mathematics

MDPI

Article

Geodesic Vector Fields on a Riemannian Manifold

Sharief Deshmukh [1,*], Patrik Peska [2] and Nasser Bin Turki [1]

1 Department of Mathematics, College of science, King Saud University, P.O. Box-2455 Riyadh 11451, Saudi Arabia; nassert@ksu.edu.sa
2 Department of Algebra and Geometry, Palacky University, 77146 Olomouc, Czech Republic; peska@seznam.cz
* Correspondence: shariefd@ksu.edu.sa

Received: 29 December 2019; Accepted: 16 January 2020; Published: 19 January 2020

Abstract: A unit geodesic vector field on a Riemannian manifold is a vector field whose integral curves are geodesics, or in other worlds have zero acceleration. A geodesic vector field on a Riemannian manifold is a smooth vector field with acceleration of each of its integral curves is proportional to velocity. In this paper, we show that the presence of a geodesic vector field on a Riemannian manifold influences its geometry. We find characterizations of n-spheres as well as Euclidean spaces using geodesic vector fields.

Keywords: geodesic vector field; eikonal equation; isometric to sphere; isometric to Euclidean space

MSC: 53C20; 53C21; 53C24

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold. We call a smooth vector field ξ on M geodesic vector field if

$$\nabla_\xi \xi = \rho \xi, \tag{1}$$

where ∇ is the covariant derivative operator with respect to the Riemannian connection on (M, g) and $\rho : M \to R$ is a smooth function called the potential function of the geodesic vector field ξ. If the potential function $\rho = 0$, then ξ is called a unit geodesic vector field (as in this case the integral curves of ξ are geodesics). By a non-trivial geodesic vector field, we mean nonzero geodesic vector field for which the potential function $\rho \neq 0$. Physically, a geodesic vector field has integral curves with an acceleration vector always proportional to the velocity vector. These fields are connected with generalized Fermi coordinates [1]. Geodesic vector fields naturally appear in many situations as seen in the following examples:

1. On Euclidean space $(R^n, \langle , \rangle)$, the position vector field $\xi = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i}$, satisfies $\nabla_\xi \xi = \xi$, therefore ξ is a geodesic vector field with potential function $\rho = 1$.
2. Consider unit hypersphere S^n in the Euclidean space $(R^{n+1}, \langle , \rangle)$. Then, the restriction of coordinate vector field $\frac{\partial}{\partial u^1}$ on R^{n+1} to S^n can be expressed as

$$\frac{\partial}{\partial u^1} = \xi + \rho N,$$

where $\rho = \left\langle \frac{\partial}{\partial u^1}, N \right\rangle$, N being unit normal to S^n and ξ is vector field on S^n, which is the tangential component of $\frac{\partial}{\partial u^1}$. Then it is easy to see that on S^n, we have $\nabla_\xi \xi = \rho \xi$, that is, ξ is a geodesic vector field on S^n.

3. Concircular vector fields on Riemannian manifolds have been introduced by A. Fialkow (cf. [2,3]). A vector field ξ on a Riemannian manifold (M, g) is said to be a concircular vector field if $\nabla_X \xi = \rho X$ for any smooth vector field X on M, where ρ is a smooth function on M. Thus, a concircular vector field ξ satisfies $\nabla_\xi \xi = \rho \xi$, that is, a concircular vector field ξ is a geodesic vector field. It is well known that concircular vector fields play a vital role in the theory of projective and conformal transformations. Moreover, concircular vector fields have applications in general relativity, as for instance trajectories of time-like concircular fields in the de Sitter space determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis (cf. [4]). Therefore, we could expect that geodesic vector fields also have the scope of applications in general relativity. For example, global questions about the existence of these vector fields were studied in [5–10].
4. Another interesting example comes from Yamabe solitons (cf. [11,12]). Let (M, g, ξ, λ) be an n-dimensional Yamabe soliton. Then the soliton field ξ satisfies

$$\frac{1}{2}\pounds_\xi g = (S - \lambda) g,$$

where $\pounds_\xi g$ is the Lie-derivative of metric g, S is the scalar curvature and λ is a constant. If the soliton field ξ is a gradient of a smooth function, then (M, g, ξ, λ) is called a gradient Yamabe soliton. On gradient Yamabe soliton the soliton field satisfies $\nabla_\xi \xi = (S - \lambda)\xi$, that is, ξ is a geodesic field with potential function $\rho = S - \lambda$.
5. Recall that an Eikonal equation is a nonlinear partial differential equation

$$\|\nabla u\| = \frac{1}{f}$$

where on a non-compact Riemannian manifold (M, g), which is encountered in problems of wave propagation, where f is a positive function (cf. [13,14]). A straight forward observation shows that, above equation gives $\nabla_{\nabla u} \nabla u = -\frac{1}{f^3} \nabla f$, which on choosing $u = \frac{1}{f}$, gives $\nabla_{\nabla u} \nabla u = u \nabla u$, that is, an Eikonal equation gives a non-trivial geodesic vector field ∇u with potential function u. Note that Eikonal equations are also used in tumor invasion margin on Riemannian manifolds of brain fibers (cf. [15]).

It is worth noting that the main tools in studying the geometry of a Riemannian manifold are geodesics, immersions, and special vector fields. For instance, geodesics give rise to the exponential mapping and Jacobi fields, which are used in proving many global theorems for a Riemannian manifold. Immersions are used to study the geometry of submanifolds. Similarly, special vector fields such as unit geodesic vector fields, Killing vector fields, concircular vector fields, conformal vector fields are used in studying geometry as well as topology of a Riemannian manifold (cf. [1–4,6–11,16–29]).

Geodesic vector fields first time appeared in [12] as generalization of unit geodesic vector fields, where they are used for finding conditions under which a Yamabe soliton is trivial. As observed through above examples, geodesic vector fields have widespread appearance as compared to Killing vector fields and conformal vector fields, which suggests that they may have a role not only in the geometry of a Riemannian manifolds, but also in theory of relativity as well as medical imaging via the Eikonal equation. In this paper, we concentrate on the first two examples of geodesic vector fields mentioned above. Example 1 shows that the Euclidean space $(R^n, \langle , \rangle)$ possesses a geodesic vector field, naturally raises a question: "Under what conditions does a Riemannian manifold have a geodesic vector field necessarily isometric to the Euclidean space?" A similar question is raised through Example 2 mentioned above. In this paper, we address these questions and find characterizations of the n-sphere $S^n(c)$ as well of the Euclidean space $(R^n, \langle , \rangle)$ using geodesic vector fields (cf. Theorems 1 and 2).

2. Preliminaries

Let ξ be a geodesic vector field on an n-dimensional Riemannian manifold (M, g) with potential function ρ. We denote by α the smooth 1-form dual to ξ. Then we have

$$d\alpha(X,Y) = g\left(\nabla_X \xi, Y\right) - g\left(\nabla_Y \xi, X\right), \tag{2}$$

$$\left(\pounds_\xi g\right)(X,Y) = g\left(\nabla_X \xi, Y\right) + g\left(\nabla_Y \xi, X\right), \qquad X, Y \in \mathfrak{X}(M), \tag{3}$$

where ∇ is the covariant derivative operator with respect the Riemannian connection on (M, g) and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. Note that the Lie derivative $\pounds_\xi g$ is symmetric, while the smooth 2-form $d\alpha$ is skew-symmetric, which give a symmetric operator B and a skew-symmetric operator ψ on M defined by

$$\left(\pounds_\xi g\right)(X,Y) = 2g(BX,Y), \qquad d\alpha(X,Y) = 2g(\psi X, Y).$$

Then using Equations (2) and (3), we conclude

$$\nabla_X \xi = BX + \psi X, \qquad X \in \mathfrak{X}(M). \tag{4}$$

Using the defining Equation (1) of geodesic vector field in Equation (4), we get

$$B\xi + \psi\xi = \rho\xi. \tag{5}$$

The curvature tensor field R and the Ricci tensor Ric of the Riemannian manifold (M, g), are given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{6}$$

and

$$Ric(X,Y) = \sum_{i=1}^{n} g\left(R(e_i, X)Y, e_i\right), \tag{7}$$

where $\{e_1, .., e_n\}$ is a local orthonormal frame on M. The Ricci operator Q of the Riemannian manifold (M, g) is a symmetric operator defined by

$$g(QX,Y) = Ric(X,Y), \qquad X, Y \in \mathfrak{X}(M).$$

The scalar curvature S of the Riemannian manifold is defined by $S = TrQ$ the trace of the Ricci operator Q. The gradient ∇S of the scalar curvature satisfies (cf, [30])

$$\frac{1}{2}\nabla S = \sum_{i=1}^{n} \left(\nabla Q\right)(e_i, e_i), \tag{8}$$

where the covariant derivative

$$\left(\nabla Q\right)(X,Y) = \nabla_X QY - Q\nabla_X Y.$$

Choosing $Y = Z = \xi$ in Equation (6) and using Equations (1) and (4), we conclude

$$\begin{aligned} R(X,\xi)\xi = {} & X(\rho)\,\xi + \rho\,(BX + \psi X) - (\nabla B)(\xi, X) - (\nabla \psi)(\xi, X) \\ & - B^2 X - \psi^2 X - (B\psi + \psi B)(X). \end{aligned} \tag{9}$$

Taking $X = e_i$ in above equation and the inner product with e_i, on summing the resulting equation over an orthonormal frame $\{e_1, .., e_n\}$, we get

$$Ric\,(\xi, \xi) = \xi\,(\rho) + \rho f - \xi\,(f) - \|B\|^2 + \|\psi\|^2\,, \tag{10}$$

where $f = TrB$ the trace of the symmetric operator B, we have used $Tr\psi = 0$ (ψ being skew-symmetric) and the fact that $B\psi + \psi B$ is a skew-symmetric operator and

$$\|B\|^2 = \sum_{i=1}^{n} g(Be_i, Be_i),\ \|\psi\|^2 = \sum_{i=1}^{n} g(\psi e_i, \psi e_i).$$

We associate one more smooth function $h : M \to R$ on a Riemannian manifold (M, g) to geodesic vector field ξ, defined by

$$h = \frac{1}{2}\,\|\xi\|^2\,. \tag{11}$$

Then, using Equation (4), we get the following expression for the gradient ∇h of the smooth function h,

$$\nabla h = B\xi - \psi\xi. \tag{12}$$

Note that for a smooth function $F : M \to R$ on a Riemannian manifold (M, g), the Hessian operator A_F and the Laplacian ΔF are defined by

$$A_F X = \nabla_X \nabla F, \qquad \Delta F = \mathrm{div}\nabla F, \tag{13}$$

where

$$\mathrm{div}X = \sum_{i=1}^{n} g\left(\nabla_{e_i} X, e_i\right).$$

The Hessian $Hess(F)$ is defined by

$$Hess(F)(X, Y) = g\,(A_F X, Y)\,, \qquad X, Y \in \mathfrak{X}(M). \tag{14}$$

3. A Characterization of Euclidean Spaces

In this section, we use a non-trivial geodesic vector field on a connected Riemannian manifold to find a characterization of the Euclidean spaces. We have seen through Example-1 in the introduction that the Euclidean space $(R^n, \langle,\rangle)$ admits a geodesic vector field ξ with potential function ρ a constant. Recall that a geodesic vector field ξ with potential function ρ is said to be a non-trivial geodesic vector field if ξ is nonzero and $\rho \neq 0$.

Theorem 1. *Let (M, g) be an n-dimensional complete and connected Riemannian manifold. The following two statements are equivalent:*

1. *There exists a non-trivial geodesic vector field ξ with potential function ρ with the properties that $Tr\pounds_\xi g$ is constant along the integral curves of ξ and Ricci curvature $Ric(\xi, \xi)$ satisfies*

$$Ric(\xi, \xi) \geq \frac{1}{4}\,\|d\alpha\|^2 + \frac{1}{4}\left(Tr\pounds_\xi g\right)\left(2\rho - \frac{1}{n} Tr\pounds_\xi g\right) + \xi\,(\rho)\,.$$

2. *(M, g) is isometric to Euclidean space $(R^n, \langle,\rangle)$.*

Proof. Suppose that ξ is a non-trivial geodesic vector field on the connected Riemannian manifold (M, g), such that $\xi(f) = 0$, where $f = \frac{1}{2} Tr\pounds_\xi g = TrB$ and the Ricci curvature $Ric\,(\xi, \xi)$ satisfies

$$Ric(\xi,\xi) \geq \frac{1}{4}\left\|d\alpha\right\|^2 + \frac{1}{4}\left(Tr\pounds_\xi g\right)\left(2\rho - \frac{1}{n}Tr\pounds_\xi g\right) + \xi\left(\rho\right).$$

Now, as $d\alpha(X,Y) = 2g\left(\psi X, Y\right)$, we get $\frac{1}{4}\left\|d\alpha\right\|^2 = \|\psi\|^2$ and the above inequality takes the form

$$Ric(\xi,\xi) \geq \|\psi\|^2 + f\left(\rho - \frac{1}{n}f\right) + \xi(\rho). \tag{15}$$

Using Equation (10) with $\xi(f) = 0$, we get

$$Ric(\xi,\xi) = \xi(\rho) + \rho f - \|B\|^2 + \|\psi\|^2,$$

that is,

$$\|B\|^2 - \frac{1}{n}f^2 = \|\psi\|^2 + f\left(\rho - \frac{1}{n}f\right) + \xi(\rho) - Ric(\xi,\xi). \tag{16}$$

Now, using the inequality (15) in the above equation, we conclude

$$\|B\|^2 - \frac{1}{n}f^2 \leq 0. \tag{17}$$

However, by Schwartz's inequality, we have $\|B\|^2 \geq \frac{1}{n}f^2$ and the equality holds if and only if $B = \frac{f}{n}I$. Thus, inequality (17), implies

$$B = \frac{f}{n}I. \tag{18}$$

Using Equations (5) and (18), we conclude

$$\psi\xi = \left(\rho - \frac{f}{n}\right)\xi,$$

and taking the inner product with ξ in the above equation and noting that ψ is skew-symmetric, we get

$$\left(\rho - \frac{f}{n}\right)\|\xi\|^2 = 0.$$

As ξ is non-trivial, $\|\xi\| \neq 0$ and consequently, on connected M above two equations give

$$\rho = \frac{f}{n}, \qquad \psi\xi = 0. \tag{19}$$

Combining Equations (12), (18) and (19), we conclude $\nabla h = \frac{f}{n}\xi$, which on using Equation (4), gives

$$A_h X = \frac{1}{n}X(f)\xi + \frac{f}{n}\left(\frac{1}{n}X + \psi X\right).$$

Thus, the Hessian $Hess(h)$ is given by

$$Hess(h)(X,Y) = \frac{1}{n}X(f)\alpha(Y) + \frac{f^2}{n^2}g(X,Y) + \frac{f}{n}g(\psi X,Y). \tag{20}$$

Now, using the facts that $Hess(h)$ is symmetric and the operator ψ is skew-symmetric in above equation, we conclude

$$0 = \frac{1}{n}\left(X(f)\alpha(Y) - Y(f)\alpha(X)\right) + \frac{2f}{n}g\left(\psi X, Y\right),$$

that is,

$$2f\psi X = \alpha\left(X\right)\nabla f - X(f)\xi, \qquad X \in \mathfrak{X}(M). \tag{21}$$

Taking $X = \xi$ in above equation and using Equation (19), we get $\|\xi\|^2 \nabla f = \xi(f)\xi = 0$ by the assumption in the statement. Since, ξ is non-trivial geodesic vector field, the equation $\|\xi\|^2 \nabla f = 0$ on connected M, implies $\nabla f = 0$, that is, f is a constant. Note that the constant f has to be a nonzero constant, for if $f = 0$, then Equation (19) would imply $\rho = 0$, which is a contradiction to the fact that ξ is a non-trivial geodesic vector field. Using this fact that f is a nonzero constant in Equation (21), we conclude $\psi = 0$. Hence, Equation (20), takes the form

$$Hess(h) = cg, \tag{22}$$

where c is a nonzero constant. Finally, we observe that the smooth function h is not a constant, for if not, then the Equation (12), would imply $\xi = 0$, a contradiction to the fact that ξ is a non-trivial geodesic vector field. Hence, Equation (22) on a complete and connected Riemannian manifold (M, g) implies that (M, g) is isometric to the Euclidean space $(R^n, \langle , \rangle)$ (cf. [31], Theorem 1, p. 778, [14]).

Conversely, on the Euclidean space $(R^n, \langle , \rangle)$, we have the position vector field

$$\xi = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i},$$

which satisfies $\nabla_X \xi = X$, $X \in \mathfrak{X}(R^n)$, where ∇ is the covariant derivative with respect to the Euclidean connection. Then, it follows that ξ is the non-trivial geodesic vector field with potential function $\rho = 1$ and corresponding operators $B = I$ and $\psi = 0$. Thus, $f = TrB = n$ is a constant and $Ric\,(\xi, \xi) = 0$, that is, we get

$$Ric(\xi, \xi) = \|\psi\|^2 + f\left(\rho - \frac{1}{n} f\right) + \xi(\rho),$$

which meet the requirements in the statement. □

4. A Characterization of n-Spheres

In this section, we use non-trivial geodesic vector field on a compact and connected Riemannian manifold to find a characterization of a n-sphere $S^n(c)$. Indeed we prove the following:

Theorem 2. *Let (M, g) be an n-dimensional compact and connected Riemannian manifold of positive Ricci curvature and constant scalar curvature. The following two statements are equivalent:*

1. *There exists a non-trivial geodesic vector field ξ with potential function ρ and Ricci curvature $Ric(\xi, \xi)$ satisfies*
$$\int_M Ric(\xi, \xi) \geq \int_M \left(\frac{n-1}{4n} \left(Tr\pounds_\xi g\right)^2 + \frac{1}{4} \|d\alpha\|^2 \right).$$
2. *(M, g) is isometric to n-sphere $S^n(c)$.*

Proof. Let ξ be a non-trivial geodesic vector field on an n-dimensional compact and connected Riemannian manifold (M, g) of constant scalar curvature, with potential function ρ satisfying the condition in the statement. Since, $f = TrB = \frac{1}{2} Tr\pounds_\xi g$ and $\|\psi\|^2 = \frac{1}{4}\|d\alpha\|^2$, the condition in the statement reads

$$\int_M Ric(\xi, \xi) \geq \int_M \left(\frac{n-1}{n} f^2 + \|\psi\|^2 \right). \tag{23}$$

Using Equation (4), we get $\operatorname{div}\xi = f$ and consequently,

$$\operatorname{div} f\xi = \xi(f) + f^2 \text{ and } \operatorname{div}\rho\xi = \xi(\rho) + \rho f.$$

Integrating these equations, we conclude

$$\int_M \xi(f) = -\int_M f^2 \text{ and } \int_M (\xi(\rho) + \rho f) = 0. \tag{24}$$

Now, integrating Equation (10) and using Equation (24), we get

$$\int_M \left(Ric(\xi,\xi) + \|B\|^2 - \|\psi\|^2 - f^2 \right) = 0,$$

which gives

$$\int_M \left(\|B\|^2 - \frac{1}{n} f^2 \right) = \int_M \left(\frac{n-1}{n} f^2 + \|\psi\|^2 - Ric(\xi,\xi) \right). \tag{25}$$

Next, we use the inequality (23) in the above equation, to conclude

$$\int_M \left(\|B\|^2 - \frac{1}{n} f^2 \right) \leq 0. \tag{26}$$

However, by Schwartz's inequality, we have $\|B\|^2 \geq \frac{1}{n} f^2$, that is,

$$\int_M \left(\|B\|^2 - \frac{1}{n} f^2 \right) \geq 0$$

and combining this inequality with inequality (26), we conclude

$$\int_M \left(\|B\|^2 - \frac{1}{n} f^2 \right) = 0.$$

Thus, using Schwartz's inequality, we get $\|B\|^2 = \frac{1}{n} f^2$ and this equality holds if and only if $B = \frac{1}{n} f$. Moreover, Equation (25) implies

$$\int_M \left(\left(\frac{n-1}{n} \right) f^2 + \|\psi\|^2 - Ric\,(\xi,\xi) \right) = 0. \tag{27}$$

Using $B = \frac{f}{n} I$, and following the proof of Theorem 1, through Equations (18)–(21), we conclude

$$2f\psi X = \alpha(X)\nabla f - X(f)\xi. \tag{28}$$

Taking $X = \xi$ in above equation and using $\psi\xi = 0$, we have $\|\xi\|^2 \nabla f = \xi(f)\xi$, which on taking the inner product with ∇f, gives

$$\|\xi\|^2 \|\nabla f\|^2 = (\xi(f))^2. \tag{29}$$

Using a local orthonormal frame $\{e_1, .., e_n\}$ on M, Equation (28), gives

$$4f^2 g\,(\psi e_i, \psi e_i) = g\,(\alpha(e_i)\nabla f - e_i(f)\xi, \alpha(e_i)\nabla f - e_i(f)\xi)$$

and summing these equations, leads to

$$4f^2 \|\psi\|^2 = 2\|\xi\|^2 \|\nabla f\|^2 - 2\,(\xi(f))^2.$$

Thus, using Equation (29), we conclude $f^2 \|\psi\|^2 = 0$. Note that if $f = 0$, then Equation (19), gives $\rho = 0$, which is contrary to our assumption that ξ is non-trivial geodesic vector field. Hence, on connected M equation $f^2 \|\psi\|^2 = 0$ implies that $\psi = 0$. Now, Equation (5) transforms to

$$\nabla_X \xi = \frac{f}{n} X, \qquad X \in \mathfrak{X}(M), \tag{30}$$

which on using Equation (6), gives the following expression for the curvature tensor

$$R(X,Y)\xi = \frac{1}{n}\left(X(f)Y - Y(f)X \right), \qquad X, Y \in \mathfrak{X}(M).$$

We use this equation to find

$$Ric(Y,\xi) = -\frac{n-1}{n}Y(f),$$

which gives

$$Q(\xi) = -\frac{n-1}{n}\nabla f. \tag{31}$$

Since, the scalar curvature S is a constant, we find divergence $\mathrm{div}Q\xi$ using Equations (8) and (30), a straight forward computation gives $\mathrm{div}Q\xi = \frac{f}{n}S$. Inserting this in Equation (31), we conclude

$$\Delta f = -\frac{S}{n-1}f. \tag{32}$$

Now, the Equation (30), gives $\mathrm{div}\xi = f$, that is,

$$\int_M f = 0.$$

If f is a constant, then above equation would imply $f = 0$, which we have seen above, gives a contradiction. Hence, Equation (32) suggests that the non-constant function f is an eigenfunction of the Laplace operator Δ on compact M with eigenvalue $\frac{S}{n-1}$, which confirms that the constant $S > 0$. Moreover, Equation (32), implies

$$\frac{1}{2}\Delta f^2 = f\Delta f + \|\nabla f\|^2 = \|\nabla f\|^2 - \frac{S}{n-1}f^2,$$

which, after integration, gives

$$\int_M \|\nabla f\|^2 = \frac{S}{n-1}\int_M f^2. \tag{33}$$

Next, using $\psi = 0$ in Equation (27), we have

$$\int_M Ric(\xi,\xi) = \frac{n-1}{n}\int_M f^2, \tag{34}$$

and taking the inner product with ∇f in Equation (31), we conclude

$$Ric(\nabla f,\xi) = -\frac{n-1}{n}\|\nabla f\|^2. \tag{35}$$

Recall that the Bochner's formula states that

$$\int_M \left(Ric(\nabla f,\nabla f) + \left\|A_f\right\|^2 - (\Delta f)^2\right) = 0. \tag{36}$$

We compute

$$\begin{aligned} Ric\left(\nabla f + \frac{S}{n-1}\xi, \nabla f + \frac{S}{n-1}\xi\right) &= Ric(\nabla f,\nabla f) + \frac{2S}{n-1}Ric\left(\nabla f,\xi\right) \\ &\quad + \frac{S^2}{(n-1)^2}Ric(\xi,\xi), \end{aligned}$$

which on integration and the use of Equations (34)–(36), leads to

$$\int_M Ric\left(\nabla f + \frac{S}{n-1}\xi, \nabla f + \frac{S}{n-1}\xi\right) = \int_M \left(-\left\|A_f\right\|^2 + (\Delta f)^2 - \frac{2S}{n}\|\nabla f\|^2 + \frac{S^2}{n(n-1)}f^2\right).$$

Using Equation (33) in above equation, we get

$$\int_M Ric\left(\nabla f + \frac{S}{n-1}\xi, \nabla f + \frac{S}{n-1}\xi\right) = \int_M \left(-\left\|A_f\right\|^2 + (\Delta f)^2 - \frac{S^2}{n(n-1)}f^2\right). \tag{37}$$

Note that Equation (32), gives

$$(\Delta f)^2 - \frac{S^2}{n(n-1)}f^2 = \frac{S^2}{n(n-1)^2}f^2 = \frac{1}{n}(\Delta f)^2.$$

Inserting this equation in Equation (37), leads to

$$\int_M Ric\left(\nabla f + \frac{S}{n-1}\xi, \nabla f + \frac{S}{n-1}\xi\right) = -\int_M \left(\left\|A_f\right\|^2 - \frac{1}{n}(\Delta f)^2\right).$$

In this equation, we use the facts that $Ric > 0$ and the Schwartz's inequality $\left\|A_f\right\|^2 \geq \frac{1}{n}(\Delta f)^2$, to conclude

$$\nabla f = -\frac{S}{n-1}\xi \text{ and } A_f = \frac{\Delta f}{n}I. \tag{38}$$

Taking the covariant derivative in the first equation of Equation (38) with respect to $X \in \mathfrak{X}(M)$ and using Equation (30), we get

$$\nabla_X \nabla f = -cfX, \qquad X \in \mathfrak{X}(M), \tag{39}$$

where c is a positive constant given by $S = n(n-1)c$. Note that, we have ruled out above that f can be a constant. Hence, the non-constant function f satisfies the Obata's differential Equation (39) (cf. [26]) and consequently, the Riemannian manifold (M, g) is isometric to the sphere $S^n(c)$.

Conversely, if (M, g) is isometric to $S^n(c)$, then the Ricci curvature for any smooth vector field X on $S^n(c)$ is given by $Ric(X, X) = (n-1)c\,\|X\|^2$. We treat $S^n(c)$ as hypersurface of the Euclidean space $\left(R^{n+1}, \langle,\rangle\right)$ with unit normal vector field N and the shape operator $A = -\sqrt{c}I$. Now, choosing a nonzero constant vector field $w \in \mathfrak{X}(R^{n+1})$, we express its restriction to the sphere $S^n(c)$ as $w = \xi + sN$, where ξ is tangential component of w to $S^n(c)$ and $s = \langle w, N\rangle$ is the smooth function on $S^n(c)$. Taking covariant derivative with respect to $X \in \mathfrak{X}(S^n(c))$ of the equation $w = \xi + sN$ and using Gauss and Weingarten formulas for the hypersurface, we get

$$0 = \nabla_X \xi - \sqrt{c}g(X, \xi)N + X(s)N + \sqrt{c}sX.$$

Equating tangential and normal components in the above equation, we get

$$\nabla_X \xi = -\sqrt{c}sX, \qquad \nabla s = \sqrt{c}\xi. \tag{40}$$

The first equation in Equation (40) gives $\nabla_\xi \xi = \rho\xi$, where $\rho = -\sqrt{c}s$. This proves that ξ is a geodesic vector field with potential function ρ. Suppose $\rho = 0$, this will mean $s = 0$ and consequently, the second equation in Equation (40) will imply that $\xi = 0$. Thus, $w = 0$ on $S^n(c)$, but as w is a constant vector field, we get $w = 0$ on R^{n+1}, contrary to our assumption that w is a nonzero constant vector field. Hence, $\rho \neq 0$. Similarly, we can show that ξ is a nonzero vector field. Hence, ξ is a non-trivial geodesic vector field on $S^n(c)$. Next, by second equation in the Equation (40), we have $c\,\|\xi\|^2 = \|\nabla s\|^2$, and that

$$\int_{S^n(c)} Ric(\xi, \xi) = (n-1)c\int_{S^n(c)} \|\xi\|^2 = (n-1)\int_{S^n(c)} \|\nabla s\|^2. \tag{41}$$

Also, by the first equation in (40), for the geodesic vector field ξ, the operators B and ψ are $B = -\sqrt{c}sI$ and $\psi = 0$, and that $f = TrB = -n\sqrt{c}s$. Moreover, using Equation (40), we find $\text{div}\xi = -n\sqrt{c}s$ and $\Delta s = -ncs$. Thus, we get

$$\int_{S^n(c)} \|\nabla s\|^2 = nc \int_{S^n(c)} s^2 = \frac{1}{n} \int_{S^n(c)} f^2. \tag{42}$$

Finally, using Equations (41) and (42), we conclude

$$\int_{S^n(c)} Ric(\xi,\xi) = \frac{n-1}{n} \int_{S^n(c)} f^2 = \int_{S^n(c)} \left(\frac{n-1}{n} f^2 + \|\psi\|^2 \right),$$

which is the equation in (23), that is, all the requirements in the statement are met. □

Author Contributions: Conceptualization, S.D., P.P. and N.B.T.; methodology, S.D. and P.P.; software, S.D. and N.B.T.; validation, S.D. and P.P.; formal analysis, P.P.; investigation, S.D. and P.P.; resources, S.D.; data curation, S.D.; writing—original draft preparation, S.D. and N.B.T.; writing—review and editing, S.D. and N.B.T.; visualization, N.B.T.; supervision, S.D. and P.P.; project administration, N.B.T.; funding acquisition, N.B.T. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciations to the Deanship of Scientific Research King Saud University for funding this work through research group no (RG-1440-142).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mikeš, J.; Vanžurová, A. Reconstruction of an affine connection in generalized Fermi coordinates. *Bull. Malays. Math. Sci. Soc.* **2017**, *40*, 205–213. [CrossRef]
2. Fialkow, A. Conformal geodesics. *Trans. Amer. Math. Soc.* **1939**, *45*, 443–473. [CrossRef]
3. Yano, K. Concircular Geometry. *Proc. Imp. Acad. Tokyo* **1940**, *16*, 195–200. [CrossRef]
4. Takeno, H. Concircular scalar field in spherically symmetric space-times I. *Tensor* **1967**, *20*, 167–176.
5. Evtushik, L.E.; Hinterleitner, I.; Guseva, N.I.; Mikeš, J. Conformal mappings onto Einstein spaces. *Russ. Math.* **2016**, *60*, 5–9. [CrossRef]
6. Hinterleitner, I.; Guseva, N.; Mikeš, J. On conformal mappings onto compact Einstein manifolds. *Geom. Integr. Quantization* **2018**, *19*, 132–139. [CrossRef]
7. Mikeš, J. Geodesic mappings of affine-connected and Riemannian spaces. *J. Math. Sci.* **1996**, *78*, 311–333. [CrossRef]
8. Mikeš, J.; Chodorová, M. On concircular and torse-forming vector fields on compact manifolds. *Acta Math. Acad. Paedagog. Nyházi.* **2010**, *26*, 329–335.
9. Mikeš, J.; Hinterleitner, I.; Guseva, N. There are no conformal Einstein rescalings of pseudo-Riemannian Einstein spaces with n complete light-like geodesics. *Mathematics* **2019**, *7*, 801. [CrossRef]
10. Mikeš, J.; Škodová, M. Concircular vector fields on compact manifolds with affine connections. *Publ. Real Soc. Mat. Esp.* **2007**, *11*, 304–309.
11. Chow, B.; Lu, P.; Ni, L. *Hamilton's Ricci Flow, Graduate Studies in Mathematics*; AMS Scientific Press: New York, NY, USA, 2010; Volume 77.
12. Deshmukh, S.; Chen, B. A note on Yamabe solitons. *Balkan J. Geom. Appl.* **2018**, *23*, 37–42.
13. Yorozu, S. Killing vector fields on noncompact Riemannian manifolds with boundary. *Kodai Math. J.* **1982**, *5*, 426–433. [CrossRef]
14. Yvonne, C.B.; Celicle, D. *Analysis, Manifolds, Physics*; Elsevier: Amsterdam, The Netherlands, 1977.
15. Mosayebi, P.; Kobzas, D.; Murtha, A.; Jagersand, M. Tumor invasion margin on Riemannian space of brain fibers. *Med. Image Anal.* **2012**, *16*, 361–373. [CrossRef] [PubMed]
16. Berestovskii, V.; Nikonorov, Y. Killing Vector Fields of Constant Length on Riemannian Manifolds. *Siberian Math. J.* **2008**, *49*, 395–407. [CrossRef]
17. Deshmukh, S. Conformal Vector Fields and Eigenvectors of Laplacian Operator. *Math. Phys. Anal. Geom.* **2012**, *15*, 163–172. [CrossRef]

18. Deshmukh, S.; Al-Solamy, F. Conformal gradient vector fields on a compact Riemannian manifold. *Colloq. Math.* **2008**, *112*, 157–161. [CrossRef]
19. Deshmukh, S. Jacobi-type vector fields and Ricci soliton. *Bull. Math. Soc. Sci. Math. Roum.* **2012**, *55*, 41–50.
20. Deshmukh, S.; Khan, V.A. Geodesic vector fields and Eikonal equation on a Riemannian manifold. *Indag. Math.* **2019**, *30*, 542–552. [CrossRef]
21. Deshmukh, S.; Turki, N.B. A note on φ-analytic conformal vector fields. *Anal. Math. Phys.* **2019**, *9*, 181–195. [CrossRef]
22. Deshmukh, S. Characterizing spheres and Euclidean spaces by conformal vector field. *Ann. Mat. Pura. Appl.* **2017**, *196*, 2135–2145. [CrossRef]
23. Mikes, J.; Stepanova, E.; Vanzurova, E. *Differential Geometry of Special Mappings*; Palacký University: Olomouc, Czech Republic, 2015.
24. Mikes, J. Equidistant Kaehler spaces. *Math. Notes* **1985**, *38*, 855–858. [CrossRef]
25. Lynge, W.C. Sufficient conditions for periodicity of a Killing vector field. *Proc. Am. Math. Soc.* **1973**, *38*, 614–616. [CrossRef]
26. Obata, M. Conformal transformations of Riemannian manifolds. *J. Diff. Geom.* **1970**, *4*, 311–333. [CrossRef]
27. Obata, M. The conjectures about conformal transformations. *J. Diff. Geom.* **1971**, *6*, 247–258. [CrossRef]
28. Xiaochun, R. Positive curvature, local and global symmetry, and fundamental groups. *Am. J. Math.* **1999**, *121*, 931–943.
29. Yorozu, S. Killing vector fields on complete Riemannian manifolds. *Proc. Am. Math. Soc.* **1982**, *84*, 115–120. [CrossRef]
30. Besse, A.L. *Einstein Manifolds*; Springer: Berlin, Germany, 1987.
31. Pigola, S.; Rimoldi, M.; Setti, A.G. Remarks on non-compact gradient Ricci solitons. *Math. Z.* **2011**, *268*, 777–790. [CrossRef]