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The Trapezoidal Fuzzy Two-Dimensional Linguistic Power Generalized Hamy Mean Operator and Its Application in Multi-Attribute Decision-Making

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Abstract: As a common information aggregation tool, the Hamy mean (HM) operator can consider the relationships among multiple input elements, but cannot adjust the effect of elements. In this paper, we integrate the idea of generalized a weighted average (GWA) operator into the HM operator, and reduce the influence of related elements by adjusting the value of the parameter. In addition, considering that extreme input data may lead to a deviation in the results, we further combine the power average (PA) operator with HM, and propose the power generalized Hamy mean (PGHM) operator. Then, we extend the PGHM operator to the trapezoidal fuzzy two-dimensional linguistic environment, and propose two new information aggregation tools, the trapezoidal fuzzy two-dimensional linguistic power generalized Hamy mean (TF2DLPGHM) operator and the weighted TF2DLPGHM (WTF2DLPGHM) operator. Some properties and special cases of these operators are discussed. Furthermore, based on the proposed WTF2DLPGHM operator, a new multi-attribute decision-making method is proposed for lean management evaluation of industrial residential projects. Finally, an example is given to show the specific steps, effectiveness, and superiority of the method.

Keywords: trapezoidal fuzzy two-dimensional linguistic variables; power generalized Hamy mean operator; multi-attribute decision-making

1. Introduction

A classic multi-attribute decision-making (MADM) problem can be described as follows: given a group of possible alternatives in advance, with the help of certain information aggregation tool, evaluate these alternatives from the perspective of multiple attributes. The purpose of decision-making is to find the most satisfactory solution from this group of alternatives. With the progress of human society, the complexity and fuzziness of decision-making environments are also increasing, which makes it difficult to deal with realistic decision-making problems with crisp numbers. In order to solve this problem, Zadeh [1] proposed the fuzzy sets (FSs) theory, which has been widely studied as an effective tool to express fuzzy and uncertain information. Later, many scholars put forward many new extended forms based on the theory to describe uncertain information more accurately, such as triangular fuzzy numbers [2], trapezoidal fuzzy numbers [3], intuitionistic fuzzy sets [4,5], interval-valued intuitionistic fuzzy sets [6,7], type-2 fuzzy sets [8], hesitant fuzzy sets [9], and so on. In view of the convenience of these fuzzy theories for expressing uncertain information, many prospective research results have been obtained [10–14].

All of these fuzzy information forms have a common feature: Decision makers (DMs) can only express their preference information quantitatively with a few crisp numbers. However, because of the hesitation and ambiguity involved in human cognitive thinking, qualitative information representation is easier than quantitative information representation. In 1975, the fuzzy linguistic evaluation method

based on linguistic variables was proposed by Zadeh [15], which allows DMs to express preference information qualitatively through single linguistic terms. With the progress in research, more advanced qualitative information representation models, such as uncertain linguistic variables [16], two-dimensional linguistic variables (2DLVs) [17], linguistic intuitionistic fuzzy numbers [18,19], and hesitant fuzzy linguistic term sets [20] have been proposed; they allow DMs to express evaluation information with two or more linguistic terms. In particular, the 2DLVs proposed by Zhu et al. [17] contain two kinds of linguistic information. The evaluation information of the first dimension is the evaluation values of alternatives given by DMs, while the second dimension is used to describe the DMs' self-judgment for the reliability of the evaluation value. On the whole, two-dimensional linguistic evaluation information covers more complete cognitive preferences, which helps DMs to express their opinions more accurately.

To further improve the accuracy of qualitative linguistic information representation, Liu [21] extended 2DLVs to two-dimensional uncertain linguistic variables (2DULVs), which is one of the most important research results. Since 2DLVs and 2DULVs were put forward, they have been widely studied and have produced many useful research results. Using the MADM method, Liu and Teng [22] extended the TODIM method to a two-dimensional uncertain linguistic environment to deal with MADM problems. Ding and Liu [23] proposed the two-dimensional uncertain linguistic DEMATEL method to determine the critical success factors of emergency management. Zhao et al. [24] extended the PROMETHEE method to a two-dimensional linguistic environment and proposed the possibility degree for 2DLVs. Wu et al. [25] proposed a risk assessment framework of waste-to-energy projects by combining the cloud model and 2DLVs. Using an aggregation operator, Liu et al. [26] extended the GWA operator and hybrid GWA operator to a two-dimensional uncertain linguistic environment to evaluate the decision-making problems in this situation. Considering the possible relationships between elements, Chu and Liu [27] proposed a new weighted generalized Heronian mean operator and weighted generalized geometric Heronian mean operator for 2DULVs. Furthermore, Liu et al. [28] studied an MSM operator in the two-dimensional uncertain linguistic environment and proposed two new MSM operators. In addition, some scholars put forward a new two-dimensional linguistic information form based on 2DLVs. Li et al. [29] proposed trapezoidal fuzzy two-dimensional linguistic variables (TF2DLVs), in which the first dimension is represented by trapezoidal fuzzy numbers, and developed two power generalized aggregation operators based on this information form. Yin et al. [30] further studied a partitioned Bonferroni mean operator in the two-dimensional uncertain linguistic environment to describe the relationships between elements. Liu [31] proposed two-dimensional uncertain linguistic generalized normalized weighted geometric Bonferroni mean.

TF2DLVs are an extended two-dimensional linguistic information form, in which the first dimension is represented by trapezoidal fuzzy numbers, and the second dimension is represented by linguistic terms. Here, the second dimension of linguistic information is used to describe the subjective confidence of DMs for giving the first-dimension evaluation information. Therefore, compared with 2DLVs, TF2DLVs represent a fusion of quantitative and qualitative evaluation information, covering more specific evaluation information, which can effectively avoid the loss of evaluation information. At present, many studies focus on 2DLVs and 2DULVs, while few focus on TF2DLVs. Therefore, this paper will further study TF2DLVs and propose a new MADM method in the trapezoidal fuzzy two-dimensional linguistic environment.

Information aggregation operators that consider the relationships between elements, such as the Heronian mean, Bonferroni mean, Maclaurin symmetric mean, and Hamy mean (HM) [32], have been widely used in various information representation models. However, the Bonferroni mean and Heronian mean can only describe the relationship between two input elements. In some decision-making scenarios, there may be relationships among multiple input elements. As a result, they cannot adapt to this decision-making situation. Instead, the HM operator can flexibly simulate the relationship among multiple input elements. In addition, the power average (PA) [33] operator is another powerful information aggregation tool. The PA operator can eliminate the influence of

unreasonable data on the sorting results by giving low weight to extreme data [34,35]. Considering the excellent characteristics of the PA operator and HM operator processing information, recently Liu et al. [36] put forward a power Hamy mean (PHM) operator by combining PA and HM operators. However, HM operator and PHM operator give the same importance to the attributes of interactive operation, so they cannot function like a Bonferroni mean operator, which can enlarge or reduce the influence of attributes by adjusting the parameters.

To make up for this deficiency, this paper integrates the idea of a GWA operator into the HM operator to flexibly adjust the impact of multiple elements. Then, combining it with a PA operator, the power generalized Hamy mean (PGHM) operator is proposed. Furthermore, we extend the PGHM operator to the trapezoidal fuzzy two-dimensional linguistic environment, and propose two new information aggregation tools, the trapezoidal fuzzy two-dimensional linguistic power generalized Hamy mean (TF2DLPGHM) operator and the weighted TF2DLPGHM (WTF2DLPGHM) operator. The proposed operators have three excellent properties. (1) They can integrate the characteristics of the PA operator, which can eliminate the influence of unreasonable data on the sorting results. (2) They can model the relationship among multiple input elements. (3) They can flexibly enlarge or reduce the influence of related elements by adjusting the parameters. Finally, based on the proposed WTF2DLPGHM operator, a new MADM method is proposed to solve the problem of lean management evaluation of industrial residential projects.

The structure of this article is as follows. In Section 2, some related concepts are explained. Section 3 proposes the PGHM operator and extends it to the trapezoidal fuzzy two-dimensional linguistic environment. Then, two new trapezoidal fuzzy two-dimensional linguistic information aggregation operators, i.e., the TF2DLPGHM operator and WTF2DLPGHM, are developed. In Section 4, based on the proposed WTF2DLPGHM operator, we develop a new method for solving MADM problems wherein attribute values are expressed in the form of TF2DLVs. In Section 5, an example is given to demonstrate the effectiveness and superiority of the method. Section 6 gives some conclusions and future research directions.

2. Preliminaries

In this section, we first introduce some basic concepts related to the work done in this paper, such as trapezoidal fuzzy numbers, linguistic term sets, trapezoidal fuzzy two-dimensional linguistic variables, the power average operator, the generalized aggregation operator, and the Hamy mean operator.

2.1. Trapezoidal Fuzzy Numbers

Definition 1 [3]. A trapezoidal fuzzy number (TRFN) can be defined as $\tilde{\sigma} = (\sigma^L, \sigma^{ML}, \sigma^{MU}, \sigma^U)$, and the membership degree function $\sigma(x): R \in [0, 1]$ is defined as follows:

$$\sigma(x) = \begin{cases} \frac{x - \sigma^L}{\sigma^{ML} - \sigma^L}, & x \in (\sigma^L, \sigma^{ML}) \\ 1, & x \in (\sigma^{ML}, \sigma^{MU}) \\ \frac{\sigma^U - x}{\sigma^U - \sigma^{MU}}, & x \in (\sigma^{MU}, \sigma^U) \\ 0, & x \in (-\infty, \sigma^L) \cup (\sigma^U, +\infty) \end{cases}, \quad (1)$$

where $\sigma^L \leq \sigma^{ML} \leq \sigma^{MU} \leq \sigma^U$. The element $x \in R$ of $\tilde{\sigma}$ is a real number, and its membership function $\sigma(x)$ is the regularly and continuous convex function, which indicates the degree to which element x belongs to the fuzzy set $\tilde{\sigma}$. In particular, when $\sigma^L < \sigma^{ML} = \sigma^{MU} < \sigma^U$, the TRFN $\tilde{\sigma}$ degenerates into a triangular fuzzy number; when $\sigma^L = \sigma^{ML} = \sigma^{MU} = \sigma^U$, the TRFN $\tilde{\sigma}$ degenerates into a real number.

Let $\tilde{\sigma} = (\sigma^L, \sigma^{ML}, \sigma^{MU}, \sigma^U)$ and $\tilde{o} = (o^L, o^{ML}, o^{MU}, o^U)$ be two TRFNs, then the operational rules between $\tilde{\sigma}$ and \tilde{o} are defined as follows [3,37]:

$$\tilde{\sigma} + \tilde{o} = (\sigma^L + o^L, \sigma^{ML} + o^{ML}, \sigma^{MU} + o^{MU}, \sigma^U + o^U); \quad (2)$$

$$\tilde{\sigma} - \tilde{o} = (\sigma^L - o^U, \sigma^{ML} - o^{MU}, \sigma^{MU} - o^{ML}, \sigma^U - o^L); \quad (3)$$

$$\tilde{\sigma}\tilde{o} = (\alpha^L\beta^L, \alpha^{ML}\beta^{ML}, \alpha^{MU}\beta^{MU}, \alpha^U\beta^U); \quad (4)$$

$$\tilde{\sigma}/\tilde{o} = (\sigma^L/o^U, \sigma^{ML}/o^{MU}, \sigma^{MU}/o^{ML}, \sigma^U/o^L); \quad (5)$$

$$k\tilde{\sigma} = (k\sigma^L, k\sigma^{ML}, k\sigma^{MU}, k\sigma^U), \quad k \geq 0; \quad (6)$$

$$(\tilde{\sigma})^k = ((\sigma^L)^k, (\sigma^{ML})^k, (\sigma^{MU})^k, (\sigma^U)^k), \quad k \geq 0. \quad (7)$$

The distance between $\tilde{\sigma}$ and \tilde{o} can be expressed as follows:

$$d(\tilde{\sigma}, \tilde{o}) = \sqrt{[(\sigma^L - o^L)^2 + (\sigma^{ML} - o^{ML})^2 + (\sigma^{MU} - o^{MU})^2 + (\sigma^U - o^U)^2]}/4. \quad (8)$$

2.2. Linguistic Term Sets

Suppose $S = \{\varsigma_0, \varsigma_1, \dots, \varsigma_{g-1}\}$ is a discrete linguistic term set (LTS) containing an odd number of linguistic terms. Generally speaking, when g is 5 or 7, it can be well adapted to the psychological cognition of DMs. In particular, if g takes a value of 5, the LTS S can be expressed as $S = \{\varsigma_0 = \text{very uncertain}, \varsigma_1 = \text{uncertain}, \varsigma_2 = \text{medium}, \varsigma_3 = \text{certain}, \varsigma_4 = \text{very certain}\}$. For \forall LTS S , ς_σ and ς_o should satisfy the following conditions [38]:

- (1) The set is ordered: $\varsigma_\sigma < \varsigma_o$, if and only if $\sigma < o$;
- (2) There is the negation operator: $\text{neg}(\varsigma_\sigma) = \varsigma_o$, such that $o = g - 1 - \sigma$;
- (3) If $\sigma \leq o$, then $\max\{\varsigma_\sigma, \varsigma_o\} = \varsigma_o$ and $\min\{\varsigma_\sigma, \varsigma_o\} = \varsigma_\sigma$.

Considering the loss of information that may occur in the calculation process, Xu [39] further proposed the continuous LTS $\hat{S} = \{\varsigma_\sigma | \sigma \in R^+\}$, which still satisfies the above conditions. Note that $\varsigma_\sigma \in S$ is usually used to evaluate alternatives; $\varsigma_\sigma \in \bar{S}$ only appears in the calculation process.

2.3. Trapezoidal Fuzzy Two-Dimensional Linguistic Variables

TF2DLVs are an extended two-dimensional linguistic information form, in which the first dimension is represented by trapezoidal fuzzy numbers, and the second dimension is represented by a linguistic term. Here, the second-dimension linguistic information is used to describe the subjective confidence of DMs for giving the first-dimension evaluation information. Therefore, TF2DLVs cover more specific evaluation information and can effectively avoid the loss of evaluation information.

Definition 2 [29]. Let $\tilde{\alpha} = (\alpha^L, \alpha^{ML}, \alpha^{MU}, \alpha^U)$ be a TRFN and ς_t be a linguistic term. If $\tilde{\alpha}$ is used to express the evaluation value of alternative given by the DM and ς_t to indicate the reliability of the evaluation value given by the DM, then a TF2DLV $\tilde{\zeta} = ([\alpha^L, \alpha^{ML}, \alpha^{MU}, \alpha^U], \varsigma_t)$ can be obtained.

Let $\tilde{\zeta}_1 = ([\alpha_1^L, \alpha_1^{ML}, \alpha_1^{MU}, \alpha_1^U], \varsigma_{t_1})$ and $\tilde{\zeta}_2 = ([\alpha_2^L, \alpha_2^{ML}, \alpha_2^{MU}, \alpha_2^U], \varsigma_{t_2})$ be any two TF2DLVs and $\alpha_1^L, \alpha_1^{ML}, \alpha_1^{MU}, \alpha_1^U, \alpha_2^L, \alpha_2^{ML}, \alpha_2^{MU}, \alpha_2^U \geq 0$, then the operational rules between $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ are expressed as follows:

$$\tilde{\zeta}_1 \oplus \tilde{\zeta}_2 = ([\alpha_1^L + \alpha_2^L, \alpha_1^{ML} + \alpha_2^{ML}, \alpha_1^{MU} + \alpha_2^{MU}, \alpha_1^U + \alpha_2^U], \varsigma_{\min(t_1, t_2)}); \quad (9)$$

$$\tilde{\zeta}_1 \otimes \tilde{\zeta}_2 = ([\alpha_1^L \alpha_2^L, \alpha_1^{ML} \alpha_2^{ML}, \alpha_1^{MU} \alpha_2^{MU}, \alpha_1^U \alpha_2^U], \varsigma_{\min(t_1, t_2)}); \quad (10)$$

$$\tilde{\zeta}_1 / \tilde{\zeta}_2 = ([\alpha_1^L / \alpha_2^L, \alpha_1^{ML} / \alpha_2^{ML}, \alpha_1^{MU} / \alpha_2^{MU}, \alpha_1^U / \alpha_2^U], \varsigma_{\min(t_1, t_2)}), \text{ where} \quad (11)$$

$$\alpha_2^L, \alpha_2^{ML}, \alpha_2^{MU}, \alpha_2^U \neq 0$$

$$k\tilde{\zeta}_1 = ([k\alpha_1^L, k\alpha_1^{ML}, k\alpha_1^{MU}, k\alpha_1^U], \varsigma_{t_1}), \quad k \geq 0; \quad (12)$$

$$(\zeta_1)^k = \left(\left[(\alpha_1^L)^k, (\alpha_1^{ML})^k, (\alpha_1^{MU})^k, (\alpha_1^U)^k \right], \varsigma_{t_1} \right), k \geq 0. \quad (13)$$

For any three TF2DLVs ζ_1 , ζ_2 and ζ_3 , they have the following relationships:

- (1) $\zeta_1 \oplus \zeta_2 = \zeta_2 \oplus \zeta_1$;
- (2) $\zeta_1 \otimes \zeta_2 = \zeta_2 \otimes \zeta_1$;
- (3) $\zeta_1 \oplus \zeta_2 \oplus \zeta_3 = \zeta_1 \oplus (\zeta_2 \oplus \zeta_3)$;
- (4) $\zeta_1 \otimes \zeta_2 \otimes \zeta_3 = \zeta_1 \otimes (\zeta_2 \otimes \zeta_3)$;
- (5) $\zeta_1 \otimes (\zeta_2 \oplus \zeta_3) = (\zeta_1 \otimes \zeta_2) \oplus (\zeta_1 \otimes \zeta_3)$;
- (6) $k_1(\zeta_1 \oplus \zeta_2) = k_1\zeta_2 \oplus k_1\zeta_1, k_1 \geq 0$;
- (7) $(k_2 + k_3)\zeta_1 = k_2\zeta_1 \oplus k_3\zeta_1, k_2, k_3 \geq 0$.

Definition 3 [29]. Let $\zeta = \left(\left[\alpha^L, \alpha^{ML}, \alpha^{MU}, \alpha^U \right], \varsigma_t \right)$ be a TF2DLV, then the expected value of ζ is expressed as follows:

$$EX(\zeta) = \frac{\alpha^L + \alpha^{ML} + \alpha^{MU} + \alpha^U}{4} \times \frac{t}{g-1} \quad (14)$$

For any two TF2DLVs ζ_1 and ζ_2 , if $EX(\zeta_1) \geq EX(\zeta_2)$, then $\zeta_1 \geq \zeta_2$, and vice versa.

Definition 4 [29]. Let $\zeta_1 = \left(\left[\alpha_1^L, \alpha_1^{ML}, \alpha_1^{MU}, \alpha_1^U \right], \varsigma_{t_1} \right)$ and $\zeta_2 = \left(\left[\alpha_2^L, \alpha_2^{ML}, \alpha_2^{MU}, \alpha_2^U \right], \varsigma_{t_2} \right)$ be any two TF2DLVs, the distance between ζ_1 and ζ_2 can be expressed as follows:

$$d(\zeta_1, \zeta_2) = \frac{1}{2(g-1)} \sqrt{(t_1\alpha_1^L - t_2\alpha_2^L)^2 + (t_1\alpha_1^{ML} - t_2\alpha_2^{ML})^2 + (t_1\alpha_1^{MU} - t_2\alpha_2^{MU})^2 + (t_1\alpha_1^U - t_2\alpha_2^U)^2} \quad (15)$$

2.4. Power Average Operator

Definition 5 [33]. Let $v_t (t = 1, 2, \dots, n)$ be a set of nonnegative real numbers, then the PA operator is expressed as follows:

$$PA(v_1, v_2, \dots, v_n) = \frac{\sum_{t=1}^n (1 + G(v_t))v_t}{\sum_{t=1}^n (1 + G(v_t))}, \quad (16)$$

where

$$G(v_t) = \sum_{l=1, l \neq t}^n \sup(v_t, v_l), \sup(v_t, v_l) = 1 - d(v_t, v_l). \quad (17)$$

$\sup(v_t, v_l)$ represents the support degree from v_t to v_l and satisfies the following properties:

- (1) $\sup(v_t, v_l) \in [0, 1]$;
- (2) $\sup(v_t, v_l) = \sup(v_l, v_t)$;
- (3) $\sup(v_t, v_l) \geq \sup(v_k, v_r)$, if $|v_t - v_l| < |v_k - v_r|$, where $1 \leq t, l, k, r \leq n$.

2.5. Generalized Aggregation Operator

Definition 6 [40]. Let $v_t (t = 1, 2, \dots, n)$ be a set of nonnegative real numbers, then the GWA operator is expressed as follows:

$$GWA(v_1, v_2, \dots, v_n) = \left(\sum_{t=1}^n \eta_t v_t^k \right)^{\frac{1}{k}}, \quad (18)$$

where $\eta = (\eta_1, \eta_2, \dots, \eta_n)^T$ is the weighting vector of $v_t (t = 1, 2, \dots, n)$, $\eta_t \in [0, 1]$ and $\sum_{t=1}^n \eta_t = 1$. k is a parameter and its range is $k \in (-\infty, 0) \cup (0, +\infty)$.

2.6. Hamy Mean Operator

Definition 7 [32]. Let $v_t (t = 1, 2, \dots, n)$ be a set of nonnegative real numbers, then the HM operator is expressed as follows:

$$HM^{(\tau)}(v_1, v_2, \dots, v_n) = \frac{\sum_{1 \leq t_1 < \dots < t_\tau \leq n} \left(\prod_{r=1}^{\tau} v_{t_r} \right)^{\frac{1}{\tau}}}{C_n^\tau}, \quad (19)$$

where $\tau (\tau = 1, 2, \dots, n)$ is a parameter of HM operator, and $(t_1, t_2, \dots, t_\tau)$ is τ -type full combination of $(1, 2, \dots, n)$. C_n^τ is called the binomial coefficient, where $C_n^\tau = \frac{n!}{\tau!(n-\tau)!}$ and $1 \leq t_1 < \dots < t_\tau \leq n$.

As seen from Definition 7, the greatest advantage of the HM operator is that it can describe the relationship among more than two input elements. In addition, it has the following ideal properties:

- (1) If $v_t = 0 (t = 1, 2, \dots, n)$, then $HM^{(\tau)}(0, 0, \dots, 0) = 0$;
- (2) If $v_t = v (t = 1, 2, \dots, n)$, then $HM^{(\tau)}(v, v, \dots, v) = v$;
- (3) If $v_t \leq \chi_t (t = 1, 2, \dots, n)$, then $HM^{(\tau)}(v_1, v_2, \dots, v_n) \leq HM^{(\tau)}(\chi_1, \chi_2, \dots, \chi_n)$;
- (4) $\min_t(v_t) \leq HM^{(\tau)}(v_1, v_2, \dots, v_n) \leq \max_t(v_t)$.

3. The Trapezoidal Fuzzy Two-Dimensional Linguistic Aggregation Operator

Compared with a Bonferroni mean operator, the HM operator can model the relationship of multiple input elements, but the attributes are given the same importance in the process of describing the attribute interaction. On the contrary, the BM operator can flexibly enlarge or reduce the influence of related elements by adjusting the values of parameters p and q . Thus, in order to make up for this deficiency, we propose the power generalized Hamy mean (PGHM) operator. The proposed PGHM operator has three functions. One is that by adjusting the value of parameter τ , the PGHM operator can model the relationship among multiple elements; the other is that by adjusting the value of parameter k , the effect of multiple elements can be enlarged or reduced; and the third is that it can eliminate the influence of unreasonable data on sorting results by giving low weight to extreme data. The mathematical expression of the PGHM operator is as follows.

3.1. The Power Generalized Hamy Mean Operator

Definition 8. Let $v_i (i = 1, 2, \dots, n)$ be a set of nonnegative real numbers and $k_1, k_2, \dots, k_\tau \geq 0$, then the PGHM operator can be expressed as follows:

$$PGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(v_1, v_2, \dots, v_n) = \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} \left(\frac{n(1+G(v_{i_r}))}{\sum_{i=1}^n (1+G(v_i))} v_{i_r} \right)^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}}{C_n^\tau}, \quad (20)$$

where $\tau (\tau = 1, 2, \dots, n)$ is a parameter of the PGHM operator, $(i_1, i_2, \dots, i_\tau)$ is a τ -tuple full combination of $(1, 2, \dots, n)$, and $C_n^\tau = \frac{n!}{\tau!(n-\tau)!}$ is called the binomial coefficient. Here $G(v_i) = \sum_{l=1, l \neq i}^n \sup(v_i, v_l)$, in which $\sup(v_i, v_l)$ represents the support degree from v_i to v_l . In addition, it is easy to prove that the PHM operator satisfies the basic properties of the HM operator.

In the following, we introduce a PGHM operator into the trapezoidal fuzzy two-dimensional linguistic environment, and further propose the trapezoidal fuzzy two-dimensional linguistic power generalized Hamy mean (TF2DLPGHM) operator and its weighted form (WTF2DLPGHM).

3.2. The TF2DLPGHM Operator

Definition 9. Let $\xi_i = ([a_i, b_i, c_i, d_i], \varsigma_{t_i}) (i = 1, 2, \dots, n)$ be a set of TF2DLVs, then the TF2DLPGHM operator can be expressed as follows:

$$TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\xi_1, \xi_2, \dots, \xi_n) = \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} \left(\frac{n(1+G(\xi_{i_r}))}{\sum_{i=1}^n (1+G(\xi_i))} \xi_{i_r} \right)^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}}{C_n^\tau}, \quad (21)$$

where $\tau (\tau = 1, 2, \dots, n)$ is a parameter of the TF2DLPGHM operator, $(i_1, i_2, \dots, i_\tau)$ is a τ -tuple full combination of $(1, 2, \dots, n)$, and $C_n^\tau = \frac{n!}{\tau!(n-\tau)!}$ is called the binomial coefficient. Here $G(\xi_i) = \sum_{l=1, l \neq i}^n \sup(\xi_i, \xi_l)$, in which $\sup(\xi_i, \xi_l)$ represents the support degree from ξ_i to ξ_l .

Let $\varepsilon_i = \frac{1+G(\xi_i)}{\sum_{i=1}^n (1+G(\xi_i))}$, obviously, $\sum_{i=1}^n \varepsilon_i = 1$. The definition of the TF2DLPGHM operator is equivalent to the following form:

$$TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\xi_1, \xi_2, \dots, \xi_n) = \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} \xi_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}}{C_n^\tau}. \quad (22)$$

Theorem 1. Let $\xi_i = ([\alpha_i^L, \alpha_i^{ML}, \alpha_i^{MU}, \alpha_i^U], \varsigma_{t_i}) (i = 1, 2, \dots, n)$ be a group of TF2DLVs, then the expansion of TF2DLPGHM operator is still a TF2DLV, then

$$TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\xi_1, \xi_2, \dots, \xi_n) = \left(\left[\frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} a_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} b_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \right. \right. \\ \left. \left. \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} c_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} d_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}} \right], \varsigma_{\min(t_i)} \right). \quad (23)$$

The detailed derivation of Equation (23) is shown in Appendix A.

Next, we will discuss some ideal properties of the TF2DLPGHM operator.

Property 1 (Idempotency). Let $\zeta = ([a, b, c, d], \varsigma_t)$ and $\zeta_i = ([a_i, b_i, c_i, d_i], \varsigma_{t_i}) (i = 1, 2, \dots, n)$ be two groups of TF2DLVs. When $\zeta = \zeta_i (i = 1, 2, \dots, n)$, we have

$$TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\zeta_1, \zeta_2, \dots, \zeta_n) = \zeta.$$

Proof. When $\zeta = \zeta_i (i = 1, 2, \dots, n)$, we can get $n\varepsilon_{i_r} = \frac{n(1+G(\zeta_{i_r}))}{\sum_{i=1}^n (1+G(\zeta_i))} = \frac{n(1+G(\zeta))}{\sum_{i=1}^n (1+G(\zeta))} = 1$. Then,

$$\begin{aligned} & TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\zeta_1, \zeta_2, \dots, \zeta_n) \\ &= \left(\left[\frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau (n\varepsilon_{i_r} a_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau (n\varepsilon_{i_r} b_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau (n\varepsilon_{i_r} c_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau (n\varepsilon_{i_r} d_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}} \right], \varsigma_{\min(t_i)} \right) \\ &= \left(\left[\frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau a^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau b^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau c^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau d^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}} \right], \varsigma_{\min(t_i)} \right) \\ &= \left(\left[\frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(a^{\frac{k_1+k_2+\dots+k_\tau}{k_1+k_2+\dots+k_\tau}} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(b^{\frac{k_1+k_2+\dots+k_\tau}{k_1+k_2+\dots+k_\tau}} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(c^{\frac{k_1+k_2+\dots+k_\tau}{k_1+k_2+\dots+k_\tau}} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(d^{\frac{k_1+k_2+\dots+k_\tau}{k_1+k_2+\dots+k_\tau}} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}} \right], \varsigma_t \right) \\ &= \left(\left[\frac{1}{C_n^\tau} (C_n^\tau a), \frac{1}{C_n^\tau} (C_n^\tau b), \frac{1}{C_n^\tau} (C_n^\tau c), \frac{1}{C_n^\tau} (C_n^\tau d) \right], \varsigma_t \right) \\ &= ([a, b, c, d], \varsigma_t). \end{aligned}$$

Thus, the proof of Property 1 is completed. \square

Property 2 (Commutativity). Let $\zeta_i = ([a_i, b_i, c_i, d_i], \varsigma_{t_i}) (i = 1, 2, \dots, n)$ and $\tilde{\beta}_j = ([e_j, f_j, g_j, h_j], \varsigma_{x_j}) (j = 1, 2, \dots, n)$ be two groups of TF2DLVs. When $\tilde{\beta}_j$ is an arbitrary arrangement of ζ_i , we have

$$TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\zeta_1, \zeta_2, \dots, \zeta_n) = TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_n).$$

Proof. When $\tilde{\beta}_j$ is an arbitrary arrangement of ζ_i , we can get $\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^\tau (n\varepsilon_{i_r} \zeta_{i_r})^{k_r} \right) =$

$$\sum_{\substack{1 \leq j_1 < \dots < j_\tau \leq n \\ < j_\tau \leq n}} \left(\prod_{r=1}^\tau (n\varepsilon_{j_r} \tilde{\beta}_{j_r})^{k_r} \right).$$

$$\text{Then, } \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} \tilde{\zeta}_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}}{C_n^\tau} = \frac{\sum_{\substack{1 \leq j_1 < \dots < j_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{j_r} \tilde{\beta}_{j_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}}{C_n^\tau}, \text{ i.e.,}$$

$$TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n) = TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_n).$$

Thus, the proof of Property 2 is completed. \square

Property 3 (Boundedness). Let $\tilde{\zeta}_i = ([a_i, b_i, c_i, d_i], \varsigma_{t_i}) (i = 1, 2, \dots, n)$ be a group of TF2DLVs, $\tilde{\beta} = \min(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n)$ and $\tilde{\varphi} = \max(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n)$, then

$$\tilde{\beta} \leq TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n) \leq \tilde{\varphi}.$$

$$\text{Proof. Since } \tilde{\beta} \leq \tilde{\zeta}_i \leq \tilde{\varphi}, \text{ then } \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} \tilde{\beta})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}}{C_n^\tau} \leq \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} \tilde{\zeta}_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}}{C_n^\tau} \leq \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \varepsilon_{i_r} \tilde{\varphi})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}}{C_n^\tau}.$$

$$\text{Therefore, } TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\tilde{\beta}, \tilde{\beta}, \dots, \tilde{\beta}) \leq TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n) \leq TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\tilde{\varphi}, \tilde{\varphi}, \dots, \tilde{\varphi}), \text{ i.e., } \tilde{\beta} \leq TF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n) \leq \tilde{\varphi}.$$

Thus, the proof of Property 3 is completed. \square

Next, we discuss some special cases of the TF2DLPGHM operator when parameters τ and k_r take different values, as described below.

Case 1. When $\tau = 1$ and $k_1 = 1$, the TF2DLPGHM operator degenerates into the trapezoidal fuzzy two-dimensional linguistic power averaging operator:

$$\begin{aligned} & TF2DLPGHM^{(1,1)}(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n) \\ &= \left(\left[\frac{1}{C_n^1} \sum_{1 \leq i_1 \leq n} (n \varepsilon_{i_1} a_{i_1})^{\frac{1}{1}}, \frac{1}{C_n^1} \sum_{1 \leq i_1 \leq n} (n \varepsilon_{i_1} b_{i_1})^{\frac{1}{1}}, \frac{1}{C_n^1} \sum_{1 \leq i_1 \leq n} (n \varepsilon_{i_1} c_{i_1})^{\frac{1}{1}}, \frac{1}{C_n^1} \sum_{1 \leq i_1 \leq n} (n \varepsilon_{i_1} d_{i_1})^{\frac{1}{1}} \right], \varsigma_{\min(t_i)} \right) \\ &= \left(\left[\frac{1}{n} \sum_{i=1}^n (n \varepsilon_i a_i)^{\frac{1}{1}}, \frac{1}{n} \sum_{i=1}^n (n \varepsilon_i b_i)^{\frac{1}{1}}, \frac{1}{n} \sum_{i=1}^n (n \varepsilon_i c_i)^{\frac{1}{1}}, \frac{1}{n} \sum_{i=1}^n (n \varepsilon_i d_i)^{\frac{1}{1}} \right], \varsigma_{\min(t_i)} \right) \\ &= \left(\left[\sum_{i=1}^n \varepsilon_i a_i, \sum_{i=1}^n \varepsilon_i b_i, \sum_{i=1}^n \varepsilon_i c_i, \sum_{i=1}^n \varepsilon_i d_i \right], \varsigma_{\min(t_i)} \right) \end{aligned}$$

Case 2. When $k_1 = k_2 = \dots = k_\tau = 1$, the TF2DLPGHM operator degenerates into the trapezoidal fuzzy two-dimensional linguistic power Hamy mean operator:

$$TF2DLPGHM^{(\tau)}(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n) = \left(\left[\frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} n \varepsilon_{i_r} a_{i_r} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} n \varepsilon_{i_r} b_{i_r} \right)^{\frac{1}{\tau}}, \right. \right. \\ \left. \left. \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} n \varepsilon_{i_r} c_{i_r} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} n \varepsilon_{i_r} d_{i_r} \right)^{\frac{1}{\tau}} \right], \varsigma_{\min(t_i)} \right).$$

Case 3. When $\tau = n$ and $k_1 = k_2 = \dots = k_n = 1$, the TF2DLPGHM operator degenerates into the trapezoidal fuzzy two-dimensional linguistic power geometric average operator:

$$\begin{aligned} & TF2DLPGHM^{(n,1,1,\dots,1)}(\xi_1, \xi_2, \dots, \xi_n) \\ &= \left(\left[\frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} n \varepsilon_{i_r} a_{i_r} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} n \varepsilon_{i_r} b_{i_r} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} n \varepsilon_{i_r} c_{i_r} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} n \varepsilon_{i_r} d_{i_r} \right)^{\frac{1}{\tau}} \right], \varsigma_{\min(i_i)} \right) \\ &= \left(\left[\frac{1}{\tau} \left(\prod_{i=1}^n n \varepsilon_i a_i \right)^{\frac{1}{\tau}}, \frac{1}{\tau} \left(\prod_{i=1}^n n \varepsilon_i b_i \right)^{\frac{1}{\tau}}, \frac{1}{\tau} \left(\prod_{i=1}^n n \varepsilon_i c_i \right)^{\frac{1}{\tau}}, \frac{1}{\tau} \left(\prod_{i=1}^n n \varepsilon_i d_i \right)^{\frac{1}{\tau}} \right], \varsigma_{\min(i_i)} \right) \\ &= \left(\left[\left(\prod_{i=1}^n n \varepsilon_i a_i \right)^{\frac{1}{n}}, \left(\prod_{i=1}^n n \varepsilon_i b_i \right)^{\frac{1}{n}}, \left(\prod_{i=1}^n n \varepsilon_i c_i \right)^{\frac{1}{n}}, \left(\prod_{i=1}^n n \varepsilon_i d_i \right)^{\frac{1}{n}} \right], \varsigma_{\min(i_i)} \right). \end{aligned}$$

3.3. The Weighted Form of TF2DLPGHM Operator

Definition 10. Let $\xi_i = ([a_i, b_i, c_i, d_i], \varsigma_{t_i}) (i = 1, 2, \dots, n)$ be a group of TF2DLVs, $w = (w_1, w_2, \dots, w_n)^T$ be the weighting vector of ξ_i , $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, then the WTF2DLPGHM operator can be expressed as follows:

$$WTF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\xi_1, \xi_2, \dots, \xi_n) = \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} \left(\frac{n w_{i_r} (1 + G(\xi_{i_r}))}{\sum_{i=1}^n w_i (1 + G(\xi_i))} \xi_{i_r} \right)^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_\tau}}}{C_n^\tau}, \quad (24)$$

where $\tau (\tau = 1, 2, \dots, n)$ is a parameter of the WTF2DLPGHM operator, $(i_1, i_2, \dots, i_\tau)$ is a τ -tuple full combination of $(1, 2, \dots, n)$, and $C_n^\tau = \frac{n!}{\tau!(n-\tau)!}$ is called the binomial coefficient. Here $G(\xi_i) = \sum_{l=1, l \neq i}^n \sup(\xi_i, \xi_l)$, in which $\sup(\xi_i, \xi_l)$ represents the support degree from ξ_i to ξ_l .

Let $\lambda_i = \frac{w_i(1+G(\xi_i))}{\sum_{i=1}^n w_i(1+G(\xi_i))}$, obviously $\sum_{i=1}^n \lambda_i = 1$. The definition of the WTF2DLPGHM operator is equivalent to the following form:

$$WTF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\xi_1, \xi_2, \dots, \xi_n) = \frac{\sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \lambda_{i_r} \xi_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_\tau}}}{C_n^\tau}. \quad (25)$$

Theorem 2. Let $\xi_i = ([\alpha_i^L, \alpha_i^{ML}, \alpha_i^{MU}, \alpha_i^U], \varsigma_{t_i}) (i = 1, 2, \dots, n)$ be a group of TF2DLVs, then the expansion of WTF2DLPGHM operator is still a TF2DLV, then

$$\begin{aligned} WTF2DLPGHM^{(\tau, k_1, k_2, \dots, k_\tau)}(\xi_1, \xi_2, \dots, \xi_n) &= \left(\left[\frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \lambda_{i_r} a_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \lambda_{i_r} b_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_\tau}}, \right. \\ &\quad \left. \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \lambda_{i_r} c_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_\tau}}, \frac{1}{C_n^\tau} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n \\ < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n \lambda_{i_r} d_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_\tau}} \right], \varsigma_{\min(i_i)} \right) \end{aligned} \quad (26)$$

The detailed derivation process of Equation (26) is shown in Appendix B.

Example 1. Let $\xi_1 = ([0, 245, 0.267, 0, 298, 0.321], c_3)$, $\xi_2 = ([0, 305, 0.343, 0.381, 0.419], c_3)$, $\xi_3 = ([0, 276, 0.322, 0.368, 0.414], c_3)$, and $\xi_4 = ([0, 309, 0.347, 0.386, 0.425], c_3)$ be four TF2DLVs and their subjective weight be $w_1 = 0.349, w_2 = 0.274, w_3 = 0.153, w_4 = 0.224$; the WTF2DLPGHM operator is used to aggregate these four elements.

Step 1. Calculate the support degree $\text{sup}(\xi_i, \xi_j)(i, j = 1, 2, 3, 4; i \neq j)$, where $\text{sup}(\xi_i, \xi_j) = 1 - d(\xi_i, \xi_j)$.

$$\begin{aligned} \text{sup}(\xi_1, \xi_2) &= 0.932, \text{sup}(\xi_1, \xi_3) = 0.931, \text{sup}(\xi_1, \xi_4) = 0.933 \\ \text{sup}(\xi_2, \xi_1) &= 0.932, \text{sup}(\xi_2, \xi_3) = 0.895, \text{sup}(\xi_2, \xi_4) = 0.896 \\ \text{sup}(\xi_3, \xi_1) &= 0.931, \text{sup}(\xi_3, \xi_2) = 0.895, \text{sup}(\xi_3, \xi_4) = 0.898 \\ \text{sup}(\xi_4, \xi_1) &= 0.933, \text{sup}(\xi_4, \xi_2) = 0.896, \text{sup}(\xi_4, \xi_3) = 0.898 \end{aligned}$$

Step 2. Calculate the $G(\xi_i)(i = 1, 2, 3, 4)$.

$$\begin{aligned} G(\xi_1) &= 0.932 + 0.931 + 0.933 = 2.796, G(\xi_2) = 0.932 + 0.895 + 0.896 = 2.723, \\ G(\xi_3) &= 0.931 + 0.895 + 0.898 = 2.724, G(\xi_4) = 0.933 + 0.896 + 0.898 = 2.727. \end{aligned}$$

Step 3. Calculate the comprehensive weight λ_i of each TF2DLV, where $\lambda_i = \frac{w_i(1+G(\xi_i))}{\sum_{i=1}^n w_i(1+G(\xi_i))}(i = 1, 2, 3, 4)$.

$$\lambda_1 = \frac{w_1(1+G(\xi_1))}{\sum_{i=1}^4 w_i(1+G(\xi_i))} = 0.353, \lambda_2 = \frac{w_2(1+G(\xi_2))}{\sum_{i=1}^4 w_i(1+G(\xi_i))} = 0.272, \lambda_3 = \frac{w_3(1+G(\xi_3))}{\sum_{i=1}^4 w_i(1+G(\xi_i))} = 0.153, \lambda_4 = \frac{w_4(1+G(\xi_4))}{\sum_{i=1}^4 w_i(1+G(\xi_i))} = 0.222$$

Step 4. Calculate the comprehensive evaluation value by the WTF2DLPGHM operator (suppose $\tau = 2, k_1 = 1, k_2 = 1$).

$$\text{WTF2DLPGHM}^{(2,2,2)}(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\sum_{1 \leq i_1 < i_2 \leq 4} \left(\prod_{r=1}^2 (4\lambda_{i_r} \xi_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2}}}{C_4^2} = ([0.274, 0.308, 0.344, 0.378], c_3).$$

Next, we discuss some special cases of the WTF2DLPGHM operator when parameters τ and k_r take different values, as described below.

Case 1. When $w_i = \frac{1}{n}$, the WTF2DLPGHM operator degenerates into the operator TF2DLPGHM operator:

$$\begin{aligned} & \text{WTF2DLPGHM}^{(\tau, k_1, k_2, \dots, k_\tau)}(\xi_1, \xi_2, \dots, \xi_n) \\ &= \left(\left[\frac{1}{C_n^{\frac{1}{\tau}}} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} a_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^{\frac{1}{\tau}}} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} b_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^{\frac{1}{\tau}}} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} c_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^{\frac{1}{\tau}}} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} d_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}} \right], \varsigma_{\min(t_i)} \right) \\ &= \left(\left[\frac{1}{C_n^{\frac{1}{\tau}}} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} \left(\frac{nw_i(1+G(\xi_{i_r}))}{\sum_{i=1}^n w_i(1+G(\xi_{i_r}))} a_{i_r} \right)^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^{\frac{1}{\tau}}} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} \left(\frac{nw_i(1+G(\xi_{i_r}))}{\sum_{i=1}^n w_i(1+G(\xi_{i_r}))} b_{i_r} \right)^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \right. \right. \\ & \quad \left. \frac{1}{C_n^{\frac{1}{\tau}}} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} \left(\frac{nw_i(1+G(\xi_{i_r}))}{\sum_{i=1}^n w_i(1+G(\xi_{i_r}))} c_{i_r} \right)^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}}, \frac{1}{C_n^{\frac{1}{\tau}}} \sum_{\substack{1 \leq i_1 < \dots < i_\tau \leq n}} \left(\prod_{r=1}^{\tau} \left(\frac{nw_i(1+G(\xi_{i_r}))}{\sum_{i=1}^n w_i(1+G(\xi_{i_r}))} d_{i_r} \right)^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_\tau}} \right], \varsigma_{\min(t_i)} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\left(\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} \left(n \frac{\frac{1}{n}(1+G(\xi_{ir}))}{\sum_{i=1}^n \frac{1}{n}(1+G(\xi_i))} a_{ir} \right)^{k_r} \right)^{\frac{k_1+k_2+\dots+k_{\tau}}{\tau}} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} \left(n \frac{\frac{1}{n}(1+G(\xi_{ir}))}{\sum_{i=1}^n \frac{1}{n}(1+G(\xi_i))} b_{ir} \right)^{k_r} \right)^{\frac{k_1+k_2+\dots+k_{\tau}}{\tau}} \right)^{\frac{1}{\tau}}, \\
&\quad \left(\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} \left(n \frac{\frac{1}{n}(1+G(\xi_{ir}))}{\sum_{i=1}^n \frac{1}{n}(1+G(\xi_i))} c_{ir} \right)^{k_r} \right)^{\frac{k_1+k_2+\dots+k_{\tau}}{\tau}} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} \left(n \frac{\frac{1}{n}(1+G(\xi_{ir}))}{\sum_{i=1}^n \frac{1}{n}(1+G(\xi_i))} d_{ir} \right)^{k_r} \right)^{\frac{k_1+k_2+\dots+k_{\tau}}{\tau}} \right)^{\frac{1}{\tau}} \right)^{\zeta_{\min(t_i)}} \\
&= \left(\left(\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n \xi_{ir} a_{ir})^{k_r} \right)^{\frac{k_1+k_2+\dots+k_{\tau}}{\tau}} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n \xi_{ir} b_{ir})^{k_r} \right)^{\frac{k_1+k_2+\dots+k_{\tau}}{\tau}} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n \xi_{ir} c_{ir})^{k_r} \right)^{\frac{k_1+k_2+\dots+k_{\tau}}{\tau}} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n \xi_{ir} d_{ir})^{k_r} \right)^{\frac{k_1+k_2+\dots+k_{\tau}}{\tau}} \right)^{\frac{1}{\tau}} \right)^{\zeta_{\min(t_i)}} \\
&= TF2DLP GHM^{(\tau, k_1, k_2, \dots, k_{\tau})}(\xi_1, \xi_2, \dots, \xi_n).
\end{aligned}$$

Case 2. When $\tau = 1$ and $k_1 = 1$, the WTF2DLP GHM operator degenerates into the weighted trapezoidal fuzzy two-dimensional linguistic power averaging operator:

$$\begin{aligned}
&WTF2DLP GHM^{(1,1)}(\xi_1, \xi_2, \dots, \xi_n) \\
&= \left(\left(\frac{1}{C_n^1} \sum_{1 \leq i_1 \leq n} (n \lambda_{i_1} a_{i_1})^{\frac{1}{1}}, \frac{1}{C_n^1} \sum_{1 \leq i_1 \leq n} (n \lambda_{i_1} b_{i_1})^{\frac{1}{1}}, \frac{1}{C_n^1} \sum_{1 \leq i_1 \leq n} (n \lambda_{i_1} c_{i_1})^{\frac{1}{1}}, \frac{1}{C_n^1} \sum_{1 \leq i_1 \leq n} (n \lambda_{i_1} d_{i_1})^{\frac{1}{1}} \right)^{\zeta_{\min(t_i)}} \right) \\
&= \left(\left(\frac{1}{n} \sum_{i=1}^n (n \lambda_i a_i)^{\frac{1}{1}}, \frac{1}{n} \sum_{i=1}^n (n \lambda_i b_i)^{\frac{1}{1}}, \frac{1}{n} \sum_{i=1}^n (n \lambda_i c_i)^{\frac{1}{1}}, \frac{1}{n} \sum_{i=1}^n (n \lambda_i d_i)^{\frac{1}{1}} \right)^{\zeta_{\min(t_i)}} \right) \\
&= \left(\left[\sum_{i=1}^n \lambda_i a_i, \sum_{i=1}^n \lambda_i b_i, \sum_{i=1}^n \lambda_i c_i, \sum_{i=1}^n \lambda_i d_i \right]^{\zeta_{\min(t_i)}} \right).
\end{aligned}$$

Case 3. When $k_1 = k_2 = \dots = k_{\tau} = 1$, the WTF2DLP GHM operator degenerates into the weighted trapezoidal fuzzy two-dimensional linguistic power Hamy mean operator:

$$\begin{aligned}
WTF2DLP GHM^{(\tau)}(\xi_1, \xi_2, \dots, \xi_n) &= \left(\left(\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} n \lambda_{i_r} a_{i_r} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} n \lambda_{i_r} b_{i_r} \right)^{\frac{1}{\tau}}, \right. \right. \\
&\quad \left. \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} n \lambda_{i_r} c_{i_r} \right)^{\frac{1}{\tau}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} n \lambda_{i_r} d_{i_r} \right)^{\frac{1}{\tau}} \right)^{\zeta_{\min(t_i)}} \right).
\end{aligned}$$

Case 4. When $\tau = n$ and $k_1 = k_2 = \dots = k_n = 1$, the TF2DLP GHM operator degenerates into the weighted trapezoidal fuzzy two-dimensional linguistic power geometric average operator:

$$\begin{aligned}
&WTF2DLP GHM^{(n,1,1,\dots,1)}(\xi_1, \xi_2, \dots, \xi_n) \\
&= \left(\left(\frac{1}{C_n^n} \sum_{\substack{1 \leq i_1 < \dots < i_n \leq n}} \left(\prod_{r=1}^n n \lambda_{i_r} a_{i_r} \right)^{\frac{1}{n}}, \frac{1}{C_n^n} \sum_{\substack{1 \leq i_1 < \dots < i_n \leq n}} \left(\prod_{r=1}^n n \lambda_{i_r} b_{i_r} \right)^{\frac{1}{n}}, \frac{1}{C_n^n} \sum_{\substack{1 \leq i_1 < \dots < i_n \leq n}} \left(\prod_{r=1}^n n \lambda_{i_r} c_{i_r} \right)^{\frac{1}{n}}, \frac{1}{C_n^n} \sum_{\substack{1 \leq i_1 < \dots < i_n \leq n}} \left(\prod_{r=1}^n n \lambda_{i_r} d_{i_r} \right)^{\frac{1}{n}} \right)^{\zeta_{\min(t_i)}} \right) \\
&= \left(\left(\frac{1}{n} \left(\prod_{i=1}^n n \lambda_i a_i \right)^{\frac{1}{n}}, \frac{1}{n} \left(\prod_{i=1}^n n \lambda_i b_i \right)^{\frac{1}{n}}, \frac{1}{n} \left(\prod_{i=1}^n n \lambda_i c_i \right)^{\frac{1}{n}}, \frac{1}{n} \left(\prod_{i=1}^n n \lambda_i d_i \right)^{\frac{1}{n}} \right)^{\zeta_{\min(t_i)}} \right) \\
&= \left(\left(\left(\prod_{i=1}^n n \lambda_i a_i \right)^{\frac{1}{n}}, \left(\prod_{i=1}^n n \lambda_i b_i \right)^{\frac{1}{n}}, \left(\prod_{i=1}^n n \lambda_i c_i \right)^{\frac{1}{n}}, \left(\prod_{i=1}^n n \lambda_i d_i \right)^{\frac{1}{n}} \right)^{\zeta_{\min(t_i)}} \right)
\end{aligned}$$

4. An Approach to MADM with the WTF2DLP GHM Operator

This paper considers a lean management evaluation problem of industrial residential projects based on TF2DLVs. It is assumed that there are n alternative projects (P_1, P_2, \dots, P_n) in lean management

evaluation of industrial residential projects. Each alternative $P_i (i = 1, 2, \dots, n)$ has σ first-class evaluation indicators $R_l (l = 1, 2, \dots, \sigma)$, whose weight vector is expressed as $W = (W_1, W_2, \dots, W_\sigma)$ and satisfies $W_l \geq 0, \sum_{l=1}^{\sigma} W_l = 1$. There are m_l second-class indicators $I_j^l (l = 1, 2, \dots, \sigma; j = 1, 2, \dots, m_l)$ under each first-class indicator R_l . $w = (w_1^l, w_2^l, \dots, w_{m_l}^l)$ represents the weight vector of each second-class indicator I_j^l and satisfies $w_j^l \geq 0, \sum_{j=1}^{m_l} w_j^l = 1$. The decision matrix is represented by $H = [h_{ij}^l]_{n \times m_l}$, where e_{ij}^l represents the evaluation value of the j th second-class indicator I_j^l of the first-class evaluation indicator R_l in alternative P_i . Evaluation values are expressed by TF2DLVs, i.e., $h_{ij}^l = ([a_{ij}^l, b_{ij}^l, c_{ij}^l, d_{ij}^l], \varsigma_{ij}^l)$ ($i = 1, 2, \dots, n; l = 1, 2, \dots, \sigma; j = 1, 2, \dots, m_l$), and satisfies $a_{ij}^l \leq b_{ij}^l \leq c_{ij}^l \leq d_{ij}^l, \varsigma_{ij}^l \in S$. Then, we can rank the alternatives according to the given information.

In the following, the proposed WTF2DLPGHM operator is applied to solve such MADM problems. The steps are as follows:

- Step 1.** Normalize the decision matrix $H = [h_{ij}^l]_{n \times m_l}$ into $E = [e_{ij}^l]_{n \times m_l}$. For the benefit attribute, we have $e_{ij}^l = h_{ij}^l$, and for the cost attribute, we have $e_{ij}^l = \text{Neg}(h_{ij}^l)$.
- Step 2.** Calculate the support degree $\text{sup}(e_{ix}^l, e_{iy}^l)$ from the second-class indicator I_{ix}^l to the second-class indicator I_{iy}^l of each alternative under the first-class indicator R_l , where $i = 1, 2, \dots, n; l = 1, 2, \dots, \sigma; x, y = 1, 2, \dots, m_l; x \neq y$.
- Step 3.** Calculate the $G(e_{ij}^l) (i = 1, 2, \dots, n; l = 1, 2, \dots, \sigma; j = 1, 2, \dots, m_l)$.
- Step 4.** Calculate the comprehensive weight λ_{ij}^l of each second-class indicator under the first-class indicator R_l , where $\lambda_{ij}^l = \frac{w_j^l(1+G(e_{ij}^l))}{\sum_{i=1}^n w_j^l(1+G(e_{ij}^l))}$.
- Step 5.** Calculate the comprehensive evaluation value $e_i^l (i = 1, 2, \dots, n; l = 1, 2, \dots, \sigma)$ of each alternative under the first-class indicator R_l by the proposed WTF2DLPGHM operator.
- Step 6.** Calculate the comprehensive weight $\lambda_i^l (i = 1, 2, \dots, n; l = 1, 2, \dots, \sigma)$ of each first-class indicator R_l , where $\lambda_i^l = \frac{W_l(1+G(e_i^l))}{\sum_{l=1}^{\sigma} W_l(1+G(e_i^l))}$.
- Step 7.** Calculate the comprehensive evaluation value $e_i (i = 1, 2, \dots, n)$ of each alternative by the proposed WTF2DLPGHM operator.
- Step 8.** Calculate the expected value $EX(e_i)$ of each alternative according to the Definition 3.
- Step 9.** Sort the alternatives according to the descending order of $EX(e_i)$.

5. A Calculation Example

Example 2. In order to promote the improvement of industrial lean management level, this paper selects three industrial residential projects (P_1, P_2, \dots, P_3) to evaluate the implementation level of lean management. Each alternative $P_i (i = 1, 2, 3)$ has five first-class evaluation indicators $R_l (l = 1, 2, \dots, 5)$ whose weight vector is expressed as $W = (W_1, W_2, W_3, W_4, W_5)$ and satisfies $W_l \geq 0, \sum_{l=1}^5 W_l = 1$. Here, the first-class indicators R_1, R_2, R_3 , and R_4 have four second-class indicators, which are expressed as $(I_1^1, I_2^1, I_3^1, I_4^1; I_1^2, I_2^2, I_3^2, I_4^2; I_1^3, I_2^3, I_3^3, I_4^3; I_1^4, I_2^4, I_3^4, I_4^4)$, and the first-class indicator R_5 have three second-class indicators (I_1^5, I_2^5, I_3^5) . In addition, $w = (w_1^l, w_2^l, w_3^l, w_4^l) (l = 1, 2, 3, 4)$ represents the weight vector of the second-class indicators $(I_1^l, I_2^l, I_3^l, I_4^l) (l = 1, 2, 3, 4)$ under the first-class indicator $R_l (l = 1, 2, 3, 4)$ and satisfies

$w_j^l \geq 0$, $\sum_{j=1}^4 w_j^l = 1$, and $w = (w_1^5, w_2^5, w_3^5)$ represents the weight vector of the second-class indicators (I_1^5, I_2^5, I_3^5) under the first-class indicator R_5 and satisfies $w_j^5 \geq 0$, $\sum_{j=1}^3 w_j^5 = 1$.

Here, the evaluation index system and index weight of lean management implementation level of industrialized residential projects are shown in Table 1. Based on the evaluation information in the form of TF2DLVs $e_{ij}^l = ([a_{ij}^l, b_{ij}^l, c_{ij}^l, d_{ij}^l], \varsigma_{ij}^l)$ ($i = 1, 2, 3; l = 1, 2, 3, 4, 5; j = 1, 2, \dots, m_l$) given by DMs, the decision matrix $H = [h_{ij}^l]_{n \times m_l}$ is constructed as shown in Table 2. In particular, DMs use the LTS $S = \{\varsigma_0, \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ with five linguistic terms to express the subjective trust in the evaluation results.

Table 1. Evaluation indicator system and indicator weight.

First-Class Indicator	Second-Class Indicator
Lean design R_1 (0.335)	Assembly design I_1^1 (0.349)
	Standardized design I_2^1 (0.274)
	Personalized design I_3^1 (0.154)
	Design applicability I_4^1 (0.224)
Component lean production and logistics R_2 (0.171)	Standardization of component production I_1^2 (0.435)
	Component Quality Control I_2^2 (0.258)
	Logistics Time Management I_3^2 (0.172)
	Nondestructive Transportation of Components I_4^2 (0.134)
Lean construction R_3 (0.234)	Construction Mechanization I_1^3 (0.169)
	Environmental protection construction I_2^3 (0.231)
	Construction Technology Management I_3^3 (0.132)
	Construction safety I_4^3 (0.468)
Organizational synergy R_4 (0.130)	Organizational integration I_1^4 (0.387)
	Organizational trust I_2^4 (0.316)
	Willingness to cooperate I_3^4 (0.118)
	Organizational Collaboration Technology I_4^4 (0.179)
Information synergy R_5 (0.130)	Accurate information I_1^4 (0.473)
	Transfer speed I_2^4 (0.383)
	Transfer cost I_3^4 (0.144)

Table 2. Trapezoidal two-dimensional linguistic decision matrix \tilde{E} .

	P_1	P_2	P_3
I_1^1	([0.245, 0.267, 0.298, 0.321], ς_3)	([0.256, 0.276, 0.281, 0.285], ς_3)	([0.203, 0.237, 0.271, 0.305], ς_4)
I_2^1	([0.305, 0.343, 0.381, 0.419], ς_3)	([0.312, 0.343, 0.365, 0.392], ς_3)	([0.076, 0.114, 0.153, 0.191], ς_4)
I_3^1	([0.276, 0.322, 0.368, 0.414], ς_3)	([0.356, 0.367, 0.373, 0.401], ς_3)	([0.184, 0.230, 0.276, 0.322], ς_4)
I_4^1	([0.309, 0.347, 0.386, 0.425], ς_3)	([0.398, 0.401, 0.412, 0.433], ς_3)	([0.155, 0.193, 0.231, 0.270], ς_4)
I_1^2	([0.175, 0.219, 0.263, 0.307], ς_4)	([0.350, 0.394, 0.438, 0.482], ς_2)	([0.175, 0.219, 0.263, 0.307], ς_3)
I_2^2	([0.345, 0.389, 0.432, 0.475], ς_4)	([0.234, 0.245, 0.259, 0.302], ς_2)	([0.086, 0.130, 0.173, 0.216], ς_3)
I_3^2	([0.253, 0.284, 0.316, 0.348], ς_4)	([0.312, 0.334, 0.345, 0.356], ς_2)	([0.253, 0.284, 0.316, 0.348], ς_3)
I_4^2	([0.231, 0.270, 0.309, 0.347], ς_4)	([0.356, 0.367, 0.381, 0.392], ς_2)	([0.309, 0.347, 0.386, 0.424], ς_3)
I_1^3	([0.305, 0.343, 0.381, 0.419], ς_3)	([0.305, 0.343, 0.381, 0.419], ς_4)	([0.076, 0.114, 0.153, 0.191], ς_3)
I_2^3	([0.277, 0.312, 0.346, 0.381], ς_3)	([0.346, 0.381, 0.381, 0.381], ς_4)	([0.139, 0.173, 0.208, 0.242], ς_3)
I_3^3	([0.162, 0.203, 0.243, 0.284], ς_3)	([0.162, 0.203, 0.243, 0.284], ς_4)	([0.324, 0.365, 0.405, 0.446], ς_3)
I_4^3	([0.276, 0.322, 0.368, 0.414], ς_3)	([0.184, 0.230, 0.276, 0.322], ς_4)	([0.184, 0.230, 0.276, 0.322], ς_3)
I_1^4	([0.222, 0.259, 0.296, 0.333], ς_2)	([0.296, 0.333, 0.370, 0.408], ς_3)	([0.222, 0.259, 0.296, 0.333], ς_3)
I_2^4	([0.253, 0.284, 0.316, 0.348], ς_2)	([0.253, 0.284, 0.316, 0.348], ς_3)	([0.253, 0.284, 0.316, 0.348], ς_3)
I_3^4	([0.250, 0.313, 0.374, 0.437], ς_2)	([0.250, 0.313, 0.374, 0.437], ς_3)	([0.116, 0.174, 0.233, 0.291], ς_3)
I_4^4	([0.324, 0.365, 0.405, 0.446], ς_2)	([0.162, 0.203, 0.243, 0.284], ς_3)	([0.162, 0.203, 0.243, 0.284], ς_3)
I_1^5	([0.198, 0.248, 0.296, 0.346], ς_3)	([0.245, 0.286, 0.327, 0.367], ς_3)	([0.010, 0.050, 0.099, 0.149], ς_2)
I_2^5	([0.327, 0.367, 0.408, 0.449], ς_3)	([0.209, 0.261, 0.312, 0.365], ς_3)	([0.082, 0.123, 0.164, 0.204], ς_2)
I_3^5	([0.364, 0.455, 0.545, 0.636], ς_3)	([0.395, 0.445, 0.494, 0.544], ς_3)	([0.209, 0.261, 0.312, 0.365], ς_2)

Note that the evaluation values of each alternative project given by DMs in Table 2 have been transformed to the same measurement standard.

5.1. The Decision-Making Steps

Step 1. Because no attribute is a cost type, there is no need to standardize matrix $H = [h_{ij}^l]_{n \times m_l}$.

Step 2. Calculate the support degree $\sup(h_{ix}^l, h_{iy}^l) (i = 1, 2, 3; x, y = 1, 2, \dots, m_l; x \neq y)$ from the second-class indicator I_{ix}^l to the second-class indicator I_{iy}^l of each alternative under the first-class indicator R_l . For example, the support degree $\sup(h_{ix}^1, h_{iy}^1) (x, y = 1, 2, 3, 4)$ from I_{ix}^1 to I_{iy}^1 under the first-class indicator R_1 is as follows:

$$\begin{aligned} \sup(h_{11}^1, h_{12}^1) &= 0.932, \sup(h_{11}^1, h_{13}^1) = 0.931, \sup(h_{11}^1, h_{14}^1) = 0.933 \\ \sup(h_{12}^1, h_{13}^1) &= 0.895, \sup(h_{12}^1, h_{14}^1) = 0.896, \sup(h_{13}^1, h_{14}^1) = 0.898 \\ \sup(h_{21}^1, h_{22}^1) &= 0.942, \sup(h_{21}^1, h_{23}^1) = 0.943, \sup(h_{21}^1, h_{24}^1) = 0.947 \\ \sup(h_{22}^1, h_{23}^1) &= 0.903, \sup(h_{22}^1, h_{24}^1) = 0.907, \sup(h_{23}^1, h_{24}^1) = 0.904 \\ \sup(h_{31}^1, h_{32}^1) &= 0.871, \sup(h_{31}^1, h_{33}^1) = 0.888, \sup(h_{31}^1, h_{34}^1) = 0.881 \\ \sup(h_{32}^1, h_{33}^1) &= 0.945, \sup(h_{32}^1, h_{34}^1) = 0.938, \sup(h_{33}^1, h_{34}^1) = 0.873 \end{aligned}$$

Step 3. Calculate the $G(h_{ij}^l) (i = 1, 2, 3; l = 1, 2, 3, 4, 5; j = 1, 2, \dots, m_l)$. For example, the $G(h_{ij}^1) (j = 1, 2, 3, 4)$ about the first-class indicator R_1 is as follows:

$$\begin{aligned} G(h_{11}^1) &= 2.796, G(h_{12}^1) = 2.723, G(h_{13}^1) = 2.724, G(h_{14}^1) = 2.727 \\ G(h_{21}^1) &= 2.832, G(h_{22}^1) = 2.752, G(h_{23}^1) = 2.750, G(h_{24}^1) = 2.758 \\ G(h_{31}^1) &= 2.640, G(h_{32}^1) = 2.754, G(h_{33}^1) = 2.706, G(h_{34}^1) = 2.692. \end{aligned}$$

Step 4. Combining with the subjective weight in Table 1, the comprehensive weight λ_{ij}^l of each second-class indicator under the first-class indicator R_l can be calculated. For example, the weight $\lambda_{ij}^1 (j = 1, 2, 3, 4)$ of each second-class indicator under the first-class indicator R_1 is as follows:

$$\begin{aligned} \lambda_{11}^1 &= 0.353, \lambda_{12}^1 = 0.272, \lambda_{13}^1 = 0.153, \lambda_{14}^1 = 0.222 \\ \lambda_{21}^1 &= 0.353, \lambda_{22}^1 = 0.272, \lambda_{23}^1 = 0.153, \lambda_{24}^1 = 0.222 \\ \lambda_{31}^1 &= 0.344, \lambda_{32}^1 = 0.278, \lambda_{33}^1 = 0.154, \lambda_{34}^1 = 0.224. \end{aligned}$$

Step 5. Calculate the comprehensive evaluation value $h_i^l (i = 1, 2, 3; l = 1, 2, 3, 4, 5)$ of each alternative under the first-class indicator R_l by the proposed WTF2DLPGHM operator ($\tau = 2, k_1 = 1, k_2 = 1$), and the results are shown in Table 3.

Table 3. Comprehensive evaluation value h_i^l of each alternative under the first-class indicator R_l .

	P_1	P_2	P_3
R_1	$([0.274, 0.308, 0.344, 0.378], \varsigma_3)$	$([0.314, 0.331, 0.342, 0.360], \varsigma_3)$	$([0.143, 0.182, 0.221, 0.259], \varsigma_4)$
R_2	$([0.226, 0.264, 0.301, 0.339], \varsigma_4)$	$([0.290, 0.311, 0.331, 0.359], \varsigma_2)$	$([0.173, 0.213, 0.252, 0.291], \varsigma_3)$
R_3	$([0.235, 0.272, 0.309, 0.346], \varsigma_3)$	$([0.220, 0.259, 0.290, 0.320], \varsigma_4)$	$([0.152, 0.191, 0.229, 0.267], \varsigma_3)$
R_4	$([0.241, 0.28, 0.318, 0.357], \varsigma_2)$	$([0.225, 0.264, 0.303, 0.342], \varsigma_3)$	$([0.179, 0.218, 0.257, 0.296], \varsigma_3)$
R_5	$([0.259, 0.368, 0.363, 0.415], \varsigma_3)$	$([0.244, 0.331, 0.332, 0.376], \varsigma_3)$	$([0.055, 0.124, 0.153, 0.199], \varsigma_2)$

Step 6. Combining with the subjective weight in Table 1, the comprehensive weight $\lambda_i^l (i = 1, 2, 3; l = 1, 2, 3, 4, 5)$ of each first-class indicator R_l can be calculated.

$$\lambda_1^1 = 0.3372, \lambda_1^2 = 0.1702, \lambda_1^3 = 0.2349, \lambda_1^4 = 0.1272, \lambda_1^5 = 0.1305$$

$$\lambda_2^1 = 0.3376, \lambda_2^2 = 0.1680, \lambda_2^3 = 0.2327, \lambda_2^4 = 0.1308, \lambda_2^5 = 0.1309$$

$$\lambda_3^1 = 0.3339, \lambda_3^2 = 0.1725, \lambda_3^3 = 0.2356, \lambda_3^4 = 0.1311, \lambda_3^5 = 0.1269$$

Step 7. Calculate the comprehensive evaluation value $h_i (i = 1, 2, 3)$ of each alternative by the proposed WTF2DLPGHM operator ($\tau = 2, k_1 = 1, k_2 = 1$).

$$h_1 = ([0.236, 0.283, 0.312, 0.350], \varsigma_2), h_2 = ([0.248, 0.286, 0.306, 0.335], \varsigma_2), h_3 = ([0.131, 0.177, 0.212, 0.250], \varsigma_2)$$

Step 8. Calculate the expected value $E(h_i)$ of each alternative according to the Definition 3.

$$E(h_1) = 0.1476, E(h_2) = 0.1469, E(h_3) = 0.0962$$

Step 9. Sort the alternatives according to the descending order of $E(h_i)$.

$$P_2 > P_1 > P_3$$

Thus, alternative project P_2 has the best level of lean management implementation, and alternative project P_3 has the worst level of lean management implementation.

5.2. Parameter Sensitivity Analysis

There are two types of parameters in the proposed WTF2DLPGHM operator, among which τ is related to the HA operator and k is related to the GWA operator. When the values of τ and k change, the final ranking results may change. Therefore, in the following, we can observe the ranking results by changing parameters τ and k . In Table 4, if k_1, k_2, \dots, k_τ are all 1, the WTF2DLPGHM operator can reduce into the weighted trapezoidal fuzzy two-dimensional linguistic power Hamy mean (WTF2LPHM) operator; then, by changing the value of τ , we only analyze the influence of correlation parameter τ on the calculation results. Table 5 shows the ranking results under different k when two attributes have interrelationships ($\tau = 2$); Table 6 shows the ranking results under different k when three attributes have interrelationships ($\tau = 3$).

In the following, let k_1, k_2, \dots, k_τ be all 1, and discuss the influence of correlation parameter τ on the calculation results. From Table 4, we can observe that when the parameter τ is assigned to different values, the ranking results are slightly different. For $\tau = 2$ and $\tau = 3$, although the values of $E(h_i)$ are different, the ranking orders are exactly the same. For $\tau = 1$, the alternative P_2 is identified as the best alternative, whereas P_2 is identified as the second-best alternative for $\tau = 2$ and $\tau = 3$. This shows that different relationship models will have a certain impact on the ranking results. In addition, from Table 4 and Figure 1, as the value of τ increases, the expected value $E(h_i)$ of the same alternative decreases. In practice, we can assign τ different values according to the actual relationship structure among the attributes.

Table 4. Ranking results with different parameter τ .

Parameters	Expected Value $E(h_i)$	Ranking
$\tau = 1, k = 1$	$EX(h_1) = 0.1627, EX(h_2) = 0.1630, EX(h_3) = 0.1099$	$P_2 > P_1 > P_3$
$\tau = 2, k_1, k_2 = 1$	$EX(h_1) = 0.1476, EX(h_2) = 0.1469, EX(h_3) = 0.0962$	$P_1 > P_2 > P_3$
$\tau = 3, k_1, k_2, k_3 = 1$	$EX(h_1) = 0.1409, EX(h_2) = 0.1396, EX(h_3) = 0.0908$	$P_1 > P_2 > P_3$

Then, we discuss the influence of parameters k_1, k_2, \dots, k_τ on the calculation results. From the data in Table 5, when k_1 is less than k_2 , the expected values of all alternatives decrease with the increase in k_2 , and the ranking results obtained by the proposed WTF2DLPGHM operator are always $P_1 > P_2 > P_3$. On the contrary, when k_1 is greater than k_2 , the expected values of all alternatives increase with the increase of k_1 , the ranking results change significantly, and alternative P_2 changes from the second best to the best. Similarly, we can get the same conclusion from Table 6. When k_1 is less than k_2 and k_3 , the expected values of all alternatives decrease with the increase of k_2 and k_3 , and the ranking results are always $P_1 > P_2 > P_3$. Instead, when k_1 is greater than k_2 and k_3 , the expected values of all alternatives increase with the increase of k_1 , and alternative P_2 changes from the second best to the best. Combined with the above analysis, we see that the changes in parameters k_1, k_2, \dots, k_τ do affect the calculation results. That is to say, by taking different k values, the influence degree of input attributes will increase or decrease and the calculated results will change accordingly. Thus, DMs can flexibly adjust the impact of each attribute according to the actual situation.

Table 5. Ranking results with different parameters k_1, k_2 .

Parameters	Expected Value $E(h_i)$	Ranking
$k_1 = 1, k_2 = 1$	$EX(h_1) = 0.1476, EX(h_2) = 0.1469, EX(h_3) = 0.0962$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 2$	$EX(h_1) = 0.1348, EX(h_2) = 0.1279, EX(h_3) = 0.0877$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 3$	$EX(h_1) = 0.1299, EX(h_2) = 0.1198, EX(h_3) = 0.0849$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 4$	$EX(h_1) = 0.1276, EX(h_2) = 0.1155, EX(h_3) = 0.0837$	$P_1 > P_2 > P_3$
$k_1 = 2, k_2 = 1$	$EX(h_1) = 0.1645, EX(h_2) = 0.1695, EX(h_3) = 0.1085$	$P_2 > P_1 > P_3$
$k_1 = 3, k_2 = 1$	$EX(h_1) = 0.1747, EX(h_2) = 0.1822, EX(h_3) = 0.1163$	$P_2 > P_1 > P_3$
$k_1 = 4, k_2 = 1$	$EX(h_1) = 0.1813, EX(h_2) = 0.1904, EX(h_3) = 0.1217$	$P_2 > P_1 > P_3$

Table 6. Ranking results with different parameters k_1, k_2, k_3 .

Parameters	Expected Value $E(h_i)$	Ranking
$k_1 = 1, k_2 = 1, k_3 = 1$	$EX(h_1) = 0.1409, EX(h_2) = 0.1396, EX(h_3) = 0.0908$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 1, k_3 = 2$	$EX(h_1) = 0.1270, EX(h_2) = 0.1213, EX(h_3) = 0.0802$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 1, k_3 = 3$	$EX(h_1) = 0.1191, EX(h_2) = 0.1108, EX(h_3) = 0.0746$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 1, k_3 = 4$	$EX(h_1) = 0.1141, EX(h_2) = 0.1042, EX(h_3) = 0.0713$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 2, k_3 = 2$	$EX(h_1) = 0.1259, EX(h_2) = 0.1194, EX(h_3) = 0.0829$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 3, k_3 = 3$	$EX(h_1) = 0.1204, EX(h_2) = 0.1116, EX(h_3) = 0.0804$	$P_1 > P_2 > P_3$
$k_1 = 1, k_2 = 4, k_3 = 4$	$EX(h_1) = 0.1175, EX(h_2) = 0.1075, EX(h_3) = 0.0792$	$P_1 > P_2 > P_3$
$k_1 = 2, k_2 = 1, k_3 = 1$	$EX(h_1) = 0.1638, EX(h_2) = 0.1690, EX(h_3) = 0.1049$	$P_2 > P_1 > P_3$
$k_1 = 3, k_2 = 1, k_3 = 1$	$EX(h_1) = 0.1801, EX(h_2) = 0.1889, EX(h_3) = 0.1164$	$P_2 > P_1 > P_3$
$k_1 = 4, k_2 = 1, k_3 = 1$	$EX(h_1) = 0.1922, EX(h_2) = 0.2030, EX(h_3) = 0.1256$	$P_2 > P_1 > P_3$

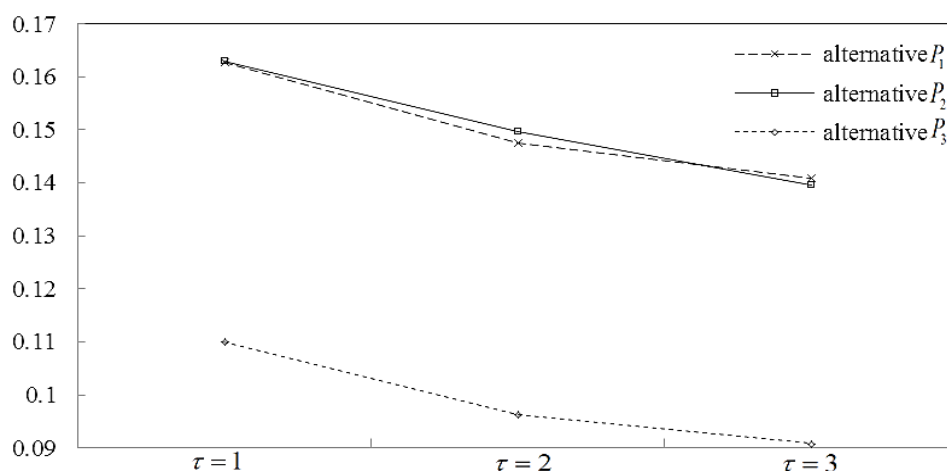


Figure 1. Expected values of the alternatives with different parameter τ .

5.3. Comparative Analysis and Discussion

5.3.1. The Validity of the Proposed Method

In order to prove the validity of the proposed method, we use two existing methods to solve the abovementioned lean management evaluation problem. The first approach is based on the trapezoidal fuzzy two-dimensional linguistic power generalized weighted aggregation (TF2DLPGWA) operator proposed by Li et al. [29], and the second approach is based on the trapezoidal fuzzy two-dimensional linguistic Bonferroni mean (TF2DLBM) operator proposed by Shi [41]. The expected values and ranking results of these different MADM methods are shown in Table 7.

The TF2DLPGWA operator proposed by Li et al. [29] integrates information under the assumption that the attributes are completely independent. For the accuracy of comparison, we set $\tau = 1$ in the proposed WTF2DLPGHM operator, then the WTF2DLPGHM operator degenerates into an operator without considering the attribute relationship. Specially, when $\tau = 1$ and $k = 1$, the proposed WTF2DLPGHM operator can reduce into the weighted trapezoidal fuzzy two-dimensional linguistic power averaging (WTF2DLPA) operator, and the TF2DLPGWA operator proposed by Li et al. [29] also degenerates into WTF2DLPA operator when $k = 1$. As shown in Table 8, under such parameter values, the expected values of the method proposed by Li et al. [29] are exactly the same as in our proposed method. Therefore, when the attributes are completely independent, the consistency of the ranking results of the proposed WTF2DLPGHM operator and the TF2DLPGWA operator [29] greatly proves the effectiveness of the proposed method.

Then, we consider the case of correlation between attributes. The TF2DLBM operator proposed by Shi [41] considers the relationship of two different attributes. In order to discuss the influence of parameter k , we first set $\tau = 2$, and then the proposed WTF2DLPGHM operator becomes an operator similar to the TF2DLBM operator, which only considers the relationship between two attributes. In case of $\tau = 2$ and $k_1, k_2 = 1$, the ranking results of the proposed WTF2DLPGHM operator are exactly the same as that of the TF2DLBM operator [41] in the case of $p = q = 1$. Therefore, the consistency the ranking results of the proposed WTF2DLPGHM operator and the TF2DLBM operator [41] greatly shows the effectiveness of our method.

Table 7. Results obtained by different MADM methods.

Methods	Parameters	Expected Value $E(h_i)$	Ranking
Method proposed by Li et al. [29] (TF2DLPGWA)	$k = 1$	$EX(h_1) = 0.1627, EX(h_2) = 0.1630, EX(h_3) = 0.1099$	$P_2 > P_1 > P_3$
The proposed method	$\tau = 1, k = 1$	$EX(h_1) = 0.1627, EX(h_2) = 0.1630, EX(h_3) = 0.1099$	$P_2 > P_1 > P_3$
Method proposed by Shi [41] (TF2DLBM)	$p = q = 1$	$EX(h_1) = 0.0082, EX(h_2) = 0.0081, EX(h_3) = 0.0053$	$P_1 > P_2 > P_3$
The proposed method	$\tau = 2, k_1, k_2 = 1$	$EX(h_1) = 0.1476, EX(h_2) = 0.1469, EX(h_3) = 0.0962$	$P_1 > P_2 > P_3$

Table 8. Results obtained by different MADM methods.

Methods	Parameters	Expected Value $E(h_i)$	Ranking
Method proposed by Li et al. [29] (TF2DLPGWA)	$k = 1$	$EX(h_1) = 0.1627, EX(h_2) = 0.1630, EX(h_3) = 0.1099$	$P_2 > P_1 > P_3$
The proposed method	$\tau = 1, k = 1$	$EX(h_1) = 0.1627, EX(h_2) = 0.1630, EX(h_3) = 0.1099$	$P_2 > P_1 > P_3$
The proposed method $\tau = 2$	$k_1 = 1, k_2 = 1$	$EX(h_1) = 0.1476, EX(h_2) = 0.1469, EX(h_3) = 0.0962$	$P_1 > P_2 > P_3$
The proposed method $\tau = 3$	$k_1 = 1, k_2 = 1, k_3 = 1$	$EX(h_1) = 0.1409, EX(h_2) = 0.1396, EX(h_3) = 0.0908$	$P_1 > P_2 > P_3$

5.3.2. A Comparison with the TF2DLPGWA Operator

As seen in Table 8, the calculation results of the proposed WTF2DLPGHM operator in the case of $\tau = 1$ and $k = 1$ are exactly the same as for the TF2DLPGWA operator [29] in the case of $k = 1$. This is because, under such parameter settings, both the proposed WTF2DLPGHM operator and the

TF2DLPGWA operator [29] degenerate into a WTF2DLPA operator without considering attribute correlation. However, for the proposed WTF2DLPGHM operator, when τ is 2 or 3, the ranking result obtained is completely different to that of the TF2DLPGWA operator [29]. The proposed WTF2DLPGHM operator ($\tau = 2$ and $\tau = 3$) thinks that alternative P_2 is better than alternative P_1 , and the TF2DLPGWA operator [29] gives the opposite result. The reasons for this are as follows. The TF2DLPGWA operator proposed by Li et al. [29] is based on the simplest PA operator and GWA operator. These operators are developed in a completely independent attributes environment, and have some limitations in terms of practical application. In the lean management evaluation problem of industrial residential projects, there is an obvious relationship between attributes. Therefore, the TF2DLPGWA operator [29] cannot give accurate evaluation results. On the contrary, the proposed WTF2DLPGHM operator combines the excellent characteristics of the Hamy mean operator, and can consider the relationship among multiple input elements flexibly by taking different values for parameter τ . For example, when τ is 3, the proposed operator considers the relationship between two attributes, and when τ is 2, the relationship among three attributes is considered. Therefore, the proposed WTF2DLPGHM operator can adapt to more complex MADM environments.

5.3.3. A Comparison with the TF2DLBM Operator

Compared with the TF2DLBM operator, the proposed WTF2DLPGHM operator has two advantages: one is to integrate the characteristics of the PA operator, which can eliminate the influence of unreasonable data on the sorting results; the other is to model the relationship among multiple input elements by adjusting the parameter τ (see Table 9 for details).

For the TF2DLBM operator, when $p = q = 1$, the ranking results are the same as for the proposed WTF2DLPGHM operator when $\tau = 2$, $k_1 = 1$ and $k_2 = 1$, but there is a gap in the ranking results when $\tau = 3$, $k_1 = 2$, $k_2 = 1$ and $k_3 = 1$. The consistency of the former shows the effectiveness of the proposed method and the inconsistency of the latter shows that the parameter τ has an important impact on the calculation results. The TF2DLBM operator can capture the relationship between two attributes ($n = 2$), and enlarge or reduce the influence degree of corresponding attributes by adjusting the values of p and q . However, the TF2DLBM operator cannot be handled for a MADM problem in which multiple attributes ($n \geq 2$) have relationships. The proposed WTF2DLPGHM operator can make up for this deficiency. On the one hand, by adjusting the value of τ , the WTF2DLPGHM operator can model the relationship among multiple attributes; on the other hand, by adjusting the value of k , the influence degree of multiple attributes can be enlarged or reduced. Thus, the proposed method has strong flexibility and can deal with complex MADM problems.

Table 9. Results obtained by different MADM methods.

Methods	Parameters	Expected Value $E(h_i)$	Ranking
Method proposed by Shi [41] (TF2DLBM)	$p = q = 1$	$EX(h_1) = 0.0082, EX(h_2) = 0.0081, EX(h_3) = 0.0053$	$P_1 > P_2 > P_3$
The proposed method $\tau = 2$	$k_1 = 1, k_2 = 1$	$EX(h_1) = 0.1476, EX(h_2) = 0.1469, EX(h_3) = 0.0962$	$P_1 > P_2 > P_3$
The proposed method $\tau = 3$	$k_1 = 2, k_2 = 1, k_3 = 1$	$EX(h_1) = 0.1638, EX(h_2) = 0.1690, EX(h_3) = 0.1049$	$P_2 > P_1 > P_3$

6. Conclusions

When describing the relationship between attributes, a traditional Hamy mean operator gives the same importance to the attributes of interactive operation—unlike a Bonferroni mean operator, which can enlarge or reduce the influence of related attributes by adjusting the parameters. In order to make up for this deficiency, we introduce the idea of a generalized weighted average operator into the Hamy mean operator and propose the power generalized Hamy mean (PGHM) operator. Furthermore, we extend the PGHM operator to the trapezoidal fuzzy two-dimensional linguistic environment, and propose two new information aggregation tools, the TF2DLPGHM operator and the WTF2DLPGHM

operator. Finally, an example is given to show the specific steps. Compared with the existing two methods, the effectiveness of this method is verified.

Compared with the general MADM methods, the superiority of the proposed method is mainly shown in four aspects: (1) The proposed method takes TF2DLVs as the information representation form, which helps to improve the accuracy of the linguistic information description and reduce the loss of information. (2) The proposed WTF2DLPGHM operator can integrate the characteristics of the PA operator, which can eliminate the influence of unreasonable data on the sorting results. (3) The proposed WTF2DLPGHM operator can model the relationship among multiple input elements. (4) The proposed WTF2DLPGHM operator can enlarge or reduce the influence of related elements by adjusting the parameters. In the future, we will extend the partition HM operator to some new fuzzy information, such as q-Rung Orthopair Fuzzy information [42–45], the trapezoidal fuzzy two-dimensional linguistic environment to further clarify the interactions between attributes. In the meantime, we can use the proposed method to solve MAGDM problems.

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Appendix A

Proof. Based on the operational rules of TF2DLVs in Definition 2, we get $\zeta_{i_r}^{k_r} = ([a_{i_r}^{k_r}, b_{i_r}^{k_r}, c_{i_r}^{k_r}, d_{i_r}^{k_r}], \varsigma_{t_{i_r}})$, $(n\varepsilon_{i_r} \zeta_{i_r})^{k_r} = ([(n\varepsilon_{i_r} a_{i_r})^{k_r}, (n\varepsilon_{i_r} b_{i_r})^{k_r}, (n\varepsilon_{i_r} c_{i_r})^{k_r}, (n\varepsilon_{i_r} d_{i_r})^{k_r}], \varsigma_{t_{i_r}})$.

$$\text{Then, } \prod_{r=1}^{\tau} (n\varepsilon_{i_r} \zeta_{i_r})^{k_r} = \left(\left[\prod_{r=1}^{\tau} (n\varepsilon_{i_r} a_{i_r})^{k_r}, \prod_{r=1}^{\tau} (n\varepsilon_{i_r} b_{i_r})^{k_r}, \prod_{r=1}^{\tau} (n\varepsilon_{i_r} c_{i_r})^{k_r}, \prod_{r=1}^{\tau} (n\varepsilon_{i_r} d_{i_r})^{k_r} \right], \varsigma_{\min(t_{i_r})} \right)$$

$$\left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} \zeta_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}} = \left(\left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} a_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}}, \left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} b_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}}, \left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} c_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}}, \left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} d_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}}, \varsigma_{\min(t_{i_r})} \right)$$

$$\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} \zeta_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}} = \left(\left[\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} a_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} b_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}}, \right. \right.$$

$$\left. \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} c_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}}, \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\varepsilon_{i_r} d_{i_r})^{k_r} \right)^{\frac{1}{k_1+k_2+\dots+k_{\tau}}} \right], \varsigma_{\min(t_i)} \right)$$

Thus, the proof of Theorem 1 is completed. \square

Appendix B

Proof. Based on the operational rules of TF2DLVs in Definition 2, we can get

$$(n\lambda_{i_r} \zeta_{i_r})^{k_r} = ([(n\lambda_{i_r} a_{i_r})^{k_r}, (n\lambda_{i_r} b_{i_r})^{k_r}, (n\lambda_{i_r} c_{i_r})^{k_r}, (n\lambda_{i_r} d_{i_r})^{k_r}], \varsigma_{t_{i_r}})$$

Then, $\prod_{r=1}^{\tau} (n\lambda_{i_r} \tilde{\zeta}_{i_r})^{k_r} = \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} a_{i_r})^{k_r}, \prod_{r=1}^{\tau} (n\lambda_{i_r} b_{i_r})^{k_r}, \prod_{r=1}^{\tau} (n\lambda_{i_r} c_{i_r})^{k_r}, \prod_{r=1}^{\tau} (n\lambda_{i_r} d_{i_r})^{k_r} \right), \varsigma_{\min(t_{i_r})}$

$$\left(\prod_{r=1}^{\tau} (n\lambda_{i_r} \tilde{\zeta}_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}} = \left[\left(\prod_{r=1}^{\tau} (n\lambda_{i_r} a_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}}, \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} b_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}}, \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} c_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}}, \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} d_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}} \right], \varsigma_{\min(t_{i_r})}$$

Therefore, $WTF2DLPGHM^{(\tau, k_1, k_2, \dots, k_{\tau})}(\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_n) = \frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} \tilde{\zeta}_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}}$

$$= \left[\left(\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} a_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}} \right), \left(\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} b_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}} \right), \right.$$

$$\left. \left(\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} c_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}} \right), \left(\frac{1}{C_n^{\tau}} \sum_{\substack{1 \leq i_1 < \dots < i_{\tau} \leq n}} \left(\prod_{r=1}^{\tau} (n\lambda_{i_r} d_{i_r})^{k_r} \right)^{\frac{1}{k_1 + k_2 + \dots + k_{\tau}}} \right) \right], \varsigma_{\min(t_i)}$$

Thus, the proof of Theorem 2 is completed. \square

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