)\rho(T\zeta, T\eta) = \alpha(0, n)\rho(0, n+1) = 2^{-(n+1)} \leq \frac{1}{2}\rho(0, n) = \psi(\rho(\zeta, \eta)).$$

Case 2. $\zeta = n$, $\eta = n + 1$, $n \in \mathbb{N}$. Thus, we obtain

$$\begin{aligned} \alpha(\zeta, \eta)\rho(T\zeta, T\eta) &= \alpha(n, n+1)\rho(Tn, T(n+1)) = \rho(n+1, n+2) \\ &= 2^{-(n+2)} = \frac{1}{2}\rho(n, n+1) = \psi(\rho(\zeta, \eta)). \end{aligned}$$

Therefore, all conditions of Theorem 1 are satisfied.

Clearly, we can also apply Theorem 2 because $\mathcal{T}$ is continuous (with respect to τ_ρ).

Now, we present an easy example where we can apply Theorem 1 but not Theorem 2.

Example 2. *Let $\mathcal{X} := \{0, \infty\} \cup \mathbb{N}$. Clearly $(\mathcal{X}, \rho)$ is a Hausdorff left K-complete quasi-metric space where (the quasi-metric) ρ is defined as follows:*

$\rho(\zeta, \zeta) = 0$ *for all* $\zeta \in \mathcal{X}$.
$\rho(0, 1/n) = 1/n$ *for all* $n \in \mathbb{N}$, *and*
$\rho(\zeta, \eta) = 1$ *otherwise.*

Now define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ as $\mathcal{T}0 = 0$, $\mathcal{T}\infty = \infty$, and $\mathcal{T}n = \infty$ for all $n \in \mathbb{N}$.

Since $\rho(0,n) \to 0$ as $n \to \infty$, but $\rho(\mathcal{T}0,\mathcal{T}n) = \rho(0,\infty) = 1$, we conclude that $\mathcal{T}$ is not continuous. However, it is obvious that $\mathcal{T}$ is an $\alpha - \psi$-SVV contractive mapping for α given by $\alpha(\infty,\infty) = 1$, and $\alpha(\zeta,y) = 0$ otherwise, and any $\psi \in \Psi$.

In our main result (Theorem 3 below), we prove that Theorem 1 characterizes left K-completeness of Hausdorff quasi-metric spaces. However, Theorem 2 does not provide such characterization even in the case of metric spaces, as Suzuki and Takahashi constructed in [24] an example of a non-complete metric space for which every continuous self map has fixed points.

Theorem 3. *A Hausdorff quasi-metric space is left K-complete if and only if every α–ψ-SVV contractive mapping has a fixed point.*

Proof of Theorem 3. Let $(\mathcal{X},\rho)$ be a Hausdorff left K-complete quasi-metric space. By Theorem 1, every α–ψ-*SVV* contractive mapping on $(\mathcal{X},\rho)$ has a fixed point.

Conversely, suppose that $(\mathcal{X},\rho)$ is a Hausdorff quasi-metric space which is not left K-complete. Then there exists a left K-Cauchy sequence $(\zeta_n)_{n\in\mathbb{N}}$ (of distinct points) in (ζ,ρ) which is not convergent for τ_ρ. Put $\mathcal{A} = \{\zeta_n : n \in \mathbb{N}\}$. Since $\rho(\zeta_1,\mathcal{A}\backslash\{\zeta_1\}) > 0$, there exists $h_1 \in \mathbb{N}$, with $h_1 > 1$, such that $\rho(\zeta_j,\zeta_k) < \rho(\zeta_1,\mathcal{A}\backslash\{\zeta_1\})/2$ whenever $h_1 \le j \le k$. Similarly, there exists $h_2 \in \mathbb{N}$, with $h_2 > \max\{2,h_1\}$, such that $\rho(\zeta_j,\zeta_k) < \rho(\zeta_2,\mathcal{A}\backslash\{\zeta_2\})/2$ whenever $h_2 \le j \le k$. In this way we obtain a subsequence $(h_n)_{n\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that $h_n > \max\{n,h_{n-1}\}$ and $\rho(\zeta_j,\zeta_k) < \rho(\zeta_n,\mathcal{A}\backslash\{\zeta_n\})/2$ whenever $h_n \le j \le k$.

Define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ as follows:

$\mathcal{T}\zeta_n = \zeta_{h_n}$ for $n \in \mathbb{N}$, and $\mathcal{T}\zeta = \zeta_1$ for $\zeta \in \mathcal{X}\backslash\mathcal{A}$, and

$\alpha(\zeta,\eta) = 1$ if $\zeta = \zeta_n$ and $\eta = \zeta_m$ for $n,m \in \mathbb{N}$ with $n < m$, and $\alpha(\zeta,\eta) = 0$ otherwise.

We first note that $\alpha(\zeta_1,\mathcal{T}\zeta_1) = 1$ because $1 < h_1$.

Moreover $\mathcal{T}$ is α-admissible. Indeed, if $\alpha(\zeta,\eta) \ge 1$, then $\zeta = \zeta_n$ and $\eta = \zeta_m$ with $n < m$. So $\alpha(\mathcal{T}\zeta,\mathcal{T}\eta) = \alpha(\zeta_{h_n},\zeta_{h_m}) = 1$ because $h_n < h_m$.

Next, we show that $\mathcal{T}$ is α–ψ-contractive for $\psi \in \Psi$ given by $\psi(t) = t/2$. Indeed, by the construction of α it suffices to check the case that $\zeta = \zeta_n$ and $\eta = \zeta_m$ with $n < m$. Thus, we obtain

$$\begin{aligned}\alpha(\zeta,\eta)\rho(\mathcal{T}\zeta,\mathcal{T}\eta) &= \alpha(\zeta_n,\zeta_m)\rho(\zeta_{h_n},\zeta_{h_m}) < \frac{1}{2}\rho(\zeta_n,\mathcal{A}\backslash\{\zeta_n\}) \\ &\le \frac{1}{2}\rho(\zeta_n,\zeta_m) = \frac{1}{2}\rho(\zeta,\eta) = \psi(\rho(\zeta,\eta)).\end{aligned}$$

Finally, note that $(\mathcal{X},\rho)$ trivially satisfies the property (A) because the only convergent sequences in $\mathcal{A}$ are those that are eventually constant.

We have shown that $\mathcal{T}$ is an α–ψ-*SVV* contractive mapping without fixed point. This contradiction concludes the proof. □

Corollary 1. *A metric space is complete if and only every α–ψ-SVV contractive mapping has a fixed point.*

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