

Value at Risk Estimation Using the GARCH-EVT Approach with Optimal Tail Selection [†]

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Abstract: A conditional Extreme Value Theory (GARCH-EVT) approach is a two-stage hybrid method that combines a Generalized Autoregressive Conditional Heteroskedasticity (GARCH) filter with the Extreme Value Theory (EVT). The approach requires pre-specification of a threshold separating distribution tails from its middle part. The appropriate choice of a threshold level is a demanding task. In this paper we use four different optimal tail selection algorithms, i.e., the path stability method, the automated Eye-Ball method, the minimization of asymptotic mean squared error method and the distance metric method with a mean absolute penalty function, to estimate out-of-sample Value at Risk (VaR) forecasts and compare them to the fixed threshold approach. Unlike other studies, we update the optimal fraction of the tail for each rolling window of the returns. The research objective is to verify to what extent optimization procedures can improve VaR estimates compared to the fixed threshold approach. Results are presented for a long and a short position applying 10 world stock indices in the period from 2000 to June 2019. Although each approach generates different threshold levels, the GARCH-EVT model produces similar Value at Risk estimates. Therefore, no improvement of VaR accuracy may be observed relative to the conservative approach taking the 95th quantile of returns as a threshold.

Keywords: Value at Risk; optimal tail selection; Extreme Value Theory; GARCH-EVT

1. Introduction

Value at Risk (VaR) is the most widely known measure of market risk. VaR indicates how big the maximum loss of an asset or a portfolio of assets is over the target horizon so that there is a low, pre-specified probability q that the actual loss will be greater. VaR can also be considered in the context of returns. Formally, we denote by r_t the return on assets at time t . The one-day-ahead Value at Risk for a long trading position at a q significance level, noted $VaR_q(r_t)$, anticipated conditionally to an information set, \mathcal{F}_t , available at time t is defined by the formula:

$$P(r_{t+1} \leq VaR_q(r_t) | \mathcal{F}_t) = q. \quad (1)$$

This definition shows that VaR is a q th conditional quantile of the returns distribution. For a short trading position VaR is a $1-q$ th conditional quantile of the returns distribution:

$$P(r_{t+1} \geq VaR_{1-q}(r_t) | \mathcal{F}_t) = q. \quad (2)$$

The main practical problem consists in the selection of an appropriate method to measure Value at Risk. The most popular and the simplest methods include historical simulation, Monte Carlo simulation or variance-covariance. Pérignon and Smith (2010) [1] reported that 73% of U.S. international banks use historical simulation, while only a minority of financial institutions use more complex parametric models. When financial markets become volatile and extreme returns appear, none of these methods are capable of appropriately measuring the risk.

The Extreme Value Theory (EVT) provides a theoretical and practical foundation for statistical models describing extreme events. There is an extensive body of literature that focuses on EVT and discusses the tail behavior of assets [2–10]. EVT models deal with i.i.d. variables, for that purpose McNeil and Frey (2000) [11] constructed a combination of the Extreme Value Theory and Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model, referred to as conditional EVT or GARCH-EVT. The model captures the most important stylized facts about financial time series, such as volatility clustering and leptokurtosis, and quickly adapts to recent market movements. The two-stage GARCH-EVT model is widely used to estimate accurately financial risk measured by the Value at Risk or Expected Shortfall (ES). McNeil and Frey (2000) [11] showed that the application of GARCH and EVT combined leads to a more accurate estimation of VaR compared with EVT methods. Fernandez (2003) [12] showed that EVT outdoes a GARCH model with normal innovations by far and that it gives similar results to a GARCH model with t innovations, as long as the innovations come from a symmetric and fat-tailed distribution. In turn, Jadhav and Ramanathan (2009) [13] estimated VaR using 14 (parametric and non-parametric) estimation procedures at a 99% confidence level. The EVT method performs better than the other methods. Gençay and Selçuk (2004) [14] reported that at the 99th and higher quantiles the Generalized Pareto Distribution (GPD) model is clearly superior to five other methods used in the study in terms of VaR forecasting. Moreover, Z. Zhang and H. K. Zhang (2016) [15] showed that the Exponential GARCH (EGARCH) model with Generalized Error Distribution (GED) combined with the EVT approach does very well in predicting critical loss for precious metal markets. Just (2014) [16] verified unconditional and conditional VaR estimation models in the agricultural commodity market. The GARCH-EVT model enables a correct VaR estimation also in this market. Tabasi et al. (2019) [17] showed that the GARCH-EVT model outperforms the simple GARCH model with Student's t and normal distributions for residuals.

Application of GARCH-EVT in empirical research requires pre-specification of a threshold which separates tails of distribution from its middle part. The appropriate choice of threshold level u is ambiguous, but crucial in the estimation of Generalized Pareto Distribution parameters and the corresponding accuracy of Value at Risk. The standard practice is to adopt as low a threshold as possible, but there is a trade-off between variance and bias. If a threshold is too low, the asymptotic basis of the model is violated leading to a high bias. However, too high a threshold generates insufficient excesses with which the model is estimated, leading to a high variance [18]. Most authors preferred to select a threshold as a fixed quantile of the data set, instead of determining a threshold value at each step, especially when they use a moving window of observation to find out-of-sample VaR estimates. McNeil and Frey (2000) [11], Karmakar and Shukla (2015) [19], Bee et al. (2016) [20], Totić and Božović (2016) [21], Li (2017) [22], Fernandez (2003) [12], Jadhav and Ramanathan (2009) [13] and Huang et al. (2017) [23] chose the 90th quantile of the loss distribution as a threshold. In contrast, a less conservative, but fixed threshold was used by Gençay and Selçuk (2004) [14], Cifter (2011) [24], Soltane et al. (2012) [25].

Traditional approaches to threshold selection are based on graphical representations. A frequently used procedure consists in the analysis of a mean residual life plot, which represents the mean of the excesses of threshold u . This method was applied by Aboura (2014) [26] and Omari et al. (2017) [27] to estimate VaR based on the GARCH-EVT approach. Another very popular procedure to threshold selection involves a graphical representation of Hill [28], Pickands [29] or Dekkers–Einmahl–de Haan estimators [30]. The graphical based threshold choice procedures require identification of stable regions in the graphs and thus they are highly subjective. There is extensive literature proposing the optimal choice of threshold, corresponding to the application of automated methods. The most common methods for the adaptive choice of the threshold are based on the

minimization of asymptotic mean squared error (AMSE) estimate (e.g., [31–34]). For example, Hall (1990) [35] and Danielsson et al. (2001) [33] use a bootstrap procedure to minimize the AMSE. Heuristic rules, e.g., Csörgő and Viharos (1998) [36], the Eye-Ball method [37] or the path stability (PS) method [38] to select the tail fraction are easy to implement, but they are arbitrary. The distance metric methods were proposed in [37]. Danielsson et al. (2016) [37] used three types of distances between empirical and theoretical quantiles, i.e., the Kolmogorov–Smirnov, average squared and mean absolute distances. A recent overview of the topic can be found in [37,38]. However, none of these conceptions outperforms other methods in all situations, thus researchers are still searching for another method to find the optimal tail (e.g., [39]).

This paper provides an empirical study of conditional EVT. The research objective of this paper is to compare the predictive ability of VaR estimates when each estimate is made with an optimal choice of the distribution tail. Five methods are applied to describe the tail, i.e., the minimization of AMSE estimate introduced in [38], the path stability algorithm [38], the distance metric method with the mean absolute penalty function [37], the automated Eye-Ball method [37] and the fixed quantile procedure. Unlike other studies we update the optimal fraction of the tail for each moving window of observations. It means that each VaR forecast is calculated on the basis of a new estimate of the threshold and in this way we can estimate the risk with the newest time horizon. We hypothesize that the optimal choice of the tail fraction makes it possible to improve the accuracy of VaR prediction. This study contributes to the literature in the following ways. Firstly, we conduct the empirical study for one day out-of-sample Value at Risk forecasts using 10 stock indices from the period 2000–June 2019. We propose to use the GARCH-EVT model with optimal tail selection. For all considered models, we allow the tail fraction and the model parameters to change over time. Secondly, we compare the VaR forecasts accuracy and the tail fraction between the investigated models. The results indicate which model may be used in practice by investors, financial and regulatory institutions. This study is an extended version of conference proceedings [40], where only path stability algorithm was analyzed.

The rest of the paper is organized as follows. Section 2 provides methodological details. It introduces the Peaks Over Threshold (POT) model and describes the tail selection problem in this model. Then the conditional GARCH-EVT model and backtesting procedures are presented. Section 3 presents empirical results for VaR forecasting, while Section 4 concludes the study.

2. Methods

2.1. Modeling Tail Using Extreme Value Theory

The Extreme Value Theory is a theory that focuses primarily on analyzing the asymptotic behavior of extreme values of a random variable. The theory provides robust tools for estimating only extreme values distribution instead of the whole distribution. The Peaks Over Threshold, next to the Block Maxima Model (BMM), is one of the two key models of EVT. It allows us to model the tail regions of the distribution instead of the entire sample. Using the POT method one can easily derive a closed-form expression for the Value at Risk or Expected Shortfall. In this study we focus on the Value at Risk. Although VaR is a primary measure of market risk for financial institutions it not a subadditive measure. ES is free from the drawback and theoretically captures the information contained in the tail in a better fashion. However, as an expected value ES requires that the distribution must have a finite first moment. Guégan and Hassani (2018) [41] evidenced infinite ES in financial data when alpha-stable and Generalized Extreme Value distributions were considered. The same problem may appear in case of GPD when the shape parameter is greater than one. We avoid this difficulty in our study since only VaR is taken into consideration. The distribution of excesses over high threshold is defined as:

$$F_u(y) = P(X - u \leq x | X > u) = \frac{F(y+u) - F(u)}{1 - F(u)} \text{ for } 0 < y < x_0 - u, \quad (3)$$

where $x_0 \leq \infty$ is the right endpoint of F .

The Pickands-Balkema-de Haan Theorem [42] is a basic result in POT and states that for a large class of underlying distributions F , there exists a function $\beta(u)$ such that:

$$\lim_{u \rightarrow x_0} \sup_{0 < y < x_0 - u} |F_u(y) - G_{\xi, \beta(u)}(y)| = 0. \quad (4)$$

$G_{\xi, \beta(u)}(y)$ is the Generalized Pareto Distribution given by:

$$G_{\xi, \beta}(y) = \begin{cases} 1 - \left(1 + \xi \frac{y}{\beta}\right)^{-\frac{1}{\xi}}, & \xi \neq 0, \\ 1 - \exp\left(-\frac{y}{\beta}\right), & \xi = 0, \end{cases} \quad (5)$$

where $\beta > 0, y \geq 0$ for $\xi \geq 0$ and $0 \leq y \leq -\frac{\beta}{\xi}$ for $\xi < 0$. The distribution has only two parameters, β is a scale parameter and ξ is a shape parameter. Heavy tail distributions (i.e., stable, Cauchy, Student's t) have $\xi > 0$ (Fréchet domain of attraction), whereas thin tail distributions like normal and log-normal have $\xi = 0$ (Gumbel domain of attraction). Distributions with a finite right endpoint have $\xi < 0$ (Weibull domain of attraction). Rearranging (3) and (5) we obtain a cumulative distribution function of returns:

$$F(x) = (1 - F(u))G_{\xi, \beta}(y)(x - u) + F(u), \quad x > u. \quad (6)$$

To obtain a useful closed form of distribution (6) it is convenient to replace $F(u)$ with the empirical estimator of exceedance over a threshold. The estimator is of the form $\hat{F}(u) = \frac{1 - N_u}{n}$ where

N_u is a number of returns that exceed threshold u and n is the number of returns. The estimator of cumulative distribution F is then as follows:

$$\hat{F}(u) = 1 - \frac{N_u}{n} \left(1 + \xi \frac{(x-u)}{\beta}\right)^{-\frac{1}{\xi}}. \quad (7)$$

VaR for a short position (right tail of distribution) is given by the x value in Equation (7) and we get:

$$\widehat{VaR}_{1-q} = u + \frac{\beta}{\xi} \left(\left(\frac{n}{N_u} q \right)^{-\xi} - 1 \right), \quad (8)$$

where $1 - q$ is the confidence level of VaR. In order to calculate VaR for a long position (left tail of distribution) it is necessary to carry out the calculations for minus returns.

2.2. Optimal Tail Selection

As referred in the Introduction, there are a large number of methods dealing with the optimal choice of the distribution tail fraction. However, not all methods perform well in finite samples, therefore they may not be used in the GARCH-EVT model with moving windows. Some methods that perform well in simulation studies, based on theoretical distributions, may not perform well in financial applications. For instance, methods based on minimizing the asymptotic MSE, especially bootstrap-based methods, do not perform very well in empirical studies [37]. The efficiency assessment is usually conducted using simulations studies but it is not the case in this work. We handle the problem of the choice of optimal tail fraction from the investors' point of view without entering into theoretical considerations including estimator properties. This is empirical research and we focus on computational methods instead of simulations ones. In our study we test several methods implemented in the R tea package [43]. Since we measure the risk in finite samples, in our study we chose only four methods that were able to converge in the optimization procedure.

2.2.1. Mean Absolute Deviation Distance Metric Method

The procedure mean absolute deviation distance metric (MINDIST) proposed in [37] minimizes the distance between the largest upper order statistics of the dataset, i.e., the empirical tail, and the theoretical tail of a Pareto distribution. The parameter α of this distribution is estimated by the Hill estimator. The distance is then minimized with respect to this k . The optimal number, denoted k_0 here, is equivalent to the number of extreme values. We use mean absolute deviation (MAD) penalty functions in the study:

$$Q_n^{MAD} = \frac{1}{T} \sum_{j=1}^T |r_{n-j,n} - q(j, k)|, \quad (9)$$

where $r_{n-j,n}$ is the empirical quantiles and $q(j, k)$ is a quantile estimated by $r_{n-j+1,n} \left(\frac{k}{j}\right)^{\frac{1}{\hat{\alpha}_k}}$, $\hat{\alpha}_k$ is the Hill estimator. The Hill estimator [28] is defined as follows:

$$\hat{\xi} = \frac{1}{\hat{\alpha}} = \frac{1}{k} \sum_{i=1}^{k-1} (\log(r_{n-i,n}) - \log(r_{n-k,n})), \quad (10)$$

where k is the number of upper order statistics used in estimation of the tail index.

2.2.2. Eye-Ball Method

The automated Eye-Ball method is a heuristic procedure proposed in [37], and it searches for a stable region in the Hill plot by defining a moving window. Inside the defined window the estimates of the Hill estimator with respect to k have to be in a predefined range around the first estimate within this window. It is sufficient to claim that only the h percentage of the estimates within this window lies within this range. The smallest k to accomplish this is then the optimal number of upper order statistics, i.e., returns in the tail. The estimator is as follows:

$$k_{eye}^* = \min \left\{ k \in \{2, \dots, n^+ - w\} \mid h < \frac{1}{w} \sum_{i=1}^w I\{\hat{\alpha}(k) - \varepsilon < \hat{\alpha}(k+i) < \hat{\alpha}(k) + \varepsilon\} \right\}, \quad (11)$$

where w is the size of the moving window, which is typically 1% of the full sample, h is typically around 90%, while ε is 0.3, n^+ is the number of positive returns.

2.2.3. Path Stability Method

The path stability (PS) method is an algorithm introduced in [38]. The algorithm searches for a stable region of the PS, i.e., the Hill plot of a tail index with respect to k . This is done in the following steps:

Step 1. Given an observed returns (r_1, \dots, r_n) , compute $T(k) := \hat{\xi}_{k,n}$ using the Hill estimator for $k = 1, \dots, n - 1$.

Step 2. Obtain j_0 as a minimum value of j , a non-negative integer, such that the rounded values, to j decimal places ($j = 1$ here), of the estimates $T(k)$ are distinct. Define $a_k^{(T)}(j) = \text{round}(T(k), j)$, $k = 1, \dots, n - 1$, the rounded values of $T(k)$ to j decimal places.

Step 3. Consider the set of k values associated to equal consecutive values of $a_k^{(T)}(j_0)$ obtained in step 2. Set $k_{min}^{(T)}$ and $k_{max}^{(T)}$ the minimum and maximum values, respectively, of the set with the largest range. The largest run size is $l := k_{max}^{(T)} - k_{min}^{(T)}$.

Step 4. Consider all those estimates, $T(k)$, $k_{min}^{(T)} \leq k \leq k_{max}^{(T)}$, now with two additional decimal places, i.e., compute $T(k) = a_k^{(T)}(j_0 + 2)$. Obtain the mode of $T(k)$ and denote \mathcal{K}_T the set of k - values associated with this mode.

Step 5. Take \hat{k}_T as the maximum value of \mathcal{K}_T .

Step 6. Compute $\hat{\xi}^{PS} = \hat{\xi}_{\hat{k}_T,n}$.

2.2.4. Minimization of Asymptotic Mean Squared Error Method

The minimization of asymptotic mean squared error (dAMSE) method is an algorithm introduced in [38]. The tail in the dAMSE method is identified by minimizing the AMSE criterion with respect to k . The optimal number, denoted k_0 , may be associated with the unknown threshold u of the tail index with respect to k . This is done in the following steps:

Step 1. Given an observed returns (r_1, \dots, r_n) , compute for the tuning parameters $\tau = 0$ and $\tau = 1$, the values of $\hat{\rho}_\tau(k) := -|3(W_{k,n}^{(\tau)} - 1)/(W_{k,n}^{(\tau)} - 3)|$, dependent on the statistics:

$$W_{k,n}^{(\tau)} := \begin{cases} \frac{(M_{k,n}^{(1)})^\tau - (M_{k,n}^{(2)}/2)^{\tau/2}}{(M_{k,n}^{(2)}/2)^{\tau/2} - (M_{k,n}^{(3)}/6)^{\tau/3}}, & \text{if } \tau \neq 0, \\ \frac{\ln(M_{k,n}^{(1)}) - \ln(M_{k,n}^{(2)}/2)/2}{\ln(M_{k,n}^{(2)}/2)/2 - \ln(M_{k,n}^{(3)}/6)/3}, & \text{if } \tau = 0, \end{cases} \quad (12)$$

where

$$M_{k,n}^{(j)} = \frac{1}{k} \sum_{i=1}^k (\log r_{n-i+1:n} - \log r_{n-k:n})^j, \quad j = 1, 2, 3. \quad (13)$$

Step 2. Consider $\mathcal{K} = ([n^{0.995}], [n^{0.999}])$. Compute the median of $\{\hat{\rho}_\tau(k)\}_{k \in \mathcal{K}}$, denoted \mathcal{K}_τ , and compute $I_\tau := \sum_{k \in \mathcal{K}} (\hat{\rho}_\tau(k) - \chi_\tau)^2$, $\tau = 0, 1$. Next choose the tuning parameter, $\tau^* = 0$, if $I_0 < I_1$; otherwise choose $\tau^* = 1$.

Step 3. Work with $\hat{\rho} = \hat{\rho}_{\tau^*}(k) = \hat{\rho}_{\tau^*}(k_{01})$ and $\hat{\beta} = \hat{\beta}_{\tau^*}(k) := \hat{\beta}_{\hat{\rho}_{\tau^*}}(k_{01})$, for $k_{01} = [n^{0.999}]$ and for

$$\hat{\beta}_{\hat{\rho}}(k) := \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{d_k(\hat{\rho})D_k(0) - D_k(\hat{\rho})}{d_k(\hat{\rho})D_k(\hat{\rho}) - D_k(2\hat{\rho})}, \quad (14)$$

dependent on the estimator $\hat{\rho} = \hat{\rho}_{\tau^*}(k_{01})$, and where, for any $\alpha \leq 0$,

$$d_k(\alpha) := \frac{1}{k} \sum_{i=1}^k (i/k)^{-\alpha}, \quad D_k(\alpha) := \frac{1}{k} \sum_{i=1}^k (i/k)^{-\alpha} U_i, \quad (15)$$

with

$$U_i = i \sum_{j=1}^k (\log r_{n-i+1:n} - \log r_{n-k:n}), \quad 1 \leq i \leq k < n, [n^{0.999}], \quad (16)$$

the scaled log-spacings.

Step 4. On the basis of estimators $\hat{\beta}$ and $\hat{\rho}$ compute:

$$\hat{k}_0^H := \left\lceil \left(\frac{(1 - \hat{\rho})^2 n^{-2\hat{\rho}}}{-2\hat{\rho}\hat{\beta}^2} \right)^{\frac{1}{1-2\hat{\rho}}} \right\rceil. \quad (17)$$

Step 5. Compute $\hat{\xi}^H := \hat{\xi}_{\hat{k}_0^H, n}^H$.

2.3. Conditional Extreme Value Theory Model

The GARCH-EVT model was introduced by McNeil and Frey [11] to VaR modeling by extending the EVT framework to dependent data. This model uses the EVT to model the tails of standardized residues e_t obtained from the GARCH model. When the estimated GARCH model is correct, the residuals of the model should be realizations of the unobserved i.i.d. noise variables. It was assumed in this study that returns are modeled using the most popular GARCH(1,1) model [44]:

$$r_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2, \quad (18)$$

where: $\omega, \alpha, \beta > 0, \alpha + \beta < 1, \varepsilon_t \sim \text{i.i.d.}(0, 1)$. VaR for a short position (right tail) is calculated using the formula:

$$\text{VaR}_{1-q}(r_t) = \sigma_t(1) \text{VaR}_{1-q}(e_t), \quad (19)$$

where: $\sigma_t(1)$ —one step ahead forecast of conditional volatility of the GARCH(1,1) model, $\text{VaR}_{1-q}(e_t)$ —calculated from Formula (8) for the standardized residuals e_t of the GARCH(1,1) model.

2.4. Backtesting

The model used for VaR estimation should be statistically verified using the backtesting procedure. VaR verification tests are based on hit function (failure process) $\{I_t(q)\}_{t=1}^T$ defined for a long trading position by the formula:

$$I_t(q) = \begin{cases} 1, & r_t < \text{VaR}_q(r_{t-1}), \\ 0, & r_t \geq \text{VaR}_q(r_{t-1}). \end{cases} \quad (20)$$

In practice the most commonly used test is the Kupiec's proportion of failures test (also known as the unconditional coverage test) [45]. According to Kupiec (1995) [45], the number of VaR violations by actual returns has a binomial distribution $T_1 \sim B(T, q)$ and the hypotheses are defined as:

$$H_0: P(I_t(q) = 1) = E[I_t(q)] = q, \quad H_1: P(I_t(q) = 1) = E[I_t(q)] \neq q. \quad (21)$$

The null hypothesis assumes that the share of VaR violations by actual returns is compliant with an assumed q . The test statistic is defined as:

$$LR_{UC} = 2(\log(\hat{q}^{T_1}(1 - \hat{q})^{T_0}) - \log(q^{T_1}(1 - q)^{T_0})), \quad (22)$$

where: $\hat{q} = T_1/(T_0 + T_1)$, T_1 —the number of VaR exceedances, T_0 —the number of unexceeded VaR. With the true null hypothesis LR_{UC} statistic has the asymptotic chi-square distribution with one degree of freedom.

The advantage of Kupiec's test is that it assesses the model taking into account either too large and too small number of exceedances. A good model used for VaR estimation should also be characterized by independence of exceedances. Test proposed by Christoffersen (1998) [46] additionally checks independence of exceedances. When the model accurately estimates VaR, then an exception today should not depend on whether or not an exception occurred on the previous day. In order to test hypothesis that the exceedances are independent (the first exceedance) an alternative is defined where the sequence of exceedances is modeled with a first order Markov chain with a matrix of transition probabilities:

$$\Pi = \begin{pmatrix} 1 - \pi_{01} & \pi_{01} \\ 1 - \pi_{11} & \pi_{11} \end{pmatrix}, \quad (23)$$

where $\pi_{ij} = P(I_t(q) = j | I_{t-1}(q) = i)$. The Markov chain reflects the existence of a order one memory in the sequence of exceedances. The hypotheses in the independence test are defined as:

$$H_0: \pi_{01} = \pi_{11}, \quad H_1: \pi_{01} \neq \pi_{11}. \quad (24)$$

Finally, Christoffersen (1998) [46] combines the two tests, i.e., the unconditional coverage test and the independence test into the conditional coverage test. In this test, the null hypothesis takes the following form:

$$H_0: \pi_{01} = \pi_{11} = q. \quad (25)$$

The test statistic is of the form:

$$LR_{CC} = 2 \left(\log(\hat{\pi}_{01}^{T_{01}}(1 - \hat{\pi}_{01})^{T_{00}}\hat{\pi}_{11}^{T_{11}}(1 - \hat{\pi}_{11})^{T_{10}}) - \log(q^{T_{01}+T_{11}}(1 - q)^{T_{00}+T_{10}}) \right), \quad (26)$$

where: $\hat{\pi}_{ij} = T_{ij}/(T_{i0} + T_{i1})$, T_{ij} —the number of days when condition j occurred assuming that condition i occurred on the previous day (1 if exceedance occurs, 0 if no exceedance appears).

Under the true null hypothesis LR_{CC} statistic has the asymptotic chi-square distribution with two degrees of freedom.

The disadvantage of Christoffersen's test is that it examines the independence of the first exceedance only. Therefore, it should be supplemented by other tests e.g., a test which analyzes whether the number of periods (days) between the violations of VaR by actual returns is independent over time [47]. Under the null hypothesis of this test, the duration of time between VaR violations should have no memory and mean duration of $1/q$. Since the only continuous distribution, which is memory free, is the exponential distribution, the test can be conducted on any distribution which

embeds the exponential as a restricted case, and a likelihood ratio test then conducted to see whether the restriction holds. Here, the Weibull distribution is used with parameter $b = 1$ representing the case of the exponential. The Weibull distribution is defined as:

$$f(d) = a^b b d^{b-1} \exp\{-(ad)^b\}, \quad (27)$$

where d —the duration of time (in days) between two VaR exceedances. Under the null hypothesis of independence we have the likelihood:

$$L(q) = \prod_{t=1}^{T_1-1} (q \exp(-q d_t)), \quad (28)$$

where T_1 —the number of periods, in which an exceedance occurred. The likelihood ratio test statistic is defined as:

$$LR_{UD} = 2(\log L(\hat{a}) - \log L(q)). \quad (29)$$

The test statistic is asymptotically distributed as chi-square with one degree of freedom. We refer to [47] for details of the test.

The basic test using both VaR values and series of exceedances is the Dynamic Quantile test of Engle and Manganelli [48]. The idea of the test is that exceedances at time t should not depend on exceedances at earlier times, or on VaR and any processed information $(\omega_{t-1,j})$ available at time $t - 1$ (e.g., past returns, square of past returns). Engle and Manganelli define $Hit_t(q) = I_t(q) - q$. In the test the regression equation is estimated:

$$Hit_t(q) = \beta_0 + \sum_{i=1}^p \beta_i Hit_{t-i}(q) + \beta_{p+1} VaR_q(r_{t-1}) + \sum_{j=1}^n \beta_{p+j+1} f(\omega_{t-1,j}) + \varepsilon_t. \quad (30)$$

VaR is well estimated if there is no reason to reject the null hypothesis:

$$H_0: \beta_i = 0, i = 0, 1, 2, \dots, p + n + 1. \quad (31)$$

The current VaR exceedances are uncorrelated with past exceedances when $\beta_i = 0$ for $i = 1, \dots, p$, whereas unconditional coverage hypothesis is verified for $\beta_0 = 0$.

In matrix notation, we have $Hit = X\beta + e$. The Wald test statistic is defined as:

$$DQ = \frac{Hit' X(X'X)^{-1} X' Hit}{q(1-q)}. \quad (32)$$

Under the true null hypothesis DQ statistic has the asymptotic chi-square distribution with $p + n + 2$ degrees of freedom.

This test identifies an incorrect VaR measurement, which is not rejected by the classic tests of the number and independence of exceedances. We refer to [48,49] for details of the test.

The loss function is another goodness-of-fit measure for VaR calculation. The loss function for a given q may be determined as in [50]:

$$Q = T^{-1} \sum_{t=1}^T (q - I_t(q))(r_t - VaR_q(r_{t-1})). \quad (33)$$

A lower Q value indicates a better goodness of fit.

3. Results of Empirical Study

To test the forecasting performance of the examined GARCH-EVT model with different thresholds we chose 10 stock indices as the basis for analysis. They were the S&P 500 (SPX), FTSE 100 (UKX), CAC 40 (CAC), DAX, OMX Stockholm 30 (OMXS), KOSPI, NIKKEI 225 (NKX), Hang Seng (HSI), Bovespa (BVP), All Ordinaries (AOR). The data consists of daily prices of the selected assets from the beginning of 2000 up to the end of June 2019. It gives from 4781 (NKX) to 4982 (CAC) percentage log-returns which are used in our calculations. The data set was obtained from the financial stock news website, stooq.pl [51]. Using rolling windows of 2000 returns in size we updated the estimates of parameters for each moving window. Most used optimization methods require a relatively long window of observation to the optimal algorithms convergence. We selected a moving window of 2000 observations. The window of the 1000 generates considerable calculation problems.

The less demanding optimization algorithm is PS, which may be used for relatively small samples. Echaust (2018) [40] used only 750 returns in the moving window.

According to the GARCH-EVT approach the tail fraction is estimated for standardized residuals of the GARCH model, not for returns. We used a GARCH(1,1) model with normal innovations. Then we calculated the tail fraction in five cases, i.e., using the optimal tail selection method, and additionally the 95th quantile of each moving window. Table 1 shows the mean, minimum, maximum and standard deviation of a threshold level for indices, while Figure 1 presents the estimated threshold level for the S&P 500 index. The PS method estimates the threshold in the most conservative way, much lower than the 95th quantile. Such a choice of threshold guarantees sufficient data in the tail to calculate VaR estimates at standard confidence levels. On the other hand, the threshold calculated with the PS algorithm has a very high range and standard deviation. It means that the threshold is highly volatile over time. It is of interest that PS is the only method that indicates a higher right threshold than the left one. The Eye-Ball algorithm is the most restrictive and it sets the threshold at a high level. In turn, the MINDIST method produces the most volatile threshold estimates compared to other methods. These two remarks are consistent with Danielson et al. (2016) [37], who argued that the automated Eye-Ball method and the MINDIST method with the Kolmogorov–Smirnov metric tend to pick a small number of data in the tails and therefore the tail close to the maximum. We use a different penalty function in the MINDIST method, but the conclusion is similar. The high threshold level may result in an insufficient number of returns lying in the tails to estimate GPD parameters in the next step (see Table 2). The dAMSE methodology uses a number of order statistics the closest to the 95th quantile and has a standard deviation the closest to the method with the 95th quantile among the optimization methods used in the study.

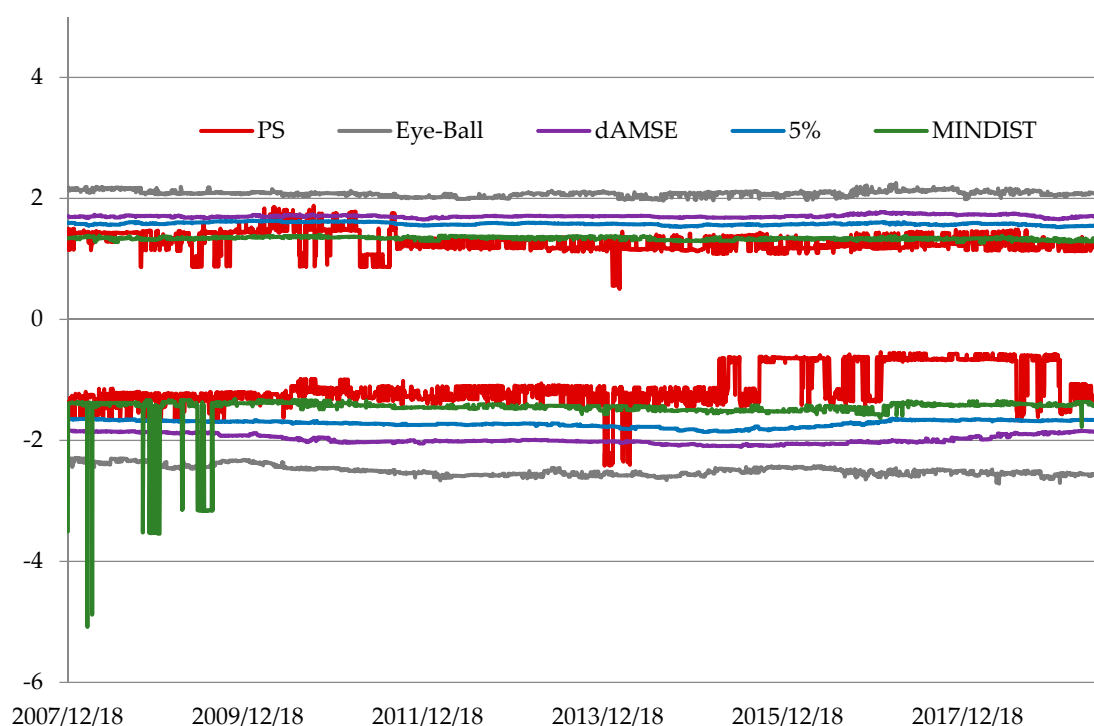


Figure 1. Threshold estimates for lower and upper tails for S&P 500 returns.

Table 1. Threshold estimates from the Conditional Extreme Value Theory model with optimal tail selection.

		Lower Tail				Upper Tail			
SPX	Max	Mean	Min	StDev	Min	Mean	Max	StDev	
PS	−2.416	−1.115	−0.541	0.308	0.504	1.284	1.880	0.164	
Eye-Ball	−2.713	−2.499	−2.290	0.080	1.963	2.081	2.253	0.045	

dAMSE	-2.103	-1.979	-1.831	0.076	1.648	1.706	1.777	0.022
5%	-1.861	-1.720	-1.638	0.055	1.525	1.580	1.631	0.023
MINDIST	-5.083	-1.483	-1.317	0.304	1.267	1.339	1.396	0.023
UKX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-1.738	-1.022	-0.423	0.303	0.444	1.194	2.010	0.435
Eye-Ball	-2.621	-2.396	-2.276	0.061	1.865	2.064	2.217	0.068
dAMSE	-2.011	-1.942	-1.875	0.035	1.633	1.702	1.757	0.029
5%	-1.784	-1.741	-1.670	0.021	1.515	1.587	1.656	0.028
MINDIST	-1.620	-1.478	-1.336	0.052	1.263	1.400	4.066	0.327
CAC	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-2.239	-1.152	-0.579	0.333	0.697	1.241	1.643	0.116
Eye-Ball	-2.594	-2.430	-2.288	0.046	1.859	2.112	2.365	0.075
dAMSE	-1.967	-1.888	-1.811	0.033	1.629	1.678	1.747	0.021
5%	-1.763	-1.718	-1.661	0.021	1.523	1.562	1.620	0.018
MINDIST	-5.506	-1.955	-1.298	0.991	1.223	1.324	3.755	0.163
DAX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-2.173	-1.260	-0.506	0.333	0.473	1.123	2.059	0.340
Eye-Ball	-2.568	-2.428	-2.238	0.074	1.975	2.095	2.322	0.075
dAMSE	-1.986	-1.926	-1.853	0.034	1.696	1.742	1.801	0.021
5%	-1.784	-1.724	-1.660	0.026	1.565	1.621	1.670	0.019
MINDIST	-4.510	-1.627	-1.378	0.521	1.275	1.365	4.614	0.071
OMXS	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-2.249	-1.170	-0.611	0.354	0.837	1.047	1.429	0.117
Eye-Ball	-2.584	-2.369	-2.178	0.082	1.996	2.128	2.304	0.057
dAMSE	-1.967	-1.904	-1.804	0.034	1.664	1.718	1.802	0.030
5%	-1.764	-1.684	-1.595	0.046	1.533	1.590	1.646	0.026
MINDIST	-5.900	-1.491	-1.312	0.440	1.278	1.348	1.482	0.042
KOSPI	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-2.285	-1.106	-0.635	0.371	0.560	1.251	1.993	0.205
Eye-Ball	-2.680	-2.481	-2.290	0.061	1.889	2.063	2.197	0.056
dAMSE	-2.105	-2.032	-1.916	0.044	1.642	1.684	1.721	0.015
5%	-1.857	-1.771	-1.684	0.044	1.530	1.578	1.627	0.020
MINDIST	-6.799	-1.584	-1.386	0.421	1.227	1.317	1.388	0.028
NKX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-2.103	-1.365	-0.623	0.257	0.648	1.368	1.905	0.223
Eye-Ball	-2.706	-2.421	-2.249	0.110	1.844	1.980	2.202	0.067
dAMSE	-1.955	-1.904	-1.821	0.030	1.624	1.682	1.761	0.027
5%	-1.780	-1.728	-1.629	0.036	1.524	1.561	1.606	0.016
MINDIST	-7.455	-2.331	-1.341	0.788	1.292	1.407	4.101	0.394
HSI	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-2.006	-1.340	-0.992	0.206	0.821	1.352	2.053	0.360
Eye-Ball	-2.687	-2.402	-2.119	0.127	1.978	2.111	2.244	0.053
dAMSE	-1.902	-1.845	-1.805	0.017	1.687	1.749	1.865	0.031
5%	-1.720	-1.680	-1.636	0.015	1.562	1.620	1.718	0.040
MINDIST	-4.866	-1.454	-1.346	0.386	1.307	1.719	4.628	0.938
BVP	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-2.358	-1.134	-0.611	0.253	1.024	1.273	1.893	0.148
Eye-Ball	-2.473	-2.312	-2.127	0.074	1.967	2.122	2.316	0.048
dAMSE	-1.882	-1.815	-1.748	0.030	1.656	1.710	1.764	0.022
5%	-1.733	-1.678	-1.585	0.038	1.560	1.597	1.643	0.018
MINDIST	-5.372	-1.710	-1.306	0.529	1.254	1.353	2.641	0.145
AOR	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	-2.213	-1.052	-0.791	0.101	0.796	1.095	1.964	0.173
Eye-Ball	-2.750	-2.461	-2.268	0.087	1.907	2.011	2.087	0.028
dAMSE	-2.015	-1.952	-1.842	0.040	1.601	1.644	1.694	0.022
5%	-1.784	-1.708	-1.612	0.036	1.496	1.545	1.603	0.025
MINDIST	-4.639	-1.786	-1.349	0.913	1.262	1.318	2.618	0.035

Note: PS—path stability method, dAMSE—minimization of asymptotic mean squared error method, 5%—fixed threshold method, MINDIST—mean absolute deviation distance metric method. The calculations were performed with the fGarch [52] and tea [43] packages in R. The following assumptions were made: GARCH—maximum log-likelihood estimation; PS— $j = 1$; Eye-Ball— $w = 0.02$ (with the exception of AOR for right tail $w = 0.021$), $h = 0.9$, $\varepsilon = 0.3$; MINDIST—size of the upper tail $ts = 0.15$.

Table 2. Number of cases where Generalized Pareto Distribution parameters are not estimated for the mean absolute deviation distance metric method.

	SPX Lower Tail	UKX Upper Tail	CAC Lower Tail	CAC Upper Tail	DAX Lower Tail	DAX Upper Tail	OMXS Lower Tail	KOSPI Lower Tail
Number of moving windows without estimating Generalized Pareto Distribution (GPD) parameters	3	22	334	9	47	1	30	8
Number of moving windows	2902	2924	2982	2982	2948	2948	2891	2811
Percentage of moving windows without estimating GPD parameters	0.10	0.75	11.20	0.30	1.59	0.03	1.04	0.28
	NKX Lower Tail	NKX Upper Tail	HSI Lower Tail	HSI Upper Tail	BVP Lower Tail	BVP Upper Tail	AOR Lower Tail	
Number of moving windows without estimating GPD parameters	552	59	41	270	19	27	198	
Number of moving windows	2781	2781	2803	2803	2821	2821	2927	
Percentage of moving windows without estimating GPD parameters	19.85	2.12	1.46	9.63	0.67	0.96	6.76	

Note: the calculations were performed with package evir [53] in R. GPD parameters were estimated by the maximum log-likelihood method.

Having specified tails of distribution the next step in calculations is to estimate GDP parameters and Value at Risk forecasts. Guégan and Hassani (2011) [54] underline the fact that the method chosen to estimate the GPD parameters affects the VaR tremendously. We use log-likelihood method described in Coles (2001) [18] which is the most popular estimation method for this model, and Formula (8) for VaR calculation. We examined the out-of-sample 99% and 99.5% VaR estimates for the left and right tails of returns distribution. Since thresholds based on the optimal choice differ substantially, it may be expected that GPD parameters and thus VaR estimation for these tail choices will behave differently. Tables 3 and 4 present results of VaR estimation and Figure 2 presents 99% VaR estimates for the S&P 500 index. We can see that the VaR estimates across methods differ very slightly despite the differences in the thresholds. We can expect that the risk is estimated correctly regardless of the method used to describe the tails. In all the cases the VaR estimates in the left tail are greater than these in the right tails. Most investors believe that the left tail risk is higher than the right one. It is a natural consequence of the crashes being perceived as much more turbulent than the booms. Crashes develop in shorter time intervals than booms and changes of prices are significantly bigger. The results from Tables 3 and 4 support this perception. Additionally, the PS algorithm highlights the risk asymmetry harder than other methods and the asymmetry tends to be greater for a more extreme risk, i.e., a higher VaR confidence.

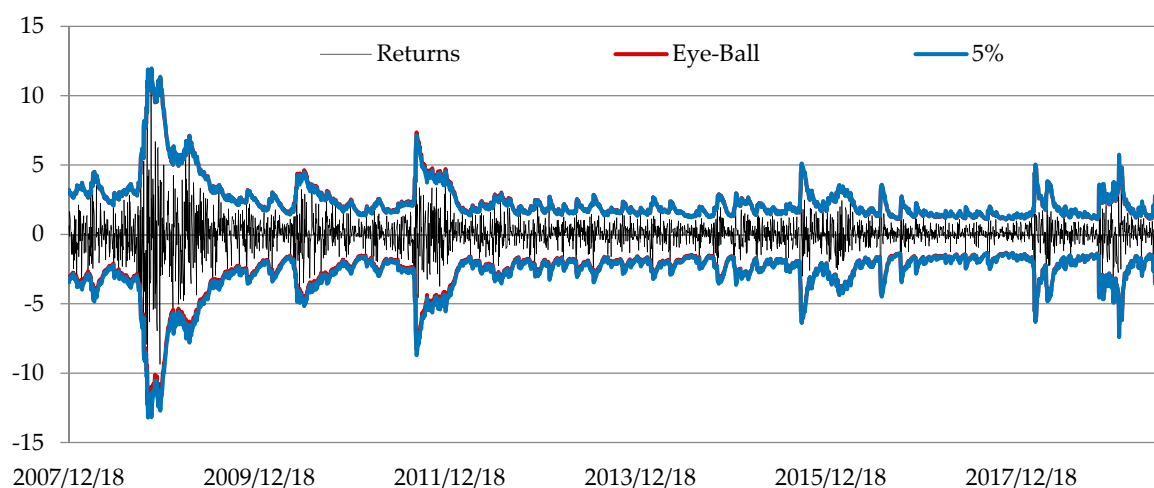
**Figure 2.** Value at Risk 0.01 and Value at Risk 0.99 estimates from the Conditional Extreme Value Theory model with Eye-Ball and fixed threshold methods for S&P 500 returns.

Table 3. Value at Risk 0.01 and Value at Risk 0.99 estimates from the Conditional Extreme Value Theory model with optimal tail selection.

Lower Tail, VaR 0.01					Upper Tail, VaR 0.99			
SPX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−13.170	−2.910	−1.312	1.695	1.072	2.449	11.897	1.517
Eye-Ball	−12.647	−2.819	−1.306	1.620	1.072	2.482	11.917	1.530
dAMSE	−13.001	−2.878	−1.313	1.669	1.078	2.472	11.959	1.529
5%	−13.192	−2.915	−1.329	1.691	1.084	2.471	11.970	1.527
MINDIST	x	x	x	x	1.076	2.458	11.934	1.520
UKX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−14.349	−2.854	−1.228	1.530	1.018	2.412	11.371	1.213
Eye-Ball	−14.191	−2.847	−1.225	1.520	1.015	2.347	11.251	1.193
dAMSE	−14.135	−2.841	−1.219	1.514	1.026	2.376	11.305	1.203
5%	−14.121	−2.842	−1.225	1.509	1.042	2.389	11.415	1.212
MINDIST	−14.207	−2.855	−1.229	1.528	x	x	x	x
CAC	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−14.763	−3.523	−1.481	1.667	1.334	3.061	12.235	1.371
Eye-Ball	−13.665	−3.369	−1.470	1.533	1.368	3.049	12.080	1.349
dAMSE	−13.624	−3.395	−1.486	1.530	1.325	3.055	12.134	1.365
5%	−13.919	−3.424	−1.479	1.567	1.326	3.048	12.161	1.362
MINDIST	x	x	x	x	x	x	x	x
DAX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−13.571	−3.417	−1.504	1.529	1.425	3.015	11.543	1.284
Eye-Ball	−12.837	−3.278	−1.474	1.438	1.348	2.966	11.032	1.252
dAMSE	−12.829	−3.316	−1.505	1.452	1.361	2.961	11.141	1.259
5%	−13.095	−3.365	−1.510	1.480	1.383	2.979	11.296	1.274
MINDIST	x	x	x	x	x	x	x	x
OMXS	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−11.838	−3.361	−1.469	1.637	1.326	2.993	10.483	1.423
Eye-Ball	−12.185	−3.358	−1.386	1.684	1.327	2.988	10.367	1.411
dAMSE	−12.115	−3.349	−1.419	1.666	1.324	3.004	10.398	1.417
5%	−12.048	−3.377	−1.425	1.675	1.324	3.000	10.374	1.410
MINDIST	x	x	x	x	1.325	3.017	10.570	1.434
KOSPI	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−15.382	−2.958	−1.484	1.569	1.216	2.465	13.476	1.381
Eye-Ball	−14.804	−2.860	−1.424	1.514	1.224	2.464	13.148	1.353
dAMSE	−15.063	−2.865	−1.448	1.535	1.226	2.470	13.282	1.368
5%	−15.412	−2.941	−1.486	1.570	1.218	2.458	13.215	1.361
MINDIST	x	x	x	x	1.217	2.472	13.371	1.382
NKX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−17.531	−3.810	−1.945	1.815	1.549	3.119	15.191	1.547
Eye-Ball	−17.391	−3.703	−1.825	1.772	1.557	3.097	14.479	1.478
dAMSE	−17.404	−3.719	−1.847	1.777	1.556	3.108	14.675	1.495
5%	−17.531	−3.763	−1.873	1.789	1.551	3.125	14.888	1.522
MINDIST	x	x	x	x	x	x	x	x
HSI	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−18.454	−3.524	−1.684	1.858	1.570	3.182	16.768	1.688
Eye-Ball	−17.963	−3.559	−1.729	1.811	1.546	3.165	16.881	1.713
dAMSE	−17.924	−3.513	−1.700	1.791	1.547	3.156	16.836	1.707
5%	−18.104	−3.516	−1.681	1.815	1.565	3.173	16.776	1.702
MINDIST	x	x	x	x	x	x	x	x
BVP	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−16.900	−4.128	−2.667	1.571	2.430	3.710	13.812	1.251
Eye-Ball	−15.850	−3.981	−2.537	1.452	2.423	3.726	13.790	1.230
dAMSE	−16.218	−3.985	−2.533	1.509	2.437	3.725	13.820	1.246
5%	−16.279	−3.993	−2.575	1.499	2.443	3.713	13.849	1.247
MINDIST	x	x	x	x	x	x	x	x
AOR	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−11.313	−2.598	−1.147	1.279	0.975	2.126	9.146	1.016
Eye-Ball	−10.784	−2.529	−1.143	1.213	0.972	2.121	8.809	0.980
dAMSE	−10.986	−2.548	−1.154	1.230	0.966	2.135	8.961	1.000
5%	−11.188	−2.602	−1.156	1.263	0.961	2.126	9.005	1.009
MINDIST	x	x	x	x	0.977	2.124	9.040	1.012

Note: PS—path stability method, dAMSE—minimization of asymptotic mean squared error method, 5%—fixed threshold method, MINDIST—mean absolute deviation distance metric method, x—VaR was not calculated. The calculations were performed with the evir package [53] in R. Generalized Pareto Distribution parameters were estimated by the maximum log-likelihood method.

Table 4. Value at Risk 0.005 and Value at Risk 0.995 estimates from the Conditional Extreme Value Theory model with optimal tail selection.

Lower tail, VaR 0.005					Upper tail, VaR 0.995			
SPX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−15.376	−3.369	−1.525	1.981	1.206	2.716	13.192	1.681
Eye-Ball	−14.556	−3.209	−1.532	1.857	1.197	2.742	13.195	1.689
dAMSE	−15.094	−3.308	−1.515	1.940	1.210	2.734	13.248	1.692
5%	−15.373	−3.367	−1.538	1.973	1.218	2.734	13.263	1.690
MINDIST	x	x	x	x	1.210	2.722	13.223	1.683
UKX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−16.329	−3.240	−1.390	1.748	1.146	2.701	12.709	1.353
Eye-Ball	−16.484	−3.261	−1.397	1.772	1.146	2.627	12.584	1.332
dAMSE	−16.169	−3.226	−1.378	1.736	1.156	2.665	12.624	1.342
5%	−16.146	−3.226	−1.385	1.729	1.176	2.681	12.766	1.353
MINDIST	−16.235	−3.239	−1.389	1.747	x	x	x	x
CAC	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−17.052	−4.055	−1.730	1.917	1.555	3.500	13.819	1.547
Eye-Ball	−16.224	−3.845	−1.694	1.776	1.603	3.476	13.545	1.497
dAMSE	−15.971	−3.922	−1.733	1.780	1.549	3.496	13.732	1.539
5%	−16.304	−3.960	−1.725	1.824	1.549	3.485	13.742	1.535
MINDIST	x	x	x	x	x	x	x	x
DAX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−15.665	−3.935	−1.720	1.769	1.620	3.365	12.789	1.422
Eye-Ball	−14.703	−3.703	−1.673	1.661	1.523	3.332	12.409	1.407
dAMSE	−14.858	−3.809	−1.720	1.682	1.548	3.308	12.385	1.396
5%	−15.171	−3.876	−1.726	1.716	1.577	3.330	12.569	1.414
MINDIST	x	x	x	x	x	x	x	x
OMXS	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−14.028	−3.887	−1.704	1.928	1.486	3.380	11.852	1.604
Eye-Ball	−14.397	−3.907	−1.586	1.992	1.508	3.363	11.889	1.596
dAMSE	−14.222	−3.853	−1.637	1.948	1.480	3.380	11.757	1.597
5%	−14.142	−3.886	−1.643	1.958	1.481	3.375	11.733	1.590
MINDIST	x	x	x	x	1.483	3.392	11.921	1.613
KOSPI	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−18.143	−3.394	−1.695	1.849	1.344	2.761	15.188	1.564
Eye-Ball	−17.295	−3.290	−1.667	1.751	1.355	2.741	14.673	1.509
dAMSE	−17.714	−3.277	−1.643	1.805	1.356	2.765	15.035	1.553
5%	−18.184	−3.370	−1.692	1.853	1.346	2.752	14.961	1.546
MINDIST	x	x	x	x	1.345	2.766	15.101	1.565
NKX	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−20.151	−4.413	−2.251	2.080	1.700	3.449	17.001	1.729
Eye-Ball	−20.613	−4.376	−2.130	2.093	1.707	3.418	16.183	1.642
dAMSE	−19.997	−4.324	−2.143	2.044	1.708	3.431	16.387	1.664
5%	−20.125	−4.369	−2.172	2.056	1.701	3.453	16.663	1.700
MINDIST	x	x	x	x	x	x	x	x
HSI	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−21.248	−4.044	−1.913	2.151	1.763	3.578	18.759	1.882
Eye-Ball	−20.707	−4.072	−1.940	2.098	1.748	3.589	19.063	1.925
dAMSE	−20.714	−4.032	−1.929	2.086	1.746	3.553	18.830	1.902
5%	−20.897	−4.033	−1.912	2.110	1.767	3.571	18.764	1.896
MINDIST	x	x	x	x	x	x	x	x
BVP	Max	Mean	Min	StDev	Min	Mean	Max	StDev
PS	−19.470	−4.757	−3.046	1.807	2.729	4.189	15.698	1.420
Eye-Ball	−17.825	−4.639	−3.006	1.653	2.743	4.213	15.798	1.402
dAMSE	−18.782	−4.636	−2.965	1.745	2.753	4.196	15.701	1.418
5%	−18.849	−4.644	−3.001	1.733	2.747	4.183	15.731	1.419
MINDIST	x	x	x	x	x	x	x	x
AOR	Max	Mean	Min	StDev	Min	Mean	Max	StDev

PS	−13.345	−3.007	−1.301	1.522	1.090	2.367	10.294	1.146
Eye-Ball	−13.021	−2.931	−1.285	1.466	1.092	2.351	9.863	1.086
dAMSE	−12.990	−2.935	−1.303	1.470	1.083	2.375	10.137	1.132
5%	−13.214	−3.002	−1.307	1.508	1.078	2.366	10.187	1.142
MINDIST	x	x	x	x	1.091	2.364	10.208	1.142

Note: PS—path stability method, dAMSE—minimization of asymptotic mean squared error method, 5%—fixed threshold method, MINDIST—mean absolute deviation distance metric method, x—VaR was not calculated. The calculations were performed with the evir package [53] in R. Generalized Pareto Distribution parameters were estimated by the maximum log-likelihood method.

Backtesting results for VaR for all the indices are summarized in Tables 5–14. Assessing the quality of the estimated VaR, based on Kupiec’s test it may be concluded that all the procedures work impeccably for the considered assets and at both confidence levels. The only one exception is the automated Eye-Ball method, which fails for S&P 500 for 99% VaR in the left tail (significance level of the 5% test). The risk is underestimated in this case and too many exceedances of VaR appear. The results of Christoffersen’s test are the same as for Kupiec’s test. This indicates the dependence of the first exceedance for the same index and the same level of confidence. A more restrictive backtest is the duration independence test. In the left tail all models (except the MINDIST method) passes the VaR duration-based procedure. In the right tail for 99.5% VaR, only ones (in the case of Hang Seng), the test rejects the null hypothesis for the PS, automated Eye-Ball and 5%-quantile methods. The dAMSE method produces accurate VaR forecasts in all the cases. The most rigorous validation procedure is the Dynamic Quantile test. It checks if a sequence of violations is time independent up to five-day lags. According to the test, violations are dependent in about 20% of cases (in nine cases for the PS, automated Eye-Ball, 5%-quantile methods and in eight cases for the dAMSE method). The test rejects the null hypothesis in the same cases regardless which method is used. If then VaR estimate is incorrect for the particular asset we can expect the same result to be reached for all methods. VaR estimates obtained using all tail selection methods have a similar goodness of fit in terms of values of the loss function and values of mean absolute deviation between the returns and the quantiles if exceedances occur for individual indices. In addition, VaR estimates for the right tail fit better to empirical quantiles for all indices than VaR estimates for the left tail.

Table 5. Backtesting for the Conditional Extreme Value Theory model with optimal tail selection, path stability method, Value at Risk 0.01 and Value at Risk 0.99.

Lower Tail, VaR 0.01										Upper Tail, VaR 0.99					
Index	ET	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss
SPX	29	35	1.168	4.381	1.860	33.083	0.597	0.037	23	1.358	1.726	0.359	6.625	0.491	0.028
			0.280	0.112	0.173	0.000	3.192			0.244	0.422	0.549	0.578	2.163	
UKX	29	27	0.178	5.239	0.000	15.823	0.580	0.034	25	0.653	1.084	0.080	1.628	0.613	0.029
			0.673	0.073	0.999	0.045	2.376			0.419	0.581	0.777	0.990	3.895	
CAC	29	23	1.710	2.068	0.146	7.090	0.910	0.042	33	0.331	1.070	3.008	16.617	0.782	0.039
			0.191	0.356	0.702	0.527	4.378			0.565	0.586	0.083	0.034	4.597	
DAX	29	22	2.102	2.433	0.001	6.398	0.809	0.040	34	0.667	1.461	1.863	6.316	0.604	0.037
			0.147	0.296	0.981	0.603	4.637			0.414	0.482	0.172	0.612	2.228	
OMXS	28	21	2.416	2.723	0.646	2.753	0.626	0.038	23	1.312	1.681	0.380	7.171	0.609	0.035
			0.120	0.256	0.422	0.949	5.144			0.252	0.431	0.538	0.518	3.648	
KOSPI	28	26	0.164	0.650	0.000	4.318	0.644	0.036	24	0.639	1.052	0.001	4.541	0.324	0.027
			0.685	0.723	1.000	0.827	2.296			0.424	0.591	0.976	0.805	1.531	
NKX	27	29	0.051	4.417	0.060	13.262	1.144	0.050	29	0.051	0.662	0.776	14.574	0.626	0.038
			0.822	0.110	0.806	0.103	4.888			0.822	0.718	0.378	0.068	2.950	
HSI	28	31	0.307	1.001	0.063	16.317	0.668	0.043	24	0.615	1.030	0.642	4.494	0.614	0.037
			0.579	0.606	0.802	0.038	2.749			0.433	0.598	0.423	0.810	3.541	
BVP	28	22	1.494	1.840	0.688	5.312	1.064	0.050	34	1.127	1.956	0.037	22.867	0.815	0.047
			0.222	0.399	0.407	0.724	6.070			0.289	0.376	0.848	0.004	4.910	
AOR	29	25	0.662	2.227	0.536	4.060	0.646	0.032	28	0.056	0.598	2.716	2.842	0.331	0.024
			0.416	0.328	0.464	0.852	3.690			0.812	0.742	0.099	0.944	1.182	

Note: ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—Engle and Manganelli’s test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p — p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 6. Backtesting for the Conditional Extreme Value Theory model with optimal tail selection, path stability method, Value at Risk 0.005 and Value at Risk 0.995.

Index	ET	T1	Lower Tail, VaR 0.005					Loss	T1	Upper Tail, VaR 0.995					Loss
			UC p	CC p	UD p	DQ p	ADmean ADmax			UC p	CC p	UD p	DQ p	ADmean ADmax	
SPX	14	17	0.407 0.524	0.607 0.738	0.506 0.477	11.597 0.170	0.658 2.213	0.021	15	0.016 0.898	0.172 0.917	0.104 0.747	15.210 0.055	0.360 1.187	0.015
UKX	14	17	0.370 0.543	3.294 0.193	0.000 0.999	21.818 0.005	0.435 1.952	0.019	17	0.370 0.543	0.569 0.752	0.345 0.557	1.015 0.998	0.565 3.437	0.017
CAC	14	14	0.057 0.811	0.189 0.910	0.553 0.457	1.992 0.981	0.945 3.861	0.025	14	0.057 0.811	0.189 0.910	0.014 0.905	1.897 0.984	1.088 4.169	0.023
DAX	14	11	1.046 0.306	1.128 0.569	2.233 0.135	1.669 0.990	0.993 4.268	0.023	19	1.133 0.287	1.380 0.502	0.011 0.918	12.161 0.144	0.576 1.731	0.020
OMXS	14	9	2.392 0.122	2.448 0.294	0.019 0.890	2.582 0.958	0.789 4.706	0.022	10	1.548 0.213	1.617 0.445	0.001 0.971	1.875 0.985	0.759 3.110	0.019
KOSPI	14	15	0.062 0.803	0.223 0.894	1.723 0.189	0.819 0.999	0.502 2.012	0.020	12	0.318 0.573	0.421 0.810	0.263 0.608	16.149 0.040	0.288 1.076	0.015
NKX	13	15	0.084 0.771	3.391 0.184	0.108 0.743	12.589 0.127	1.403 3.905	0.030	17	0.646 0.421	0.855 0.652	0.075 0.784	25.689 0.001	0.632 2.513	0.021
HSI	14	13	0.076 0.783	0.197 0.906	0.041 0.839	16.087 0.041	0.838 2.371	0.024	11	0.704 0.401	0.791 0.673	4.737 0.030	1.195 0.997	0.763 2.901	0.021
BVP	14	13	0.089 0.765	0.210 0.900	0.075 0.785	14.323 0.074	1.067 5.642	0.029	19	1.539 0.215	1.797 0.407	0.016 0.899	12.720 0.122	0.741 3.781	0.026
AOR	14	13	0.191 0.662	4.150 0.126	1.788 0.181	15.481 0.050	0.639 2.933	0.018	18	0.724 0.395	0.947 0.623	0.552 0.457	4.654 0.794	0.260 0.976	0.013

Note: ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—Engle and Manganelli’s test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p — p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 7. Backtesting for the Conditional Extreme Value Theory model with optimal tail selection, Eye-Ball method, Value at Risk 0.01 and Value at Risk 0.99.

Index	ET	T1	Lower Tail, VaR 0.01					Loss	T1	Upper Tail, VaR 0.99					Loss
			UC p	CC p	UD p	DQ p	ADmean ADmax			UC p	CC p	UD p	DQ p	ADmean ADmax	
SPX	29	41	4.428 0.035	6.644 0.036	1.863 0.172	34.470 0.000	0.592 3.443	0.037	20	3.178 0.075	3.456 0.178	0.042 0.837	8.756 0.363	0.528 2.147	0.028
UKX	29	29	0.002 0.964	4.543 0.103	0.514 0.474	14.371 0.073	0.551 2.442	0.034	31	0.105 0.746	0.770 0.681	1.568 0.210	2.202 0.974	0.560 3.894	0.029
CAC	29	31	0.047 0.829	0.698 0.705	0.101 0.751	8.827 0.357	0.793 4.326	0.042	34	0.566 0.452	1.351 0.509	2.468 0.116	17.324 0.027	0.785 4.503	0.039
DAX	29	30	0.009 0.924	0.626 0.731	0.202 0.653	3.258 0.917	0.705 4.746	0.040	36	1.361 0.243	2.251 0.324	0.471 0.493	7.359 0.498	0.649 2.563	0.037
OMXS	28	20	3.109 0.078	3.388 0.184	0.013 0.910	3.430 0.905	0.666 5.142	0.038	24	0.894 0.344	1.296 0.523	0.272 0.602	6.145 0.631	0.594 3.635	0.035
KOSPI	28	28	0.000 0.983	0.564 0.754	0.229 0.632	9.140 0.331	0.709 2.322	0.036	24	0.639 0.424	1.052 0.591	0.001 0.972	4.783 0.781	0.317 1.533	0.027

NKX	27	31	0.356	4.252	0.276	12.514	1.196	0.051	34	1.300	2.142	0.283	17.632	0.563	0.038
			0.551	0.119	0.599	0.130	5.023			0.254	0.343	0.595	0.024	3.402	
HSI	28	28	0.000	0.565	0.035	17.671	0.698	0.043	25	0.343	0.793	0.634	4.072	0.595	0.037
			0.995	0.754	0.851	0.024	2.714			0.558	0.673	0.426	0.851	3.440	
BVP	28	24	0.668	1.080	0.077	7.404	1.114	0.049	34	1.127	1.956	0.037	22.342	0.822	0.047
			0.414	0.583	0.782	0.494	6.160			0.289	0.376	0.848	0.004	5.081	
AOR	29	29	0.003	1.107	0.052	3.223	0.627	0.032	28	0.056	0.598	3.758	2.716	0.338	0.024
			0.960	0.575	0.820	0.920	3.804			0.812	0.742	0.053	0.951	1.153	

Note: ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—Engle and Manganelli’s test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p — p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 8. Backtesting for the Conditional Extreme Value Theory model with optimal tail selection, Eye-Ball method, Value at Risk 0.005 and Value at Risk 0.995.

Lower Tail, VaR 0.005										Upper Tail, VaR 0.995					
Index	ET	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss
SPX	14	21	2.561 0.110	2.868 0.238	0.216 0.642	12.667 0.124	0.672 2.610	0.021	15	0.016 0.898	0.172 0.917	0.257 0.612	14.876 0.062	0.332 1.184	0.015
UKX	14	17	0.370 0.543	3.294 0.193	0.000 0.999	21.918 0.005	0.436 2.012	0.019	20	1.784 0.182	2.059 0.357	0.122 0.727	3.030 0.932	0.547 3.434	0.017
CAC	14	16	0.078 0.780	0.251 0.882	0.058 0.810	13.447 0.097	0.996 3.808	0.025	16	0.078 0.780	0.251 0.882	0.011 0.915	1.291 0.996	1.031 4.063	0.023
DAX	14	16	0.105 0.746	0.280 0.869	1.303 0.254	2.611 0.956	0.821 4.358	0.023	18	0.677 0.411	0.898 0.638	0.085 0.770	10.843 0.211	0.684 1.880	0.021
OMXS	14	8	3.459 0.063	3.503 0.173	0.001 0.977	3.064 0.930	0.892 4.653	0.022	10	1.548 0.213	1.617 0.445	0.001 0.971	1.873 0.985	0.779 3.046	0.019
KOSPI	14	16	0.259 0.611	0.442 0.802	3.686 0.055	1.021 0.998	0.587 1.974	0.020	11	0.722 0.396	0.808 0.668	0.267 0.605	17.951 0.022	0.305 1.143	0.015
NKX	13	16	0.302 0.582	3.363 0.186	0.144 0.704	12.617 0.126	1.397 3.923	0.030	16	0.302 0.582	0.488 0.784	0.002 0.967	24.677 0.002	0.722 2.476	0.021
HSI	14	13	0.076 0.783	0.197 0.906	0.084 0.772	15.863 0.044	0.793 2.438	0.024	11	0.704 0.401	0.791 0.673	4.737 0.030	1.195 0.997	0.746 2.730	0.021
BVP	14	14	0.001 0.978	0.140 0.932	0.117 0.732	13.457 0.097	1.165 5.672	0.029	19	1.539 0.215	1.797 0.407	0.016 0.899	12.535 0.129	0.737 3.908	0.026
AOR	14	16	0.124 0.725	3.279 0.194	1.642 0.200	11.548 0.173	0.605 3.150	0.018	18	0.724 0.395	0.947 0.623	0.078 0.781	4.130 0.845	0.283 0.938	0.013

Note: ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—Engle and Manganelli’s test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p — p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 9. Backtesting for the Conditional Extreme Value Theory model with optimal tail selection, minimization of asymptotic mean squared error method, Value at Risk 0.01 and Value at Risk 0.99.

Table 1: Summary of the results of the empirical analysis																
Index	ET	T1	Lower Tail, VaR 0.01						Loss	T1	Upper Tail, VaR 0.99					
			UC	CC	UD	DQ	ADmean	UC			CC	UD	DQ	ADmean		
			<i>p</i>	<i>p</i>	<i>p</i>	<i>p</i>	ADmax	<i>p</i>			<i>p</i>	<i>p</i>	<i>p</i>	ADmax		
SPX	29	37	2.039	4.890	2.400	36.292	0.593	0.037	21	2.477	2.783	0.000	7.932	0.511	0.028	
			0.153	0.087	0.121	0.000	3.267			0.116	0.249	0.991	0.440	2.100		
UKX	29	29	0.002	4.543	0.514	14.377	0.557	0.034	28	0.054	0.596	1.663	1.529	0.591	0.029	
			0.964	0.103	0.474	0.072	2.435			0.816	0.742	0.197	0.992	3.874		
CAC	29	30	0.001	0.611	0.049	9.012	0.800	0.042	33	0.331	1.070	3.008	16.609	0.795	0.039	

			0.974	0.737	0.825	0.341	4.314			0.565	0.586	0.083	0.034	4.530	
DAX	29	27	0.217	0.716	0.014	3.751	0.745	0.040	36	1.361	2.251	0.471	7.528	0.641	0.037
			0.641	0.699	0.906	0.879	4.699			0.243	0.324	0.493	0.481	2.437	
OMXS	28	21	2.416	2.723	0.018	2.774	0.636	0.038	23	1.312	1.681	0.380	7.202	0.597	0.035
			0.120	0.256	0.893	0.948	5.100			0.252	0.431	0.538	0.515	3.650	
KOSPI	28	27	0.045	0.569	0.028	4.637	0.708	0.036	23	1.000	1.380	0.003	4.820	0.327	0.027
			0.832	0.752	0.867	0.796	2.307			0.317	0.502	0.954	0.777	1.506	
NKX	27	31	0.356	4.252	0.276	12.522	1.169	0.050	32	0.608	1.353	1.254	14.410	0.582	0.038
			0.551	0.119	0.599	0.129	5.006			0.435	0.508	0.263	0.072	3.309	
HSI	28	30	0.137	0.786	0.034	16.670	0.702	0.043	26	0.152	0.639	0.325	3.834	0.580	0.037
			0.712	0.675	0.854	0.034	2.722			0.696	0.726	0.568	0.872	3.487	
BVP	28	24	0.668	1.080	0.183	7.201	1.104	0.049	34	1.127	1.956	0.037	22.890	0.802	0.047
			0.414	0.583	0.669	0.515	6.213			0.289	0.376	0.848	0.004	4.986	
AOR	29	29	0.003	1.107	0.052	3.214	0.603	0.031	27	0.183	0.686	3.140	2.948	0.337	0.024
			0.960	0.575	0.820	0.920	3.760			0.669	0.710	0.076	0.938	1.164	

Note: ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—Engle and Manganelli’s test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p — p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 10. Backtesting for the Conditional Extreme Value Theory model with optimal tail selection, minimization of asymptotic mean squared error method, Value at Risk 0.005 and Value at Risk 0.995.

Index	ET	T1	Lower tail, VaR 0.005						Upper tail, VaR 0.995						Loss
			UC	CC	UD	DQ	ADmean	Loss	T1	UC	CC	UD	DQ	ADmean	
			p	p	p	p	ADmax			p	p	p	p	ADmax	
SPX	14	19	1.272	1.522	0.063	12.079	0.651	0.021	14	0.018	0.154	0.169	16.629	0.361	0.015
			0.259	0.467	0.802	0.148	2.331			0.893	0.926	0.681	0.034	1.137	
UKX	14	17	0.370	3.294	0.000	21.872	0.455	0.019	19	1.204	1.453	0.188	2.197	0.544	0.017
			0.543	0.193	0.999	0.005	2.012			0.272	0.484	0.664	0.974	3.417	
CAC	14	16	0.078	0.251	0.058	13.544	0.930	0.025	16	0.078	0.251	0.011	1.498	0.964	0.023
			0.780	0.882	0.810	0.094	3.819			0.780	0.882	0.915	0.993	4.107	
DAX	14	11	1.046	1.128	2.233	1.631	1.084	0.023	19	1.133	1.380	0.011	11.775	0.666	0.021
			0.306	0.569	0.135	0.990	4.330			0.287	0.502	0.918	0.162	1.847	
OMXS	14	11	0.905	0.989	0.060	1.643	0.669	0.022	10	1.548	1.617	0.001	1.875	0.761	0.019
			0.341	0.610	0.806	0.990	4.649			0.213	0.445	0.971	0.985	3.106	
KOSPI	14	17	0.581	0.788	2.955	1.475	0.543	0.020	11	0.722	0.808	0.267	17.962	0.300	0.015
			0.446	0.674	0.086	0.993	2.035			0.396	0.668	0.605	0.022	1.049	
NKX	13	17	0.646	3.479	0.009	12.105	1.331	0.030	15	0.084	0.247	0.062	24.310	0.746	0.021
			0.421	0.176	0.925	0.147	4.024			0.771	0.884	0.803	0.002	2.513	
HSI	14	13	0.076	0.197	0.041	16.130	0.832	0.024	13	0.076	0.197	0.580	14.609	0.661	0.021
			0.783	0.906	0.839	0.041	2.344			0.783	0.906	0.446	0.067	2.854	
BVP	14	14	0.001	0.140	0.117	13.181	1.095	0.029	19	1.539	1.797	0.016	12.848	0.723	0.026
			0.978	0.932	0.732	0.106	5.765			0.215	0.407	0.899	0.117	3.857	
AOR	14	15	0.009	0.164	0.202	1.258	0.613	0.018	18	0.724	0.947	0.552	4.790	0.254	0.013
			0.924	0.921	0.653	0.996	3.011			0.395	0.623	0.457	0.780	0.959	

Note: ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—Engle and Manganelli’s test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p — p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 11. Backtesting for the Conditional Extreme Value Theory model with optimal tail selection, mean absolute deviation distance metric method, Value at Risk 0.01 and Value at Risk 0.99.

Index	ET	T1	Lower tail, VaR 0.01						Upper tail, VaR 0.99						Loss
			UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	
SPX	29	x	x	x	x	x	x	x	21	2.477 0.116	2.783 0.249	0.000 0.991	7.942 0.439	0.526 2.096	0.028
UKX	29	29	0.002 0.964	4.543 0.103	0.514 0.474	14.355 0.073	0.539 2.390	0.034	x	x	x	x	x	x	x
CAC	29	x	x	x	x	x	x	x	x	x	x	x	x	x	x
DAX	29	x	x	x	x	x	x	x	x	x	x	x	x	x	x
OMXS	28	x	x	x	x	x	x	x	23	1.312 0.252	1.681 0.431	0.380 0.538	7.156 0.520	0.583 3.653	0.035
KOSPI	28	x	x	x	x	x	x	x	23	1.000 0.317	1.380 0.502	0.003 0.954	4.822 0.776	0.328 1.502	0.027
NKX	27	x	x	x	x	x	x	x	x	x	x	x	x	x	x
HSI	28	x	x	x	x	x	x	x	x	x	x	x	x	x	x
BVP	28	x	x	x	x	x	x	x	x	x	x	x	x	x	x
AOR	29	x	x	x	x	x	x	x	28	0.056 0.812	0.598 0.742	2.716 0.099	2.858 0.943	0.333 1.181	0.024

Note: x—VaR was not calculated, ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—Engle and Manganelli’s test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), *p*—*p* value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function *Q* described by Formula (33). The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 12. Backtesting for the Conditional Extreme Value Theory model with optimal tail selection, mean absolute deviation distance metric method, Value at Risk 0.005 and Value at Risk 0.995.

Index	ET	T1	Lower Tail, VaR 0.005						Upper Tail, VaR 0.995						Loss
			UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	
SPX	14	x	x	x	x	x	x	x	14	0.018 0.893	0.154 0.926	0.169 0.681	16.631 0.034	0.377 1.132	0.015
UKX	14	17	0.370 0.543	3.294 0.193	0.000 0.999	21.844 0.005	0.440 1.966	0.019	x	x	x	x	x	x	x
CAC	14	x	x	x	x	x	x	x	x	x	x	x	x	x	x
DAX	14	x	x	x	x	x	x	x	x	x	x	x	x	x	x
OMXS	14	x	x	x	x	x	x	x	9	2.392 0.122	2.448 0.294	0.398 0.528	2.662 0.954	0.830 3.114	0.019
KOSPI	14	x	x	x	x	x	x	x	12	0.318 0.573	0.421 0.810	0.263 0.608	16.144 0.040	0.281 1.048	0.015
NKX	13	x	x	x	x	x	x	x	x	x	x	x	x	x	x
HSI	14	x	x	x	x	x	x	x	x	x	x	x	x	x	x
BVP	14	x	x	x	x	x	x	x	x	x	x	x	x	x	x
AOR	14	x	x	x	x	x	x	x	18	0.724 0.395	0.947 0.623	0.552 0.457	4.645 0.795	0.260 0.978	0.013

Note: x—VaR was not calculated, ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—

Engle and Manganelli's test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p - p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 13. Backtesting for the Conditional Extreme Value Theory model with 95th quantile for tail selection, Value at Risk 0.01 and Value at Risk 0.99.

Lower Tail, VaR 0.01										Upper Tail, VaR 0.99					
Index	ET	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss
SPX	29	35	1.168	4.381	1.860	33.129	0.595	0.037	21	2.477	2.783	0.000	7.923	0.513	0.028
			0.280	0.112	0.173	0.000	3.200			0.116	0.249	0.991	0.441	2.087	
UKX	29	28	0.054	4.849	0.163	14.999	0.569	0.034	27	0.178	0.681	0.998	1.513	0.603	0.029
			0.816	0.089	0.686	0.059	2.425			0.673	0.711	0.318	0.992	3.863	
CAC	29	29	0.023	0.593	0.061	5.811	0.795	0.042	33	0.331	1.070	3.008	16.617	0.798	0.039
			0.879	0.743	0.804	0.668	4.337			0.565	0.586	0.083	0.034	4.580	
DAX	29	25	0.725	1.153	0.112	4.540	0.754	0.040	35	0.985	1.826	1.145	8.090	0.630	0.037
			0.394	0.562	0.738	0.805	4.668			0.321	0.401	0.285	0.425	2.304	
OMXS	28	20	3.109	3.388	0.056	3.433	0.643	0.038	24	0.894	1.296	0.513	6.189	0.582	0.035
			0.078	0.184	0.814	0.904	5.085			0.344	0.523	0.474	0.626	3.681	
KOSPI	28	25	0.361	0.810	0.005	4.719	0.686	0.036	24	0.639	1.052	0.001	4.764	0.324	0.027
			0.548	0.667	0.942	0.787	2.258			0.424	0.591	0.972	0.782	1.515	
NKX	27	30	0.170	4.295	0.196	12.848	1.161	0.050	29	0.051	0.662	0.776	14.507	0.621	0.038
			0.680	0.117	0.658	0.117	4.950			0.822	0.718	0.378	0.069	3.115	
HSI	28	31	0.307	1.001	0.063	16.341	0.675	0.043	25	0.343	0.793	0.634	4.075	0.587	0.037
			0.579	0.606	0.802	0.038	2.730			0.558	0.673	0.426	0.850	3.517	
BVP	28	23	1.037	1.416	0.137	7.877	1.140	0.049	34	1.127	1.956	0.037	22.737	0.817	0.047
			0.308	0.493	0.711	0.446	6.143			0.289	0.376	0.848	0.004	4.950	
AOR	29	25	0.662	2.227	0.536	4.062	0.647	0.032	27	0.183	0.686	3.140	2.870	0.343	0.024
			0.416	0.328	0.464	0.851	3.696			0.669	0.710	0.076	0.942	1.193	

Note: ET (T1)—expected (actual) number of exceedances, UC—Kupiec's test statistic, CC—Christoffersen's test statistic, UD—Christoffersen and Pelletier's test statistic, DQ—Engle and Manganelli's test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p - p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

Table 14. Backtesting for the Conditional Extreme Value Theory model with 95th quantile for tail selection, Value at Risk 0.005 and Value at Risk 0.995.

Lower Tail, VaR 0.005										Upper Tail, VaR 0.995					
Index	ET	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss	T1	UC <i>p</i>	CC <i>p</i>	UD <i>p</i>	DQ <i>p</i>	ADmean ADmax	Loss
SPX	14	18	0.783	1.008	0.292	11.586	0.623	0.021	14	0.018	0.154	0.169	16.621	0.363	0.015
			0.376	0.604	0.589	0.171	2.224			0.893	0.926	0.681	0.034	1.119	
UKX	14	17	0.370	3.294	0.000	21.832	0.446	0.019	18	0.731	0.954	0.203	1.545	0.558	0.017
			0.543	0.193	0.999	0.005	2.002			0.392	0.621	0.653	0.992	3.402	
CAC	14	15	0.001	0.152	0.197	13.759	0.960	0.025	16	0.078	0.251	0.011	1.507	0.975	0.023
			0.981	0.927	0.657	0.088	3.840			0.780	0.882	0.915	0.993	4.159	
DAX	14	11	1.046	1.128	2.233	1.630	1.025	0.023	19	1.133	1.380	0.011	11.987	0.624	0.021
			0.306	0.569	0.135	0.990	4.294			0.287	0.502	0.918	0.152	1.809	
OMXS	14	10	1.548	1.617	0.000	1.892	0.708	0.022	10	1.548	1.617	0.001	1.858	0.776	0.019
			0.213	0.445	0.991	0.984	4.630			0.213	0.445	0.971	0.985	3.139	
KOSPI	14	15	0.062	0.223	1.723	0.830	0.521	0.020	12	0.318	0.421	0.263	16.133	0.285	0.015
			0.803	0.894	0.189	0.999	1.981			0.573	0.810	0.608	0.041	1.059	
NKX	13	16	0.302	3.363	0.039	12.101	1.368	0.030	15	0.084	0.247	0.062	24.262	0.714	0.021

			0.582	0.186	0.844	0.147	3.972			0.771	0.884	0.803	0.002	2.493	
HSI	14	13	0.076	0.197	0.041	16.098	0.835	0.024	11	0.704	0.791	4.737	1.195	0.758	0.021
			0.783	0.906	0.839	0.041	2.357			0.401	0.673	0.030	0.997	2.884	
BVP	14	13	0.089	0.210	0.075	14.319	1.170	0.029	19	1.539	1.797	0.016	12.764	0.742	0.026
			0.765	0.900	0.785	0.074	5.703			0.215	0.407	0.899	0.120	3.821	
AOR	14	15	0.009	3.410	0.760	12.367	0.560	0.018	18	0.724	0.947	0.552	4.656	0.259	0.013
			0.924	0.182	0.383	0.136	2.934			0.395	0.623	0.457	0.794	0.986	

Note: ET (T1)—expected (actual) number of exceedances, UC—Kupiec’s test statistic, CC—Christoffersen’s test statistic, UD—Christoffersen and Pelletier’s test statistic, DQ—Engle and Manganelli’s test statistic (the DQ test included constant, VaR, first five lagged exceedances and one lagged squared return), p — p value, ADmean (ADmax)—mean (maximum) absolute deviation between the returns and the quantiles if exceedances occur as in McAleer and Da Veiga (2008) [55], Loss—loss function Q described by Formula (33), bold—rejection of the null hypothesis at the significance level of 0.05. The calculations were performed with the rugarch [56] and GAS [57] packages in R.

4. Conclusions

The GARCH-EVT approach allows us to model the tails of the time-varying conditional return distribution. The main problem in using the method in practice relates to the selection of the appropriate tail fraction of the distribution above which the asymptotic properties hold. In this paper we extend the state-of-the-art by conducting a comparative study of accuracy of VaR forecasts, when each VaR estimate is calculated with an optimal choice of the distribution tail fraction. We used four different optimization procedures and compared the results to an approach based on a fixed, 95th quantile of distribution as a threshold. Such a choice of threshold is seen as standard in the conditional EVT approach. The GARCH-EVT model performs relatively well in estimating the risk for all choices of threshold. Backtesting procedures indicate that regardless of the choice of the tail, approximately the same accuracy of VaR prediction is provided. This conclusion presents a real dilemma to the importance of an appropriate selection of a threshold in financial applications. We found that the optimal tail selection methods do not improve accuracy of the VaR prediction relative to the standard method and hence our research hypothesis is not confirmed. Investors may then use the conditional EVT approach, taking the 95th percentile of the sample as a threshold, to obtain the accurate estimate of tail risk.

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