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An Elementary Proof of a Theorem of Graham on Finite Semigroups

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Abstract: The purpose of this note is to give a very elementary proof of a theorem of Graham that provides a structural description of finite 0-simple semigroups and its idempotent-generated subsemigroups.

Keywords: finite semigroup; regular semigroup; 0-simple semigroup

MSC: 20M10; 20M17

1. Introduction

The aim of this short note is to present a very elementary proof of a classical theorem of Graham that gives a complete and useful description of a finite 0-simple semigroup and its idempotent-generated subsemigroup.

Therefore, all semigroups considered in the sequel will be finite.

0-simple semigroups were first introduced and studied by Rees in his seminar paper [1]. They play a significant role in semigroup theory: every regular $\mathcal{J}$ -class can be seen as a 0-simple semigroup, and every semigroup may be obtained from 0-simple semigroups by a sequence of ideal extensions. In addition, a detailed study of the idempotent-generated subsemigroup of a given semigroup turns out to be crucial for understanding complexity, which is one of the most famous problems in semigroup theory. In this context, Graham's theorem has become one of the most important basic results.

In [1], Rees proved a theorem that gives to the class of 0-simple semigroups a transparent characterization. It states that, up to isomorphism, 0-simple semigroups are precisely the Rees matrix semigroups, so called because they were introduced by him in the same paper. The proof of this theorem relies on a bunch of important structural results concerning 0-simple semigroups. In 1968, Graham published an influential contribution to the structural study of a 0-simple semigroup. He showed how to apply graph theory to obtain a description of the idempotent-generated subsemigroup of a 0-simple semigroup [2]. Graham's result was republished ten years later by Howie [3] and Houghton [4]. This last author added topological techniques and cohomology that have had a strong influence in the proof of the theorem presented in [5].

In the present note, we give a very elementary proof of Graham's result. Our concern here is in applying the basic results on regularity to the method used by Rees to prove his isomorphism theorem.

2. Preliminaries

The aim of this section is to collect some well-known results that turn out to be crucial in our proof of Graham's theorem. Most of them can be found in [6]. This book will be the main reference for notation and terminology.

Let S be a semigroup. As usual, denote by S^1 and S^0 the semigroups obtained from S by adjoining an identity and a zero if necessary, respectively. For a subset X of S , denote by $E(X)$ the subset of all idempotents of S contained in X and by $\langle X \rangle$ the subsemigroup generated by X .

Given a semigroup S and $e \in E(S)$, we denote by S_e the maximal subgroup in S with e as an identity element. This nonstandard notation is due to the fact that sometimes we consider the same idempotent belonging to several subsemigroups.

The following equivalence relations introduced by Green are fundamental:

$$\begin{aligned} s \mathcal{R} t & \text{ if and only if } sS^1 = tS^1 \\ s \mathcal{L} t & \text{ if and only if } S^1s = S^1t \\ s \mathcal{J} t & \text{ if and only if } S^1sS^1 = S^1tS^1. \end{aligned}$$

We will use $\mathcal{K}_u$ to denote the $\mathcal{K}$ -class of the element $u \in S$ (with $\mathcal{K} = \mathcal{R}, \mathcal{L}, \mathcal{J}$).

An element x of a semigroup S is called *regular* (in the sense of von Neumann) if $x \in xSx$. S is called *regular* if every element of S is regular. Note that if $x \in S$ is regular, there exists $a \in S$ such that xa and ax belong to $E(S)$ and $(xa)x(ax) = x$. Therefore, every $\mathcal{J}$ -class of a regular semigroup has an idempotent.

Proposition 1 ([7]). *Let S be a semigroup (not necessarily finite). Then:*

1. *If S is regular, then $\langle E(S) \rangle$ is a regular subsemigroup of S .*
2. *If T is a regular subsemigroup of S . Then, $\mathcal{K}_T = \mathcal{K}_S \cap (T \times T)$, where $\mathcal{K} = \mathcal{L}$ or $\mathcal{R}$.*

Definition 1 ([1]). *A semigroup S with zero is called 0-simple if $SxS = S$ for all $0 \neq x \in S$.*

Note that every 0-simple semigroup S has a unique nonzero $\mathcal{J}$ -class, $S \setminus \{0\}$, i.e., $s \mathcal{J} t$ for all nonzero $s \neq t \in S$.

Proposition 2 ([1]). *Let S be a 0-simple semigroup. Then*

1. *S is regular.*
2. *S_e is isomorphic to S_f for all $e, f \in E(S) \setminus \{0\}$.*
3. *$S_e = (eSe) \setminus \{0\}$, for all $0 \neq e \in E(S)$.*

Let A, B be nonempty sets and let G be a group. A *Rees matrix* C is a map $C : B \times A \rightarrow G^0$. We say that the Rees matrix is *regular* if every row and every column has a nonzero entry.

The *Rees matrix semigroup* with sandwich matrix C is the semigroup $\mathcal{M}^0(G, A, B, C)$ with underlying set $(A \times G \times B) \cup \{0\}$ and the operation: $0 \cdot (a, g, b) = (a, g, b) \cdot 0 = 0$, and

$$(a, g, b) \cdot (a', g', b') = \begin{cases} (a, gC(b, a')g', b') & \text{if } C(b, a') \in G, \\ 0, & \text{if } C(b, a') = 0, \end{cases}$$

for all $a, a' \in A, b, b' \in B, g \in G$.

Theorem 1 ([1]). *Every regular Rees matrix semigroup is a 0-simple semigroup. Conversely, every 0-simple semigroup S is isomorphic to a regular Rees matrix semigroup $\mathcal{M}^0(G, A, B, C)$, where G is isomorphic to the maximal subgroups S_e , for all $e \neq 0$.*

Graham's Theorem. *Let S be a 0-simple semigroup. Then there exists an isomorphism:*

$$\psi : S \rightarrow \mathcal{M}^0(G, A, B, C)$$

from S to a Rees matrix semigroup $\mathcal{M}^0(G, A, B, C)$ such that:

- The matrix C is the direct sum of the n matrices $C_1, \dots, C_n$ as shown below:

$$\begin{matrix} & A_1 & A_2 & \cdots & A_n \\ \begin{matrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{matrix} & \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix} \end{matrix}.$$

- Each matrix $C_i : B_i \times A_i \rightarrow G^0$ is regular and:

$$\langle E(S) \rangle = \bigcup_{i=1}^n \mathcal{M}^0(G_i, A_i, B_i, C_i)$$

where G_i is the subgroup of G generated by all nonzero entries of C_i , for $i = 1, \dots, n$.

3. Two Key Lemmas and Their Corollaries

In the sequel, S will denote a 0-simple semigroup and $T := \langle E(S) \rangle$.

The next two lemmas proved in [1] and their corollaries are absolutely essential in our approach.

Lemma 1 ([1] (Lemmas 2.61, 2.62, 2.63)). For each pair of nonzero idempotents e and f of S , eSf is nonzero and there exist $0 \neq x \in eSf$ and $0 \neq y \in fSe$ such that $xy = e$ and $yx = f$.

Lemma 2 ([1] (Lemma 2.7)). Let $e, f \in E(S) \setminus \{0\}$. The sets eS and fS have either no nonzero elements in common or are identical. Similarly for the sets Se and Sf and, consequently, for the sets eSf and $e'Sf'$.

Corollary 1. Let $0 \neq ef \in T$ with $e, f \in E(S)$. Then $e \mathcal{R}_T ef \mathcal{L}_T f$. In particular, $e \mathcal{J}_T f \mathcal{J}_T(ef)$.

Proof. Since S is regular, T is regular by Proposition 1. Then, there exists $0 \neq a \in T$ such that $(ef)a(ef) = ef$ and $0 \neq (ef)a =: g$ is idempotent. Hence, $gS = (ef)S \subseteq eS$ and, by Key Lemma 2, $gS = (ef)S = eS$, i.e., $e \mathcal{R}(ef)$. Now, we can apply Proposition 1 to conclude $e \mathcal{R}_T ef$.

Analogously, we have $(ef) \mathcal{L}_T f$ and therefore $e \mathcal{J}_T f \mathcal{J}_T(ef)$. $\square$

Corollary 2. Assume that $0 \neq e_1 \cdots e_r \in T$ for some $e_i \in E(S)$, $1 \leq i \leq r$. Then $e_1 \mathcal{J}_T \cdots \mathcal{J}_T e_r \mathcal{J}_T(e_1 \cdots e_r)$.

Proof. Note that $e_i e_{i+1} \neq 0$ for all $1 \leq i \leq r-1$. Then, by Corollary 1, we have $e_i \mathcal{J}_T e_{i+1}$, and $e_i \mathcal{R}_T e_i e_{i+1}$, for all $1 \leq i \leq r-1$. Hence, $e_1 \mathcal{J}_T \cdots \mathcal{J}_T e_r$ and there exists $t_i \in T$ with $e_i e_{i+1} t_i = e_i$, for each $1 \leq i \leq r-1$.

Write $t = t_{r-1} t_{r-2} \cdots t_1$. Then

$$(e_1 \cdots e_r)t = (e_1 \cdots e_r)(t_{r-1} \cdots t_1) = e_1$$

and therefore $T^1(e_1 \cdots e_r)T^1 = T^1 e_1 T^1$, i.e. $e_1 \mathcal{J}_T(e_1 \cdots e_r)$.

$\square$

4. Proof of Graham's Theorem

We split the proof into the following steps.

Step 1. Let $(\mathcal{J}_T)_{e_1}, \dots, (\mathcal{J}_T)_{e_n}$ be the nonzero $\mathcal{J}$ -classes in T . Since T is regular, we may assume that $e_1, \dots, e_n \in E(S)$. For each $k \in \{1, \dots, n\}$, we write:

$$T^{(k)} = (\mathcal{J}_T)_{e_k}^0 \subseteq T.$$

Corollary 2 ensures us that $T^{(k)}$ is a 0-simple subsemigroup of T , for each $1 \leq k \leq n$.

Step 2. Since S is regular, we have that:

$$S = \bigcup_{i,j} r_i S l_j \quad \text{where } r_i, l_j \in E(S) \setminus \{0\}, \quad i = 1, \dots, m, j = 1, \dots, l.$$

Moreover, by Key Lemma 2, we can choose the idempotents r_i, l_j such that $r_i S l_j \cap r_{i'} S l_{j'} = 0$ if $r_i S l_j \neq r_{i'} S l_{j'}$, and $r_1 = l_1 = e_1$.

Set $A := \{1, \dots, m\}$ and $B := \{1, \dots, l\}$. For each $1 \leq k \leq n$, we define:

$$A_k := \{i \in A : r_i \mathcal{J}_T e_k\}, \quad B_k := \{j \in B : l_j \mathcal{J}_T e_k\}.$$

Then $\{A_k\}_{k=1}^n, \{B_k\}_{k=1}^n$ are partitions of A and B , respectively.

Step 3. Let $k \in \{1, \dots, n\}$. Applying Key Lemma 1 we have:

(i) There exist nonzero elements $x_{1k} \in e_1 S e_k, x_{k1} \in e_k S e_1$, such that $x_{1k} x_{k1} = e_1$ and $x_{k1} x_{1k} = e_k$. Hence:

$$\begin{array}{ccc} \varphi_{1k}: e_1 S e_1 & \rightarrow & e_k S e_k \\ s & \mapsto & x_{k1} s x_{1k} \end{array} \quad \begin{array}{ccc} \varphi_{k1}: e_k S e_k & \rightarrow & e_1 S e_1 \\ s & \mapsto & x_{1k} s x_{k1} \end{array} \quad (1)$$

are isomorphisms.

(ii) Since $T^{(k)}$ is 0-simple, for all $i \in A_k, j \in B_k$, there exist nonzero elements $\bar{p}_{ik} \in r_i T^{(k)} e_k, \bar{q}_{kj} \in e_k T^{(k)} l_j$, such that

$$0 \neq x_{1k} \bar{q}_{kj} \in e_1 S l_j, \quad 0 \neq \bar{p}_{ik} x_{k1} \in r_i S e_1.$$

Then, for all $i \in A$ and $j \in B$, we define:

$$\begin{array}{ll} 0 \neq p_{i1} := \bar{p}_{ik} x_{k1} \in r_i S e_1 & \text{if } i \in A_k, \\ 0 \neq q_{1j} := x_{1k} \bar{q}_{kj} \in e_1 S l_j & \text{if } j \in B_k. \end{array}$$

5. Conclusions

According to Proposition 2, the maximal subgroups S_e , for all $0 \neq e \in E(S)$, are all isomorphic. Let $G^0 := e_1 S e_1 = (S_{e_1})^0$ and consider the Rees $(B \times A)$ -matrix given by:

$$C(j, i) := \begin{cases} q_{1j} p_{i1} & \text{if } q_{1j} p_{i1} \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad j \in B, i \in A.$$

The proof of Theorem 2.93 in [1] gives us an isomorphism

$$\psi: S \longrightarrow \mathcal{M}^0(G, A, B, C).$$

Moreover, by Corollary 1, if $j \in B_k$ and $i \in A_{k'}$ and $k \neq k'$, it follows that $l_j r_i = 0$. Therefore $C(j, i) = 0$ and then

$$C = \begin{matrix} & A_1 & A_2 & \cdots & A_n \\ \begin{matrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{matrix} & \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix} \end{matrix}.$$

This proves the first statement of the theorem.

Fix $k \in \{1, \dots, n\}$. Then, by Equation (1), φ_{k1} is an isomorphism between $e_k S e_k$ and $e_1 S e_1$ such that $\varphi_{k1}(\bar{q}_{kj} \bar{p}_{ik}) = q_{1j} p_{i1}$, for all $i \in A_k, j \in B_k$, and φ_{k1} restricted to $T_{e_k}^{(k)}$ defines an isomorphism between $T_{e_k}^{(k)}$ and a subgroup H_k of G . Since $T^{(k)}$ is 0-simple and $\bar{p}_{ik} \in r_i T^{(k)} e_k$ and $\bar{q}_{kj} \in e_k T^{(k)} l_j$, for each $i \in A_k$ and $j \in B_k$, we can follow the proof of Theorem 2.93 in [1] to conclude that the restriction of ψ to $T^{(k)}$ defines an isomorphism between $T^{(k)}$ and $\mathcal{M}^0(H_k, A_k, B_k, \tilde{C}_k)$, where $\tilde{C}_k$ is defined by:

$$\tilde{C}_k(j, i) := \begin{cases} \varphi_{k1}(\bar{q}_{kj} \bar{p}_{ik}) & \text{if } \bar{q}_{kj} \bar{p}_{ik} \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad j \in B_k, i \in A_k.$$

It is clear that $\bar{q}_{kj} \bar{p}_{ik} \neq 0$ if and only if $q_{1j} p_{i1} \neq 0$. Therefore $\tilde{C}_k = C_k$.

Since $T = \bigcup_{k=1}^n (T^k)$ and $\psi(T^k)$ is isomorphic to $\mathcal{M}^0(H_k, A_k, B_k, C_k)$, we have that $\psi(T)$ can be described as $\bigcup_{k=1}^n \mathcal{M}^0(H_k, A_k, B_k, C_k)$.

Moreover, $(i, g, j) \in \mathcal{M}^0(H_k, A_k, B_k, C_k)$ is a nonzero idempotent if, and only if, $C_k(j, i) \neq 0$ and $g = C_k(j, i)^{-1}$. Since every element of $T^{(k)}$ is a product of idempotents, it follows that $H_k = \langle \{0 \neq C_k(j, i) : j \in B_k, i \in A_k\} \rangle$.

The second statement of Graham's result now holds and the proof of the theorem is complete.

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