

## Article

# High Convergence Order Iterative Procedures for Solving Equations Originating from Real Life Problems

Ramandeep Behl <sup>1,\*</sup>, Ioannis K. Argyros <sup>2</sup> and Ali Saleh Alshomrani <sup>1</sup><sup>1</sup> Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia<sup>2</sup> Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA

\* Correspondence: ramanbehl87@yahoo.in

Received: 1 August 2019; Accepted: 4 September 2019; Published: 17 September 2019



**Abstract:** The foremost aim of this paper is to suggest a local study for high order iterative procedures for solving nonlinear problems involving Banach space valued operators. We only deploy suppositions on the first-order derivative of the operator. Our conditions involve the Lipschitz or Hölder case as compared to the earlier ones. Moreover, when we specialize to these cases, they provide us: larger radius of convergence, higher bounds on the distances, more precise information on the solution and smaller Lipschitz or Hölder constants. Hence, we extend the suitability of them. Our new technique can also be used to broaden the usage of existing iterative procedures too. Finally, we check our results on a good number of numerical examples, which demonstrate that they are capable of solving such problems where earlier studies cannot apply.

**Keywords:** iterative method; local convergence; banach space; lipschitz constant; order of convergence

**MSC:** 65G99; 65H10; 47J25; 47J05; 65D10; 65D99

## 1. Introduction

One of the most primary and principal problems in numerical analysis associate with how to approximate a locally unique zero  $\lambda_*$  of

$$S(\lambda) = 0, \quad (1)$$

where  $S : \Delta \subset \mathbb{E}_1 \rightarrow \mathbb{E}_2$  is a Fréchet-differentiable operator. In addition,  $\mathbb{E}_1, \mathbb{E}_2$  are two Banach spaces and  $\Delta$  is a convex subset of Banach space  $\mathbb{E}_1$ . We denote  $\ell(\mathbb{E}_1, \mathbb{E}_2)$  as the space of bounded linear operators from  $\mathbb{E}_1$  to  $\mathbb{E}_2$ .

Approximating a unique solution  $\lambda_*$  is vital, since several problems can be transform to Equation (1) by adopting mathematical modeling [1–8]. However, it is not always possible to get  $\lambda_*$  in a closed form. Therefore, most of the schemes to solve such problems are iterative. The convergence study of iterative schemes involves the information about  $\lambda_*$  is known as local convergence. Convergence domain of an iterative method is an important task to guarantee convergence. Hence, it is very essential to suggest the radius of convergence.

We are interested in the local study of multi-point high-order convergent method [1] given by

$$\begin{aligned} \eta_l &= \lambda_l - \beta S'(\lambda_l)^{-1} S(\lambda_l), \\ \theta_l &= \eta_l - S'(\lambda_l)^{-1} S(\eta_l), \\ \lambda_{l+1} &= \theta_l - \left[ \frac{1}{\beta} S'(\eta_l)^{-1} - \left( 1 - \frac{1}{\beta} \right) S'(\lambda_l)^{-1} \right] S(\theta_l), \quad \beta \neq 0, \quad l = 0, 1, 2, \dots \end{aligned} \quad (2)$$

where  $\lambda_0 \in \Delta$  is the starting point; for  $\beta \neq \pm 1$ , the method reaches at least fourth order and, for  $\beta = \pm 1$ , fifth order. The hypotheses on the derivatives of  $S$  restrict the suitability of the scheme in Equation (2). As a motivational example, we suggest a function  $S$  on  $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{R}$ ,  $\Delta = [-\frac{3}{2}, \frac{1}{2}]$  by

$$S(t) = \begin{cases} t^3 \ln t^2 + t^5 - t^4, & t \neq 0 \\ 0, & t = 0 \end{cases}.$$

Then, we have that

$$\begin{aligned} S'(t) &= 3t^2 \ln t^2 + 5t^4 - 4t^3 + 2t^2, \\ S''(t) &= 6t \ln t^2 + 20t^3 - 12t^2 + 10t \end{aligned}$$

and

$$S'''(t) = 6 \ln t^2 + 60t^2 - 24t + 22.$$

Then, obviously third-order derivative  $S'''(t)$  is unbounded on  $\Delta$ . There is a plethora of research articles on iterative schemes [2–22]. The initial guess  $\lambda_0$  must be close enough to the required solution for guaranteed convergence. However, it is not giving us any idea of: how to choose  $\lambda_0$ , find a convergence radius, the bounds on  $\|\lambda_l - \lambda_*\|$  and the uniqueness results. We deal with these problems for the method in Equation (2) in Section 2.

We enlarge the suitability of the scheme in Equation (2) by adopting only hypotheses on the first-order derivative of  $S$  and generalized conditions. In addition, we avoid the use of Taylor series expansions. In this way, there is no need to use the higher-order derivatives to illustrate the convergence order of the scheme in Equation (2). We adopt COC and ACOC for the order of convergence, which avoid higher-order derivatives (see Remark 1 (d)). When the generalized conditions are specialized to the Lipschitz case (see Remark 1 (a)), the Hölder case [1] (see Remark 1 (c)) or the advantages mentioned in the Introduction are obtained.

## 2. Convergence Analysis

The local convergence analysis stand on some parameters and scalar functions. Let us assume  $\beta \in T - \{0\}$ , and let  $w_0$  be a non-decreasing continuous function on  $[0, +\infty)$  having values in  $[0, +\infty)$  with  $w_0(0) = 0$ , where  $T = \mathbb{R}$  or  $T = \mathbb{C}$ .

Suppose equation

$$w_0(\alpha) = 1, \tag{3}$$

has a minimal positive solution  $\tau_0$ .

Consider that functions  $w, v$  on  $[0, \tau_0)$  are continuous and increasing with  $w(0) = 0$ . Moreover, we choose functions  $\phi_1$  and  $h_1$  on the interval  $[0, \tau_0)$  as follows:

$$\phi_1(\alpha) = \frac{\int_0^1 w((1-v)\alpha)dv + |1-\beta| \int_0^1 v(v\alpha)dv}{1-w_0(\alpha)},$$

and

$$h_1(\alpha) = \phi_1(\alpha) - 1.$$

Suppose that

$$|1-\beta|v(0) < 1. \tag{4}$$

From Equation (4), we have  $h_1(0) = |1-\beta|v(0) - 1 < 0$  and  $h_1(\alpha) \rightarrow +\infty$  as  $\alpha \rightarrow \tau_0^-$ . Then, by the intermediate value theorem, we know that the function  $h_1$  has zeros in  $(0, \tau_0)$ . Denote by

$\tau_1$  the smallest such zero of function  $h_1$ . Assume equations  $w_0(\phi_1(\alpha)\alpha) = 1$  and  $w_0(\phi_2(\alpha)\alpha) = 1$  have minimal positive solutions  $\bar{\tau}_0$  and  $\bar{\bar{\tau}}_0$ , respectively. Set

$$\rho = \min\{\tau_0, \bar{\tau}_0, \bar{\bar{\tau}}_0\}.$$

Furthermore, define some functions  $\phi_2$ ,  $h_2$ ,  $\phi_3$  and  $h_3$  on  $I = [0, \rho)$  in the following way:

$$\phi_2(\alpha) = \frac{\int_0^1 w((1-\nu)\phi_1(\alpha)\alpha)d\nu}{1-w_0(\phi_1(\alpha)\alpha)} + \frac{w((1+\phi_1(\alpha))\alpha) \int_0^1 \nu(\nu\phi_1(\alpha)\alpha)d\nu}{(1-w_0(\alpha))(1-w_0(\phi_1(\alpha)\alpha))} \phi_1(\alpha),$$

$$h_2(\alpha) = \phi_2(\alpha) - 1,$$

$$\begin{aligned} \phi_3(\alpha) = & \left[ \frac{\int_0^1 w((1-\nu)\phi_2(\alpha)\alpha)d\nu}{1-w_0(\phi_2(\alpha)\alpha)} + \frac{w((1+\phi_1(\alpha))\alpha) \int_0^1 \nu(\nu\phi_2(\alpha)\alpha)d\nu}{|\beta|(1-w_0(\alpha))(1-w_0(\phi_1(\alpha)\alpha))} \right. \\ & \left. + \frac{w((1+\phi_1(\alpha))\alpha) \int_0^1 \nu(\nu\phi_2(\alpha)\alpha)d\nu}{(1-w_0(\alpha))(1-w_0(\phi_2(\alpha)\alpha))} \right] \phi_2(\alpha), \end{aligned}$$

and

$$h_3(\alpha) = \phi_3(\alpha) - 1,$$

We obtain again  $h_2(0) = -\phi_3(0) = -1 < 0$ , and  $h_2(\alpha) \rightarrow +\infty$ ,  $h_3(\alpha) \rightarrow +\infty$  as  $\alpha \rightarrow \rho^-$ . Let us denote  $\tau_2$  and  $\tau_3$  as the smallest zero of the functions  $h_2$  and  $h_3$ , respectively, on the interval  $(0, \rho)$ . Finally, we define the convergence radius  $\tau$  as follows:

$$\tau = \min\{\tau_i\}, \quad i = 1, 2, 3. \quad (5)$$

Then, we have

$$0 < \tau < \tau_0, \quad (6)$$

$$0 \leq \phi_i(\alpha) < 1, \quad (7)$$

$$0 \leq w_0(\alpha) < 1, \quad (8)$$

$$0 \leq w_0(\phi_1(\alpha)\alpha) < 1, \quad (9)$$

and

$$0 \leq w_0(\phi_2(\alpha)\alpha) < 1, \text{ for each } \alpha \in [0, \tau). \quad (10)$$

Let  $U(z, \rho)$  and  $\bar{U}(z, \rho)$  stand, respectively, for the open and closed balls in  $\mathbb{E}_1$  with center  $z \in \mathbb{E}_1$  and radius  $\rho > 0$ .

Next, we present the local convergence analysis of the method in Equation (2) using the preceding notations.

**Theorem 1.** Let  $S : \Delta \subseteq \mathbb{E}_1 \rightarrow \mathbb{E}_2$  be a Fréchet-differentiable operator. We assume that  $v, w_0, w : [0, \infty) \rightarrow [0, \infty)$  are non-decreasing continuous functions with  $w_0(0) = w(0) = 0$ . Let  $\beta \in T - \{0\}$  be such that Equation (4) is satisfied and  $\tau_0, \bar{\tau}_0, \bar{\bar{\tau}}_0$  exist. In addition, we consider the zero  $\lambda_* \in \Delta$  is well defined, such that, for each  $\lambda \in \Delta$ ,

$$S(\lambda_*) = 0, \quad S'(\lambda_*)^{-1} \in L(\mathbb{E}_2, \mathbb{E}_1) \quad (11)$$

and

$$\|S'(\lambda_*)^{-1}(S'(\lambda) - S'(\lambda_*))\| \leq w_0(\|x - \lambda_*\|). \quad (12)$$

Further, we consider that, for each  $\lambda, \eta \in \Delta_0 := \Delta \cap U(\lambda_*, \tau_0)$ ,

$$\|S'(\lambda_*)^{-1}(S'(\lambda) - S'(\eta))\| \leq w(\|\lambda - \eta\|), \quad (13)$$

$$\|S'(\lambda_*)^{-1}S'(\lambda)\| \leq v(\|\lambda - \lambda_*\|), \quad (14)$$

and

$$\bar{U}(\lambda_*, \tau) \subseteq \Delta, \quad (15)$$

where the convergence radius  $\tau$  is given by Equation (5). Then, the sequence  $\{\lambda_l\}$  obtained for  $\lambda_0 \in U(\lambda_*, \tau) - \{\lambda_*\}$  by the scheme in Equation (2) is well defined, remains in  $U(\lambda_*, \tau)$  for each  $l = 0, 1, 2, \dots$ , and converges to  $\lambda_*$ . Moreover, the following estimates hold

$$\|\eta_l - \lambda_*\| \leq \phi_1(\|\lambda_l - \lambda_*\|)\|\lambda_l - \lambda_*\| \leq \|\lambda_l - \lambda_*\| < \tau, \quad (16)$$

$$\|\theta_l - \lambda_*\| \leq \phi_2(\|\lambda_l - \lambda_*\|)\|\lambda_l - \lambda_*\| \leq \|\lambda_l - \lambda_*\| \quad (17)$$

and

$$\|\lambda_{l+1} - \lambda_*\| \leq \phi_3(\|\lambda_l - \lambda_*\|)\|\lambda_l - \lambda_*\| \leq \|\lambda_l - \lambda_*\|, \quad (18)$$

where the functions  $\phi_i$ ,  $i = 1, 2, 3$  are defined previously. Furthermore, if

$$\int_0^1 w_0(\theta R) d\theta < 1, \text{ for } R \geq \tau, \quad (19)$$

then the point  $\lambda_*$  is the unique zero of  $S(\lambda) = 0$  in  $\Delta_1 := \Delta \cap \bar{U}(\lambda_*, R)$ .

**Proof.** We adopt the mathematical induction technique in order to demonstrate that the sequence  $\{\lambda_l\}$  is well defined in  $U(\lambda_*, \tau)$  and also converges toward  $\lambda_*$ . By the hypothesis  $\lambda_0 \in U(\lambda_*, \tau) - \{\lambda_*\}$ , and Equations (3), (5) and (12), we obtain

$$\|S'(\lambda_*)^{-1}(S'(\lambda_0) - S'(\lambda_*))\| \leq w_0(\|\lambda_0 - \lambda_*\|) < w_0(\tau) < 1. \quad (20)$$

In view of Equation (20) and Banach Lemma on non-singular operators [2,3],  $S'(\lambda_0)^{-1} \in L(\mathbb{E}_2, \mathbb{E}_1)$ ,  $\eta_0$  and  $\theta_0$  are well defined by the first two sub steps of the method in Equation (2) and

$$\|S'(\lambda_0)^{-1}S'(\lambda_*)\| \leq \frac{1}{1 - w_0(\|\lambda_0 - \lambda_*\|)}. \quad (21)$$

Adopting the first sub step of the scheme in Equations (2), (5), (7) (for  $l = 1$ ), (11), (13), (14) and (21), we yield

$$\begin{aligned} \|\eta_0 - \lambda_*\| &= \|(\lambda_0 - \lambda_* - S'(\lambda_0)^{-1}S(\lambda_0)) + (1 - \beta)S'(\lambda_0)^{-1}S(\lambda_0)\| \\ &\leq \|S'(\lambda_0)^{-1}S(\lambda_*)\| \left\| \int_0^1 S'(\lambda_*)^{-1}(S'(\lambda_* + \nu(\lambda_0 - \lambda_*)) - S'(\lambda_0))(\lambda_0 - \lambda_*) d\nu \right\| \\ &\quad + |1 - \beta| \|S'(\lambda_0)^{-1}S'(\lambda_*)\| \|S'(\lambda_0)^{-1}S(\lambda_0)\| \\ &\leq \frac{\int_0^1 w((1 - \nu)\|\lambda_0 - \lambda_*\|) d\nu \|\lambda_0 - \lambda_*\| + |1 - \beta| \int_0^1 v(\nu\|\lambda_0 - \lambda_*\|) d\nu \|\lambda_0 - \lambda_*\|}{1 - w_0(\|\lambda_0 - \lambda_*\|)} \\ &= \phi_1(\|\lambda_0 - \lambda_*\|)\|\lambda_0 - \lambda_*\| \leq \|\lambda_0 - \lambda_*\| < \tau, \end{aligned} \quad (22)$$

which implies Equation (16) for  $l = 0$  and  $\eta_0 \in U(\lambda_*, \tau)$ . We can write by the second sub step of the method in Equation (2)

$$\theta_0 - \lambda_* = \eta_0 - \lambda_* - S'(\eta_0)^{-1}S(\eta_0) + S'(\eta_0)^{-1}(S'(\eta_0) - S'(\lambda_0))S'(\lambda_0)^{-1}S(\eta_0). \quad (23)$$

Notice that  $\eta_0 \in U(\lambda_*, \tau)$  with  $\eta_0 = \lambda_0$  in Equation (20),  $S'(\eta_0)^{-1} \in L(\mathbb{E}_2, \mathbb{E}_1)$  and  $\lambda_1$  are well defined and

$$\begin{aligned} \|S'(\eta_0)^{-1}S'(\lambda_*)\| &\leq \frac{1}{1-w_0(\|\eta_0-\lambda_*\|)} \\ &\leq \frac{1}{1-w_0(\phi_1(\|\lambda_0-\lambda_*\|)\|\lambda_0-\lambda_*\|)} \leq \frac{1}{1-w_0(\|\lambda_0-\lambda_*\|)}. \end{aligned} \quad (24)$$

By Equations (5), (6) (for  $l = 2$ ), (9), (11), (13), (14), and (21)–(24), we get

$$\begin{aligned} \|\theta_0 - \lambda_*\| &= \|\eta_0 - \lambda_* - S'(\eta_0)^{-1}S(\eta_0)\| \\ &\quad + \|S'(\eta_0)^{-1}S'(\lambda_*)\| \|S'(\lambda_*)^{-1}(S'(\lambda_0) - S'(\eta_0))\| \|S'(\lambda_0)^{-1}S'(\lambda_*)\| \|S'(\lambda_*)^{-1}S'(\eta_0)\| \\ &\leq \frac{\int_0^1 w((1-\nu)\|\eta_0 - \lambda_*\|)d\nu \|\eta_0 - \lambda_*\|}{1-w_0(\|\eta_0 - \lambda_*\|)} + \frac{w(\|\eta_0 - \lambda_*\|) \int_0^1 v(\nu\|\eta_0 - \lambda_*\|)d\nu \|\eta_0 - \lambda_*\|}{(1-w_0(\|\lambda_0 - \lambda_*\|))(1-w_0(\|\eta_0 - \lambda_*\|))} \\ &\leq \phi_2(\|\lambda_0 - \lambda_*\|)\|\lambda_0 - \lambda_*\| \leq \|\lambda_0 - \lambda_*\| < \tau, \end{aligned} \quad (25)$$

which implies Equation (17) for  $l = 0$  and  $\theta_0 \in U(\lambda_*, \tau)$ .

Then, by the third sub step of the method in Equations (2), (5), (7) (for  $l = 3$ ), (9), (10), and (21)–(25) (for  $\eta_0 = \theta_0$ ), we yield

$$\begin{aligned} \|\lambda_1 - \lambda_*\| &\leq \|\theta_0 - \lambda_* - S'(\theta_0)^{-1}S(\theta_0)\| + \frac{1}{|\beta|} \|S'(\eta_0)^{-1}(S'(\lambda_0) - S'(\eta_0))S'(\lambda_0)^{-1}S'(\lambda_*)\| \|S'(\lambda_*)^{-1}S(\theta_0)\| \\ &\quad + \|S'(\theta_0)^{-1}(S'(\lambda_0) - S'(\theta_0))S'(\lambda_0)^{-1}S'(\lambda_*)\| \|S'(\lambda_*)^{-1}S(\theta_0)\| \\ &\leq \frac{\int_0^1 w((1-\nu)\|\theta_0 - \lambda_*\|)\|\theta_0 - \lambda_*\|d\nu}{1-w_0(\|\theta_0 - \lambda_*\|)} + \frac{1}{|\beta|} \frac{w(\|\eta_0 - \lambda_*\|) \int_0^1 v(\nu\|\theta_0 - \lambda_*\|)d\nu \|\theta_0 - \lambda_*\|}{(1-w_0(\|\eta_0 - \lambda_*\|))(1-w_0(\|\lambda_0 - \lambda_*\|))} \\ &\quad + \frac{w(\|\theta_0 - \lambda_*\|) \int_0^1 v(\nu\|\theta_0 - \lambda_*\|)d\nu \|\theta_0 - \lambda_*\|}{(1-w_0(\|\theta_0 - \lambda_*\|))(1-w_0(\|\lambda_0 - \lambda_*\|))} \\ &\leq \phi_3(\|\lambda_0 - \lambda_*\|)\|\lambda_0 - \lambda_*\| \leq \|\lambda_0 - \lambda_*\| < \tau, \end{aligned} \quad (26)$$

which shows Equation (18) for  $l = 0$  and  $\theta_0 \in U(\lambda_*, \tau)$ . By changing  $\lambda_0, \eta_0, \theta_0, \lambda_1$  by  $\lambda_l, \eta_l, \theta_l, \lambda_{l+1}$  in the preceding estimates, we attain at Equations (16)–(18). Therefore, in view of the estimates

$$\|\lambda_{l+1} - \lambda_*\| \leq c\|\lambda_l - \lambda_*\| < \tau, \quad c = \phi_3(\|\lambda_0 - \lambda_*\|) \in [0, 1), \quad (27)$$

we deduce that  $\lim_{l \rightarrow \infty} \lambda_l = \lambda_*$  and  $\lambda_{l+1} \in U(\lambda_*, \tau)$ .

Finally, we have to illustrate the uniqueness part. We assume that  $\eta_* \in \Delta_1$  with  $S(\eta_*) = 0$  and define  $Q = \int_0^1 S'(\lambda_* + \theta(\lambda_* - \eta_*))d\theta$ . Using Equations (12) and (19), we get

$$\begin{aligned} \|S'(\lambda_*)^{-1}(Q - S'(\lambda_*))\| &\leq \int_0^1 w_0(\theta\|\eta_* - \lambda_*\|)d\nu \\ &\leq \int_0^1 w_0(\theta R)d\theta < 1. \end{aligned} \quad (28)$$

It is confirmed from Equation (28) that  $Q$  is an invertible operator. Then, in view of the identity

$$0 = S(\lambda_*) - S(\eta_*) = Q(\lambda_* - \eta_*), \quad (29)$$

we deduce that  $\lambda_* = \eta_*$ .  $\square$

**Remark 1.**

(a) It is clear from Equation (12) that the condition in Equation (14) can be released and adopted as follow:

$$v(\alpha) = 1 + w_0(\alpha) \text{ or } v(\alpha) = 1 + w_0(\tau_0), \quad (30)$$

since,

$$\begin{aligned} \|S'(\lambda_*)^{-1} [(S'(\lambda) - S'(\lambda_*)) + S'(\lambda_*)]\| &= 1 + \|S'(\lambda_*)^{-1}(S'(\lambda) - S'(\lambda_*))\| \\ &\leq 1 + w_0(\|\lambda - \lambda_*\|) \\ &= 1 + w_0(\alpha) \text{ for } \|\lambda - \lambda_*\| \leq \tau_0. \end{aligned} \quad (31)$$

Further, Singh et al. [1] considered the following conditions for each  $\lambda, \eta \in \Delta$  in the Hölder case

$$\|S'(\lambda_*)^{-1}(S'(\lambda) - S'(\lambda_*))\| \leq w_0 \|\lambda - \lambda_*\|^p, \quad (32)$$

$$\|S'(\lambda_*)^{-1}(S'(\lambda) - S'(\eta))\| \leq \bar{w} \|\lambda - \eta\|^p, \quad (33)$$

for  $\beta \in \left(\frac{4}{5}, \frac{5}{4}\right)$  (corresponding to Equation (4)).

In our case, we have

$$\|S'(\lambda_*)^{-1}(S'(\lambda) - S'(\lambda_*))\| \leq w_0 \|\lambda - \lambda_*\|^p, \text{ for each } \lambda, \eta \in \Delta_0, \quad (34)$$

thus

$$w \leq \bar{w} \quad (35)$$

holds, since  $\Delta_0 \subseteq \Delta$ . Hence, the improvements, as stated in the Abstract of this paper, hold for  $w < \bar{w}$  (see the numerical examples too).

Estimates

$$\|S'(\lambda_*)^{-1}S'(\lambda)\| \leq 1 + w_0 \|\lambda - \lambda_*\|^p, \quad (36)$$

$$\|S'(\lambda_*)^{-1}S'(\lambda_* + \theta(\lambda - \lambda_*))\| \leq 1 + w_0 \|\lambda - \lambda_*\|^p, \quad (37)$$

used in [1,23] are not better than ours

$$\|S'(\lambda_*)^{-1}S'(\lambda)\| \leq v(\|\lambda - \lambda_*\|), \text{ for each } \lambda \in \Delta_0. \quad (38)$$

Indeed, use  $S(\lambda) = \sin \lambda$ , then  $v(\alpha) = 1$ . However, the corresponding one in Equations (36) and (37) are less tight than Equation (38). Hence, the results using Equation (38) instead of Equations (36) and (37) provide advantages, as stated in the Introduction.

(b) If  $w_0$  is a strictly increasing function, then we can consider

$$\tau_0 = w_0^{-1}(1) \quad (39)$$

instead of Equation (3).

(c) If  $w_0, w$  are constants functions,  $p = 1$  and  $\beta = 1$ , then, we showed in [2,13] using only Equations (12) and (13) for the case of Newton's method (see the definition of function  $\phi_1$  too)

$$\tau_1 = \frac{2}{2w_0 + w}, \quad (40)$$

thus

$$\tau \leq \tau_1. \quad (41)$$

Therefore, the convergence radius  $\tau$  has maximum value  $\tau_1$  and  $\tau_1$  is the convergence radius of Newton's method

$$\lambda_{l+1} = \lambda_l - S'(\lambda_l)^{-1}S(\lambda_l). \quad (42)$$

Rheindoldt [22] and Traub [8] provided the following convergence radius instead of  $\tau_1$

$$\tau_{TR} = \frac{2}{3w_1}. \quad (43)$$

On the other hand, Argyros [2,3] proposed the following convergence radius

$$\tau_A = \frac{2}{2w_0 + w_1}, \quad (44)$$

where  $w_1$  is the Lipschitz constant for Equation (8) on  $\Delta$ . However, we have

$$w \leq w_1, \quad w_0 \leq w_1, \quad (45)$$

thus

$$\tau_{TR} \leq \tau_A \leq \tau_1 \quad (46)$$

and

$$\frac{\tau_{TR}}{\tau_A} \rightarrow \frac{1}{3} \quad \text{as} \quad \frac{w_0}{w} \rightarrow 0. \quad (47)$$

The convergence radius  $q$  adopted in [24] is smaller than the radius  $\tau_{DS}$  proposed by Dennis and Schnabel [3]

$$q < \tau_{DS} = \frac{1}{2w_1} < \tau_{TR}. \quad (48)$$

However,  $q$  cannot be computed using the Lipschitz constants.

- (d) By adopting fifth-order derivative of  $S$ , the convergence order of the scheme in Equation (2) was demonstrated in [24]. On the other hand, our approach required only hypotheses on first-order derivative of  $S$ . To obtain the convergence order, we adopt the following techniques for the computational order of convergence COC

$$\xi = \frac{\ln \frac{\|\lambda_{n+2} - \lambda_*\|}{\|\lambda_{l+1} - \lambda_*\|}}{\ln \frac{\|\lambda_{l+1} - \lambda_*\|}{\|\lambda_l - \lambda_*\|}}, \quad \text{for each } l = 0, 1, 2, \dots \quad (49)$$

or the approximate computational order of convergence (ACOC) [19],

$$\xi^* = \frac{\ln \frac{\|\lambda_{l+2} - \lambda_{l+1}\|}{\|\lambda_{l+1} - \lambda_l\|}}{\ln \frac{\|\lambda_{l+1} - \lambda_n\|}{\|\lambda_n - \lambda_{n-1}\|}}, \quad \text{for each } l = 1, 2, \dots \quad (50)$$

Neither technique  $t$  requires any kind of derivative(s). It is also vital to note that there is no need of exact zero  $\lambda_*$  in the case of  $\xi^*$ .

- (e) Consider operator  $S$  satisfying the autonomous differential equation [2,3]

$$S'(\lambda) = T(S(\lambda)) \quad (51)$$

where  $T$  is a given continuous operator. By  $S'(\lambda_*) = T(S(\lambda_*)) = P(0)$ , we can use our results without the prior knowledge of required solution  $\lambda_*$ . For example,  $S(\lambda) = e^x - 1$ . Therefore, we obtain  $T(\lambda) = x + 1$ .

(f) In view of estimates

$$\begin{aligned}\|S'(\lambda_*)^{-1}(S'(\lambda_0) - S'(\eta_0))\| &= \|S'(\lambda_*)^{-1}(S'(\lambda_0) - S'(\lambda_*))\| + \|S'(\lambda_*)^{-1}(S'(\eta_0) - S'(\lambda_0))\| \\ &\leq w_0(\|\lambda_0 - \lambda_*\|) + w_0(\|\eta_0 - \lambda_*\|) \\ &\leq w_0(\|\lambda_0 - \lambda_*\|) + w_0(\phi_1(\|\lambda_0 - \lambda_*\|)\|\lambda_0 - \lambda_*\|) \\ &\leq w_0(\tau) + w_0(\phi_1(\tau)\tau),\end{aligned}\quad (52)$$

and similarly

$$\begin{aligned}\|S'(\lambda_*)^{-1}(S'(\lambda_0) - S'(\theta_0))\| &\leq w_0(\|\lambda_0 - \lambda_*\|) + w_0(\|\theta_0 - \lambda_*\|) \\ &\leq w_0(\tau) + w_0(\phi_2(\tau)\tau),\end{aligned}\quad (53)$$

we can replace the terms  $w((1 + \phi_1(\alpha))\alpha)$ ,  $w((1 + \phi_2(\alpha))\alpha)$  in the definition of functions  $\phi_2$  and  $\phi_3$  by  $w_0(\alpha) + w_0(\phi_1(\alpha)\alpha)$ ,  $w_0(\alpha) + w_0(\phi_2(\alpha)\alpha)$ , respectively. If

$$w_0(\alpha) \leq w(\alpha), \quad \alpha \in [0, \tau_0)$$

and say  $w_0$ ,  $w$  are constants, then the new functions  $\phi_2$  and  $\phi_3$  are tighter than the old one leading to larger  $\tau$  and tighter error bounds on the distances  $\|\lambda_l - \lambda_*\|$  (if  $w_0 < w$ ).

### 3. Concrete Examples

Here, we test the convergence conditions using concrete examples.

**Example 1.** Here, we assume one of the well-known Hammerstein integral equations (see pp. 19–20, [25]) defined by:

$$x(s) = \frac{1}{5} \int_0^1 S(s, t)x(t)^3 dt, \quad x \in C[0, 1], \quad s, t \in [0, 1], \quad (54)$$

where the kernel  $S$  is:

$$S(s, t) = \begin{cases} s(1-t), & s \leq t, \\ (1-s)t, & t \leq s. \end{cases}$$

We use  $\int_0^1 \phi(t)dt \simeq \sum_{k=1}^8 w_k \phi(t_k)$  in Equation (54), where  $t_k$  and  $w_k$  are the abscissas and weights, respectively. Denoting the approximations of  $x(t_i)$  with  $x_i$  ( $i = 1, 2, 3, \dots, 8$ ), then it yields the following  $8 \times 8$  system of nonlinear equations:

$$5x_i - 5 - \sum_{k=1}^8 a_{ik}x_k^3 = 0, \quad i = 1, 2, 3, \dots, 8,$$

$$a_{ik} = \begin{cases} w_k t_k (1 - t_i), & k \leq i, \\ w_k t_i (1 - t_k), & i < k. \end{cases}$$

By Gauss–Legendre quadrature formula, we obtained the values of  $t_k$  and  $w_k$  when  $k = 8$ , which are depicted in Table 1.

The required approximate root is:

$$\lambda_* = (1.002096 \dots, 1.009900 \dots, 1.019727 \dots, 1.026436 \dots, 1.026436 \dots, \\ 1.019727 \dots, 1.009900 \dots, 1.002096 \dots)^T.$$

Then, we get  $w_0(t) = w(t) = \frac{3}{40}t$  and  $v(t) = 1 + w_0(t)$ . We have the following radii of convergence for this problem in Table 2.



**Table 1.** Abscissas and weights for  $k = 8$ .

$j$	$t_j$	$w_j$
1	0.01985507175123188415821957...	0.05061426814518812957626567...
2	0.10166676129318663020422303...	0.11119051722668723527217800...
3	0.23723379504183550709113047...	0.15685332293894364366898110...
4	0.40828267875217509753026193...	0.18134189168918099148257522...
5	0.59171732124782490246973807...	0.18134189168918099148257522...
6	0.76276620495816449290886952...	0.15685332293894364366898110...
7	0.89833323870681336979577696...	0.11119051722668723527217800...
8	0.98014492824876811584178043...	0.05061426814518812957626567...

**Table 2.** Distinct convergence radii for Example 1.

$\beta$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau$	$\lambda_0$	$l$	$\rho$
1	8.88889	7.12148	6.73523	6.73523	(0.5, 0.5, ..., 0.5 (8 times))	3	4.9998
$\frac{1}{3}$	2.42424	3.1914	2.51312	2.42424	(0.5, 0.5, ..., 0.5 (8 times))	4	3.9999
$\frac{1}{4}$	1.77778	2.93819	2.14571	1.77778	(0.25, 0.25, ..., 0.25 (8 times))	4	3.9999

We follow in all examples the stopping criteria for programming of: (i)  $\|F(\lambda_l)\| < 10^{-100}$ ; and (ii)  $\|\lambda_{l+1} - \lambda_l\| < 10^{-100}$ .

**Example 2.** Consider the nonlinear integral Equations (13) and (16), when  $\mathbb{E}_1 = \mathbb{E}_2 = C[0, 1]$  as

$$\lambda(\gamma_1) = \int_0^1 G(\gamma_1, \gamma_2) \left( \lambda(\gamma_2)^{\frac{3}{2}} + \frac{\lambda(\gamma_2)^2}{2} \right) d\gamma_2, \quad (55)$$

where the kernel  $G : [0, 1] \times [0, 1]$  is

$$G(\gamma_1, \gamma_2) = \begin{cases} (1 - \gamma_2)\gamma_2, & \gamma_2 \leq \gamma_1, \\ \gamma_1(1 - \gamma_2), & \gamma_1 \leq \gamma_2. \end{cases} \quad (56)$$

The solution  $\lambda_*(\gamma_1) = 0$  is the same as the solution of Equation (1), where  $S : C[0, 1] \rightarrow C[0, 1]$  is defined by

$$[S(\lambda)](\gamma_1) = \lambda(\gamma_1) - \int_0^{\gamma_2} G(\gamma_1, \gamma_2) \left( \lambda(\gamma_2)^{\frac{3}{2}} + \frac{\lambda(\gamma_2)^2}{2} \right) d\gamma_2. \quad (57)$$

We get

$$\left\| \int_0^{\gamma_2} G(\gamma_1, \gamma_2) d\gamma_2 \right\| \leq \frac{1}{8}. \quad (58)$$

Moreover,

$$[S'(\lambda)\eta](\gamma_1) = \eta(\gamma_1) - \int_0^{\gamma_2} G(\gamma_1, \gamma_2) \left( \frac{3}{2}\lambda(\gamma_2)^{\frac{1}{2}} + \lambda(\gamma_2) \right) \eta(\gamma_2) d\gamma_2,$$

thus, since  $S'(\lambda_*(\gamma_1)) = I$ ,

$$\left\| S'(\lambda_*)^{-1} (S'(\lambda) - S'(\eta)) \right\| \leq \frac{1}{8} \left( \frac{3}{2} \|\lambda - \eta\|^{\frac{1}{2}} + \|\lambda - \eta\| \right). \quad (59)$$

Hence, we have

$$w_0(\alpha) = w(\alpha) = \frac{1}{8} \left( \frac{3}{2} \alpha^{\frac{1}{2}} + \alpha \right),$$

thus, by Remark 1 (a), we can choose

$$v(\alpha) = 1 + w_0(\alpha).$$

Therefore, our results can be utilized but not the ones in [1] because  $S'$  is unbounded on  $\Delta$ . We have the following radii of convergence for the problem in Example 2 mentioned in Table 3.

**Table 3.** Distinct convergence radii for Example 2.

$\beta$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau$
1	2.6303	1.72877	1.53391	1.53391
$\frac{1}{2}$	0.77579	0.678555	0.450781	0.450781
$\frac{1}{4}$	0.231922	0.454561	0.157970	0.157970

**Example 3.** Consider a system of differential equations,

$$\begin{aligned} s_1'(\lambda) - s_1(\lambda) - 1 &= 0 \\ s_2'(\eta) - (e-1)\eta - 1 &= 0 \\ s_3'(\theta) - 1 &= 0 \end{aligned} \quad (60)$$

that describes the movement of a particle in three dimensions with  $\lambda, \eta, \theta \in \Delta$  for  $s_1(0) = s_2(0) = s_3(0) = 0$ . Then, the solution  $v = (\lambda, \eta, \theta)^T$  relates to  $S := (s_1, s_2, s_3) : \Delta \rightarrow \mathbb{R}^3$ , given as

$$S(v) = \left( e^\lambda - 1, \frac{e-1}{2}\eta^2 + \eta, \theta \right)^T. \quad (61)$$

It follows from Equation (61) that

$$S'(v) = \begin{bmatrix} e^\lambda & 0 & 0 \\ 0 & (e-1)\eta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have that  $w_0(\alpha) = L_0\alpha$ ,  $w(\alpha) = L\alpha$  and  $v(\alpha) = M$ , where  $L_0 = e - 1 < L = e^{\frac{1}{L_0}} = 1.789572397$  and  $M = e^{\frac{1}{L_0}} = 1.7896$ . We have the following radii of convergence for Example 3, depicted in Tables 4 and 5.

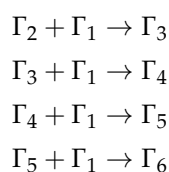
**Table 4.** Convergence radii for Example 3.

$\beta$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau$	$\lambda_0$	$l$	$\rho$
1	0.377542	0.16544	0.134375	0.134375	(0.09, 0.09, 0.09)	3	4.9996

**Table 5.** Convergence radii for Example 3 with  $w_0(t) = w(t) = et$ ,  $v(t) = e$  (call them barfunctions).

$\beta$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau$	$\lambda_0$	$l$	$\rho$
1	0.245253	0.0650807	0.0497009	0.0497009	(0.03, 0.03, 0.03)	3	4.9994

**Example 4.** The chemical reaction [26] illustrated in this case shows how  $\Gamma_1$  and  $\Gamma_2$  are utilized at rates  $q_* - Q_*$  and  $Q_*$ , respectively, for a tank reactor (known as CSTR), given by:



Douglas [27] analyzed the CSTR problem for designing simple feedback control systems. The following mathematical formulation was adopted:

$$K_C \frac{2.98(\lambda + 2.25)}{(\lambda + 1.45)(\lambda + 2.85)^2(\lambda + 4.35)} = -1,$$

where the parameter  $K_C$  has a physical meaning and is described in [26,27]. For the particular value of choice  $K_C = 0$ , we obtain the corresponding equation:

$$S(\lambda) = \lambda^4 + 11.50\lambda^3 + 47.49\lambda^2 + 83.06325\lambda + 51.23266875. \quad (62)$$

The function  $S$  has four solutions  $\lambda_* = (-2.85, -1.45, -2.85, -4.35)$ . Nonetheless, the desired zero is  $\lambda_* = -4.35$  for Equation (62). We assume  $\Delta = [-4.5, -4]$ . Then, we have  $w_0(\alpha) = w(\alpha) = 0.644828\alpha$  and  $v(\alpha) = 0.238439$ .

The radii of convergence for the method (2) on the basis of Example 4 are mentioned in Table 6.

**Table 6.** Different radii of convergence for Example 4.

$\beta$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau$	$\lambda_0$	$l$	$\rho$
1	1.03387	0.963489	0.942188	0.942188	−4.6	4	5.0000
−1	0.829407	0.81933	0.806711	0.806711	−4.5	4	5.0000
$\frac{1}{3}$	0.955364	0.906428	0.872289	0.872289	−4.4	4	4.0000

**Example 5.** By the example in the Introduction, we get  $L = L_0 = 96.662907$  and  $M = 2$ . The radii of convergence of the method (2) for Example 5 are described in Table 7.

**Table 7.** Different radii of convergence for Example 5.

$\beta$	$\tau_1$	$\tau_2$	$\tau_3$	$\tau$	$\lambda_0$	$l$	$\rho$
1	0.00689682	0.000104857	0.000074428	0.000074428	1.00001	2	5.0000
1.1	0.00551746	0.0104211	0.000459821	0.000459821	1.0001	3	4.0000

**Author Contributions:** R.B. and I.K.A.: Conceptualization; Methodology; Validation; Writing—Original Draft Preparation; Writing—Review & Editing. A.S.A.: Review & Editing.

**Funding:** Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, under Grant No. G: 424-130-1440.

**Acknowledgments:** This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, under Grant No. G: 424-130-1440. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Singh, S.; Gupta, D.K.; Badoni, R.P.; Martínez, E.; Hueso, J.L. Local convergence of a parameter based iteration with Hölder continuous derivative in Banach spaces. *Calcolo* **2017**, *54*, 527–539. [\[CrossRef\]](#)
2. Argyros, I.K. *Convergence and Application of Newton-Type Iterations*; Springer: Berlin/Heidelberg, Germany, 2008.
3. Argyros, I.K.; Hilout, S. *Computational Methods in Nonlinear Analysis*; World Scientific Publishing Company: Hackensack, NJ, USA, 2013.
4. Cordero, A.; Torregrosa, J.R.; Vassileva, M.P. Increasing the order of convergence of iterative schemes for solving nonlinear system. *J. Comput. Appl. Math.* **2012**, *252*, 86–94. [\[CrossRef\]](#)
5. Hernández, M.A.; Martínez, E. On the semilocal convergence of a three steps Newton-type process under mild convergence conditions. *Numer. Algor.* **2015**, *70*, 377–392. [\[CrossRef\]](#)

6. Potra, F.A.; Pták, V. Nondiscrete introduction and iterative process. *Res. Notes Math.* **1984**, *103*.
7. Sharma, J.R.; Ghua, R.K.; Sharma, R. An efficient fourth-order weighted-Newton method for system of nonlinear equations. *Numer. Algor.* **2013**, *62*, 307–323. [[CrossRef](#)]
8. Traub, J.F. *Iterative Methods for the Solution of Equations*; Prentice-Hall Series in Automatic Computation; Prentice-Hall: Englewood Cliffs, NJ, USA, 1964.
9. Amat, S.; Busquier, S.; Plaza, S.; Gutiérrez, J.M. Geometric constructions of iterative functions to solve nonlinear equations. *J. Comput. Appl. Math.* **2003**, *157*, 197–205. [[CrossRef](#)]
10. Amat, S.; Busquier, S.; Plaza, S. Dynamics of the King and Jarratt iterations. *Aequationes Math.* **2005**, *69*, 212–223. [[CrossRef](#)]
11. Amat, S.; Hernández, M.A.; Romero, N. A modified Chebyshev's iterative method with at least sixth order of convergence. *Appl. Math. Comput.* **2008**, *206*, 164–174. [[CrossRef](#)]
12. Argyros, I.K.; Magreñán, Á.A. Ball convergence theorems and the convergence planes of an iterative methods for nonlinear equations. *SeMA* **2015**, *71*, 39–55.
13. Argyros, I.K.; George, S. Local convergence of some higher-order Newton-like method with frozen derivative. *SeMa* **2015**, *70*, 47–59. [[CrossRef](#)]
14. Cordero, A.; Torregrosa, J.R. Variants of Newton's method using fifth-order quadrature formulas. *Appl. Math. Comput.* **2007**, *190*, 686–698. [[CrossRef](#)]
15. Cordero, A.; Torregrosa, J.R. Variants of Newton's method for functions of several variables. *Appl. Math. Comput.* **2006**, *183*, 199–208. [[CrossRef](#)]
16. Ezquerro, J.A.; Hernández, M.A. New iterations of R-order four with reduced computational cost. *BIT Numer. Math.* **2009**, *49*, 325–342. [[CrossRef](#)]
17. Ezquerro, J.A.; Hernández, M.A. A uniparametric halley type iteration with free second derivative. *Int. J. Pure and Appl. Math.* **2003**, *6*, 99–110.
18. Gutiérrez, J.M.; Hernández, M.A. Recurrence relations for the super-Halley method. *Comput. Math. Appl.* **1998**, *36*, 1–8. [[CrossRef](#)]
19. Kou, J. A third-order modification of Newton method for systems of nonlinear equations. *Appl. Math. Comput.* **2007**, *191*, 117–121.
20. Montazeri, H.; Soleymani, F.; Shateyi, S.; Motsa, S.S. On a new method for computing the numerical solution of systems of nonlinear equations. *J. Appl. Math.* **2012**, *2012*. [[CrossRef](#)]
21. Petkovic, M.S.; Neta, B.; Petkovic, L.; Džunić, J. *Multipoint Methods for Solving Nonlinear Equations*; Elsevier: Amsterdam, The Netherlands, 2013.
22. Rheinboldt, W.C. *An Adaptive Continuation Process for Solving Systems of Nonlinear Equations*; Banach Center Publications: Warszawa, Poland, 1978; Volume 3, pp. 129–142.
23. Martínez, E.; Singh, S.; Hueso, J.L.; Gupta, D.K. Enlarging the convergence domain in local convergence studies for iterative methods in Banach spaces. *Appl. Math. Comput.* **2016**, *281*, 252–265. [[CrossRef](#)]
24. Xiao, X.Y.; Yin, H.W. Increasing the order convergence for iterative methods to solve nonlinear systems. *Calcolo* **2016**, *53*, 285–300. [[CrossRef](#)]
25. Ortega, J.M.; Rheinboldt, W.C. *Iterative Solution of Nonlinear Equations in Several Variables*; Academic Press: New York, NY, USA, 1970.
26. Constantinides, A.; Mostoufi, N. *Numerical Methods for Chemical Engineers with MATLAB Applications*; Prentice Hall PTR: Upper Saddle River, NJ, USA, 1999.
27. Douglas, J.M. *Process Dynamics and Control*; Prentice Hall: Englewood Cliffs, NJ, USA, 1972; Volume 2.

