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Hyers–Ulam–Rassias Stability of Set Valued Additive and Cubic Functional Equations in Several Variables

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Abstract: In this paper, we establish Hyers–Ulam–Rassias stability results belonging to two different set valued functional equations in several variables, namely additive and cubic. The results are obtained in the contexts of Banach spaces. The work is in the domain of set valued analysis.

Keywords: Hyers–Ulam–Rassias stability; set valued functional equation; convex set; Banach space; Hausdorff distance

1. Introduction

In this paper, we consider the stability properties of two set valued functional equations, one of which is additive and the other is cubic. The single valued versions of these two equations are in the works of Ebadian et al. [1] and Chang et al. [2], respectively. The type of stability which we investigate here is Hyers–Ulam–Rassias stability. This concept of stability was for the first time mathematically formulated by Ulam [3], which was partly solved by Hyers [4] and further extended by Rassias [5]. Over the following years, this type of stability was considered by many researchers and has come to be known as Hyers–Ulam–Rassias (H–U–R) stability. The basic notion of this stability can be illustrated by looking at the question on a linear equation: “Does an approximately linear equation have a linear approximation?” Today, it has many extended forms and has been studied in several domains of mathematics including differential equations [6], functional equations [7], isometries [8], etc. In the course of our investigation, we use the Hausdorff distance between two appropriate subsets of a Banach space. We consider real vector spaces and real Banach spaces on which our functional equations are defined. It may be mentioned that Hyers–Ulam–Rassias stability studied for set valued functional equations was initiated by Lu et al. [9] and was followed in several works (e.g., [10–15]). We should emphasize the importance of the problem we consider in the present context. Set valued functions and functional equations are used in areas of applications of mathematics such as vector optimization problems [16]. Stability is a concept which cannot be ignored when we want to apply any algorithm to some specified problem. Among different types of stabilities, the speciality of H–U–R stability is that when it occurs we have an assurance that we have an approximation to some desired degree of accuracy by the same type of mathematical concept. By this assurance, we can proceed with the same type of algorithms without any change.

2. Preliminaries

The following are some notations and definitions which we use in the present work.

Let Y be a real Banach space, the set of all non-empty closed bounded subsets of Y be $C_b(Y)$, the set of all non-empty closed convex subsets of Y be $C_c(Y)$, and the set of all non-empty closed bounded convex subsets of Y be $C_{cb}(Y)$. We denote $A \oplus B = \overline{A+B}$, where $A, B \in C_c(Y)$. Then, $sA + sB = s(A+B)$ and $(s+t)A \subseteq sA + tA$, where $s, t \in \mathbb{R}^+$ and A, B are subsets of Y . When A is convex, $(s+t)A = sA + tA$, where $s, t \in \mathbb{R}^+$.

For $A, A' \in C_b(Y)$, the Hausdorff distance $h(A, A')$ between A and A' is defined by

$$h(A, A') := \inf\{s \geq 0 | A \subseteq A' + sB_Y, A' \subseteq A + sB_Y\},$$

where B_Y is the closed unit ball in Y , that is, $B_Y = \{y \in Y : \|y\|_Y \leq 1\}$. Then, $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup [17]. For the subsets $A_1, A_2, \dots, A_n \in C_c(Y)$, we denote $A_1 \oplus A_2 \oplus \dots \oplus A_n = \overline{A_1 + A_2 + \dots + A_n}$.

The following lemma can be found in [17] and is used to establish results in the following sections.

Lemma 1. Let $P, P', Q, Q', S \in C_{cb}(Y)$ and $t > 0$. Then,

(a) $h(P \oplus P', Q \oplus Q') \leq h(P, Q) + h(P', Q')$;

(b) $h(tP, tQ) = th(P, Q)$; and

(c) $h(P, Q) = h(P \oplus S, Q \oplus S)$.

We consider two set valued functional equations, one of which is additive in m -variables

$$\sum_{i=1}^m f\left(mz_i + \sum_{j=1, j \neq i}^m z_j\right) \oplus f\left(\sum_{i=1}^m z_i\right) = 2m \sum_{i=1}^m f(z_i) \quad (1)$$

where $m \geq 2$ is a fixed positive integer and the other is cubic in n -variables

$$\begin{aligned} & 2f\left(\sum_{i=1}^{n-1} z_i + 2z_n\right) \oplus 2f\left(\sum_{i=1}^{n-1} z_i - 2z_n\right) \oplus 2 \sum_{i=1}^{n-1} f(2z_i) \oplus \\ & 7(n-1)[f(z_1) \oplus f(-z_1)] = 4f\left(\sum_{i=1}^{n-1} z_i\right) \oplus 8 \sum_{i=1}^{n-1} [f(z_i + z_n) \oplus f(z_i - z_n)] \end{aligned} \quad (2)$$

where $n \geq 3$ is a fixed integer and $f : X \rightarrow C_{cb}(Y)$ is a function where X is a real vector space and Y is real a Banach space.

3. Stability of Equation

The following is one of our main theorems.

Theorem 1. Let X be a real vector space, Y be a real Banach space and $m \geq 2$ be a positive integer. Let $\phi : X^m \rightarrow [0, \infty)$ be such that

$$\Phi(x_1, x_2, \dots, x_m) := \sum_{i=0}^{\infty} \frac{1}{m^i} \phi(m^i x_1, m^i x_2, \dots, m^i x_m) < \infty \text{ for all}$$

$x_1, x_2, \dots, x_m \in X$. Let $f : X \rightarrow C_{cb}(Y)$ be a mapping which satisfies $f(0) = \{0\}$ and the functional inequality

$$\begin{aligned} & h\left(\sum_{i=1}^m f\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) \oplus f\left(\sum_{i=1}^m x_i\right), 2m \sum_{i=1}^m f(x_i)\right) \\ & \leq \phi(x_1, x_2, \dots, x_m) \end{aligned} \quad (3)$$

for all $x_1, x_2, \dots, x_m \in X$. Then, we can find a unique set valued additive mapping $A : X \rightarrow C_{cb}(Y)$ defined by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{m^n} f(m^n x) \quad (4)$$

satisfying the following inequality

$$h(A(x), f(x)) \leq \frac{1}{m} \Phi(x, 0, \dots, 0) \quad (5)$$

for all $x \in X$.

Proof. Putting $x_1 = x, x_j = 0, j = 2, 3, \dots, m$ in Inequality (3), we get

$$h\left(\frac{1}{m}f(mx), f(x)\right) \leq \frac{1}{m}\theta(x) \quad (6)$$

where $\theta(x) = \phi(x, 0, \dots, 0)$ for all $x \in X$. Replacing x by $m^{n-1}x$ in Inequality (6), we obtain

$$h\left(\frac{1}{m^n}f(m^n x), \frac{1}{m^{n-1}}f(m^{n-1}x)\right) \leq \frac{1}{m^n}\theta(m^{n-1}x) \quad (7)$$

for all $x \in X, n \in \mathbb{N}$.

Now, for all $x \in X, n \in \mathbb{N}$

$$\begin{aligned} & h\left(\frac{1}{m^n}f(m^n x), f(x)\right) \\ & \leq h\left(\frac{1}{m^n}f(m^n x), \frac{1}{m^{n-1}}f(m^{n-1}x)\right) + h\left(\frac{1}{m^{n-1}}f(m^{n-1}x), \frac{1}{m^{n-2}}f(m^{n-2}x)\right) \\ & \quad + \dots + h\left(\frac{1}{m}f(mx), f(x)\right) \\ & \leq \sum_{i=1}^n \frac{1}{m^i}\theta(m^{i-1}x) \quad [\text{by (7)}]. \end{aligned} \quad (8)$$

Again, for all $x \in X, n, p \in \mathbb{N}$, we get

$$\begin{aligned} & h\left(\frac{1}{m^{n+p}}f(m^{n+p}x), \frac{1}{m^n}f(m^n x)\right) = \frac{1}{m^n}h\left(\frac{1}{m^p}f(m^p(m^n x)), f(m^n x)\right) \\ & \leq \frac{1}{m} \sum_{i=n}^{n+p-1} \frac{1}{m^i}\theta(m^i x) \quad [\text{by (8)}]. \end{aligned}$$

It implies that the sequence $\{\frac{1}{m^n}f(m^n x)\}$ is a Cauchy sequence in $(C_{cb}(Y), h)$ by the virtue of convergence of the series $\sum_{i=0}^{\infty} \frac{1}{m^i}\phi(m^i x_1, m^i x_2, \dots, m^i x_m)$ for all $x_1, x_2, \dots, x_m \in X$. Therefore, the completeness of $(C_{cb}(Y), h)$ assures that there exists a set valued function $A : X \rightarrow C_{cb}(Y)$ satisfying Equation (4).

Replacing x_i by $m^n x_i, i = 1, 2, \dots, m$ in Inequality (3), we have for all $x_1, x_2, \dots, x_m \in X, n \in \mathbb{N}$

$$\begin{aligned} & h\left(\sum_{i=1}^m \frac{1}{m^n}f\left(m^n\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right)\right) \oplus \frac{1}{m^n}f\left(m^n\left(\sum_{i=1}^m x_i\right)\right), \right. \\ & \left. 2m \sum_{i=1}^m \frac{1}{m^n}f(m^n x_i)\right) \leq \frac{1}{m^n}\phi(m^n x_1, m^n x_2, \dots, m^n x_m). \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we have for all $x_1, x_2, \dots, x_m \in X$

$$h\left(\sum_{i=1}^m A\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) \oplus A\left(\sum_{i=1}^m x_i\right), 2m \sum_{i=1}^m A(x_i)\right) = 0.$$

This shows that A satisfies Equation (1), that is, A is an additive set valued mapping. Taking limit as $n \rightarrow \infty$ in Inequality (8), we have the Inequality (5).

From the definition in Equation (4) of A , we get for all $x \in X, k \in \mathbb{N}$

$$A(x) = \frac{1}{m^k} A(m^k x). \quad (9)$$

For the purpose of proving the uniqueness of the set valued additive mapping A , we assume that $A' : X \rightarrow C_{cb}(Y)$ be another additive set valued mapping satisfying Equation (4) and Inequality (5). Then, Equation (9) is also satisfied by A' . Now, using Equation (9), we get for all $x \in X$

$$\begin{aligned} h(A(x), A'(x)) &\leq \frac{1}{m^k} \left(h(A(m^k x), f(m^k x)) + h(f(m^k x), A'(m^k x)) \right) \\ &\leq \frac{2}{m^{k+1}} \Phi(m^k x, 0, \dots, 0) \quad [\text{by (5)}] \\ &= \frac{2}{m} \sum_{i=k}^{\infty} \frac{1}{m^i} \theta(m^i x) \end{aligned}$$

which tends to 0 as $k \rightarrow \infty$ due to convergence of the series $\sum_{i=0}^{\infty} \frac{1}{m^i} \phi(m^i x_1, m^i x_2, \dots, m^i x_m)$ for all $x_1, x_2, \dots, x_m \in X$. That is, $A(x) = A'(x)$ for every $x \in X$. This shows that A is unique. $\square$

We note special cases of the above theorem in the following corollaries.

Corollary 1. Let X and Y be real normed spaces, of which Y is a real Banach space and $m \geq 2$ is a positive integer. Let $f : X \rightarrow C_{cb}(Y)$ be a mapping which satisfies $f(0) = \{0\}$ and the functional inequality

$$h\left(\sum_{i=1}^m f\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) \oplus f\left(\sum_{i=1}^m x_i\right), 2m \sum_{i=1}^m f(x_i)\right) \leq \epsilon \sum_{j=1}^m \|x_j\|^{r_j}$$

for all $x_1, x_2, \dots, x_m \in X$ and $\epsilon \geq 0, 0 < r_j < 1, j = 1, 2, \dots, m$. Then, we can find a unique additive mapping $A : X \rightarrow C_{cb}(Y)$ defined by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{m^n} f(m^n x)$$

satisfying the following inequality

$$h(f(x), A(x)) \leq \frac{\epsilon \|x\|^{r_1}}{m - m^{r_1}}$$

for all $x \in X$.

Proof. Define $\phi(x_1, x_2, \dots, x_m) = \epsilon \sum_{j=1}^m \|x_j\|^{r_j}$. Then, clearly, $\Phi(x_1, x_2, \dots, x_m) = \sum_{j=1}^m \frac{m\epsilon \|x_j\|^{r_j}}{m - m^{r_j}} < \infty$.

By the application of Theorem 1, the proof is completed. $\square$

Corollary 2. Let X be a real vector space, Y be a real Banach space and $m \geq 2$ be a positive integer. Let $f : X \rightarrow C_{cb}(Y)$ be a mapping which satisfies $f(0) = \{0\}$ and the functional inequality

$$h\left(\sum_{i=1}^m f\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) \oplus f\left(\sum_{i=1}^m x_i\right), 2m \sum_{i=1}^m f(x_i)\right) \leq \epsilon$$

for all $x_1, x_2, \dots, x_m \in X$ and $\epsilon \geq 0$. Then, we can find a unique additive mapping $A : X \rightarrow C_{cb}(Y)$ defined by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{m^n} f(m^n x)$$

satisfying the following inequality

$$h(f(x), A(x)) \leq \frac{\epsilon}{m-1}$$

for all $x \in X$.

Proof. Define $\phi(x_1, x_2, \dots, x_m) = \epsilon$. Then, clearly, $\Phi(x_1, x_2, \dots, x_m) = \frac{m\epsilon}{m-1} < \infty$. By the application of Theorem 1, the proof is completed. $\square$

Corollary 3. Let X and Y be real normed spaces of which Y is a real Banach space. Let $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be some mapping satisfying the following properties

- (i) $\psi(0) = 0$;
- (ii) $\psi(st) \leq \psi(s)\psi(t)$, $s, t \geq 0$;
- (iii) $\psi(t) < t$, $t > 1$; and
- (iv) $\lim_{t \rightarrow \infty} \frac{\psi(t)}{t} = 0$.

Let $m \geq 2$ be a positive integer and $f : X \rightarrow C_{cb}(Y)$ be a mapping which satisfies $f(0) = \{0\}$ and the functional inequality

$$h\left(\sum_{i=1}^m f\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) \oplus f\left(\sum_{i=1}^m x_i\right), 2m \sum_{i=1}^m f(x_i)\right) \leq \epsilon \sum_{j=1}^m \psi(\|x_j\|)$$

for all $x_1, x_2, \dots, x_m \in X$ and $\epsilon \geq 0$. Then, we can find a unique additive mapping $A : X \rightarrow C_{cb}(Y)$ defined by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{m^n} f(m^n x)$$

satisfying the following inequality

$$h(f(x), A(x)) \leq \frac{\epsilon \psi(\|x\|)}{m - \psi(m)}$$

for all $x \in X$.

Proof. Define $\phi(x_1, x_2, \dots, x_m) = \epsilon \sum_{j=1}^m \psi(\|x_j\|)$. Then, clearly, $\Phi(x_1, x_2, \dots, x_m) \leq \frac{m\epsilon}{m-\psi(m)} \sum_{j=1}^m \psi(\|x_j\|) < \infty$. By the application of Theorem 1, the proof is completed. $\square$

The following is a stability result with another control function.

Theorem 2. Let X be a real vector space, Y be a real Banach space and $m \geq 2$ be a positive integer. Let $\phi : X^m \rightarrow [0, \infty)$ be a function such that

$$\Phi(x_1, x_2, \dots, x_m) := \sum_{i=0}^{\infty} m^i \phi\left(\frac{x_1}{m^i}, \frac{x_2}{m^i}, \dots, \frac{x_m}{m^i}\right) < \infty \text{ for all } x_1, x_2, \dots, x_m \in X. \text{ Let } f : X \rightarrow C_{cb}(Y)$$

be a mapping which satisfies $f(0) = \{0\}$ and the Inequality (3) for all $x_1, x_2, \dots, x_m \in X$. Then, we can find a unique set valued additive mapping $A : X \rightarrow C_{cb}(Y)$ defined by

$$A(x) := \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right)$$

satisfying the following inequality

$$h(A(x), f(x)) \leq \Phi\left(\frac{x}{m}, 0, \dots, 0\right)$$

for all $x \in X$.

Proof. Theorem 2 is a restatement of Theorem 1 only by making a minor change in the control function ϕ and therefore the proof remains almost the same as that of Theorem 1. The proof is omitted. $\square$

We note a special case of the above theorem in the following corollary.

Corollary 4. Let X and Y be normed spaces of which Y is a real Banach space and $m \geq 2$ be a positive integer. Let $f : X \rightarrow C_{cb}(Y)$ be a mapping satisfying $f(0) = \{0\}$ and the functional inequality

$$h\left(\sum_{i=1}^m f\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) \oplus f\left(\sum_{i=1}^m x_i\right), 2m \sum_{i=1}^m f(x_i)\right) \leq \epsilon \sum_{j=1}^m \|x_j\|^{r_j}$$

for all $x_1, x_2, \dots, x_m \in X$ and $\epsilon \geq 0, r_j > 1, j = 1, 2, \dots, m$. Then, we can find a unique additive mapping $A : X \rightarrow C_{cb}(Y)$ defined by

$$A(x) := \lim_{n \rightarrow \infty} m^n f\left(\frac{x}{m^n}\right)$$

satisfying the following inequality

$$h(f(x), A(x)) \leq \frac{\epsilon \|x\|^{r_1}}{m^{r_1} - m}$$

for all $x \in X$.

Proof. Define $\phi(x_1, x_2, \dots, x_m) = \epsilon \sum_{j=1}^m \|x_j\|^{r_j}$. Then, clearly, $\Phi(x_1, x_2, \dots, x_m) = \sum_{j=1}^m \frac{\epsilon \|x_j\|^{r_j}}{1 - m^{1-r_j}}$. By the application of Theorem 2, the proof is completed. $\square$

4. Stability of Equation

The following is an H-U-R stability result for the cubic equation.

Theorem 3. Let X be assumed to be a real vector space and Y be a real Banach space. Let $n \geq 3$ be a positive integer. Let $\phi : X^n \rightarrow [0, \infty)$ be a function such that $\Phi(x_1, x_2, \dots, x_n) := \sum_{k=0}^{\infty} \frac{1}{2^{3k}} \phi(2^k x_1, 2^k x_2, \dots, 2^k x_n) < \infty$ for all $x_1, x_2, \dots, x_n \in X$. Suppose that $f : X \rightarrow C_{cb}(Y)$ is a mapping satisfying $f(0) = \{0\}$ and the functional inequality

$$\begin{aligned} & h\left(2f\left(\sum_{i=1}^{n-1} x_i + 2x_n\right) \oplus 2f\left(\sum_{i=1}^{n-1} x_i - 2x_n\right) \oplus 2 \sum_{i=1}^{n-1} f(2x_i) \oplus 7(n-1) \right. \\ & \quad \left. [f(x_1) \oplus f(-x_1)], 4f\left(\sum_{i=1}^{n-1} x_i\right) \oplus 8 \sum_{i=1}^{n-1} [f(x_i + x_n) \oplus f(x_i - x_n)]\right) \\ & \leq \phi(x_1, x_2, \dots, x_n) \end{aligned} \quad (10)$$

for all $x_1, x_2, \dots, x_n \in X$. Then, we can find a unique cubic mapping $c : X \rightarrow C_{cb}(Y)$ defined by

$$c(x) := \lim_{k \rightarrow \infty} \frac{1}{2^{3k}} f(2^k x) \quad (11)$$

satisfying the following inequality

$$h(f(x), c(x)) \leq \frac{1}{2^4(n-2)} \Phi(0, x, \dots, x, 0) \quad (12)$$

for all $x \in X$.

Proof. Putting $x_1 = 0, x_2 = x_3 = \dots = x_{n-1} = x, x_n = 0$ in Inequality (10), we get

$$h\left(\frac{1}{2^3}f(2x), f(x)\right) \leq \frac{1}{2^4(n-2)}\Theta(x) \quad (13)$$

where $\Theta(x) = \phi(0, x, \dots, x, 0)$ for all $x \in X$. Replacing x by $2^{k-1}x$ in Inequality (13), we obtain

$$h\left(\frac{1}{2^{3k}}f(2^k x), \frac{1}{2^{3(k-1)}}f(2^{k-1}x)\right) \leq \frac{1}{2^{3k+1}(n-2)}\Theta(2^{k-1}x) \quad (14)$$

for all $x \in X, k \in \mathbb{N}$.

Now, for all $x \in X, k \in \mathbb{N}$

$$\begin{aligned} & h\left(\frac{1}{2^{3k}}f(2^k x), f(x)\right) \\ & \leq h\left(\frac{1}{2^{3k}}f(2^k x), \frac{1}{2^{3(k-1)}}f(2^{k-1}x)\right) + h\left(\frac{1}{2^{3(k-1)}}f(2^{k-1}x), \frac{1}{2^{3(k-2)}}f(2^{k-2}x)\right) \\ & \quad + \dots + h\left(\frac{1}{2^3}f(2x), f(x)\right) \\ & \leq \frac{1}{2^4(n-2)} \sum_{i=0}^{k-1} \frac{1}{2^{3i}}\Theta(2^i x) \quad [\text{by (14)}]. \end{aligned} \quad (15)$$

Again, for all $x \in X, k, p \in \mathbb{N}$, we get

$$\begin{aligned} & h\left(\frac{1}{2^{3(k+p)}}f(2^{k+p}x), \frac{1}{2^{3k}}f(2^k x)\right) = \frac{1}{2^{3k}}h\left(\frac{1}{2^{3p}}f(2^p(2^k x)), f(2^k x)\right) \\ & \leq \frac{1}{2^4(n-2)} \sum_{i=k}^{k+p-1} \frac{1}{2^{3i}}\Theta(2^i x) \quad [\text{by (15)}]. \end{aligned}$$

This shows that the sequence $\{\frac{1}{2^{3k}}f(2^k x)\}$ is a Cauchy sequence in $(C_{cb}(Y), h)$ by the virtue of convergence of the series $\sum_{k=0}^{\infty} \frac{1}{2^{3k}}\phi(2^k x_1, 2^k x_2, \dots, 2^k x_n)$ for all $x_1, x_2, \dots, x_n \in X$. From the completeness of $(C_{cb}(Y), h)$, there exists a set valued function $c : X \rightarrow C_{cb}(Y)$ satisfying Equation (11).

Replacing x_i by $2^k x_i, i = 1, 2, \dots, n$ in Inequality (10), we have for all $x_1, x_2, \dots, x_n \in X, k \in \mathbb{N}$

$$\begin{aligned} & \frac{1}{2^{3k}}h\left(2f\left(2^k\left(\sum_{i=1}^{n-1}x_i + 2x_n\right)\right) \oplus 2f\left(2^k\left(\sum_{i=1}^{n-1}x_i - 2x_n\right)\right) \oplus \right. \\ & 2\sum_{i=1}^{n-1}f(2^k(2x_i)) \oplus 7(n-1)[f(2^k x_1) \oplus f(-2^k x_1)], 4f\left(2^k\sum_{i=1}^{n-1}x_i\right) \oplus \\ & \left. 8\sum_{i=1}^{n-1}[f(2^k(x_i + x_n)) \oplus f(2^k(x_i - x_n))]\right) \\ & \leq \frac{1}{2^{3k}}\phi(2^k x_1, 2^k x_2, \dots, 2^k x_n). \end{aligned}$$

Taking limit as $k \rightarrow \infty$, we get for all $x_1, x_2, \dots, x_n \in X$

$$h\left(2c\left(\sum_{i=1}^{n-1}x_i + 2x_n\right) \oplus 2c\left(\sum_{i=1}^{n-1}x_i - 2x_n\right) \oplus 2\sum_{i=1}^{n-1}c(2x_i) \oplus \right.$$

$$7(n-1)[c(x_1) \oplus c(-x_1)], 4c\left(\sum_{i=1}^{n-1} x_i\right) \oplus 8 \sum_{i=1}^{n-1} [c(x_i + x_n) \oplus c(x_i - x_n)] = 0.$$

This shows that c satisfies Equation (2), that is, c is a cubic set valued mapping. Taking limit as $k \rightarrow \infty$ in Inequality (15), we have the Inequality (12). From the definition in Equation (11) of c , we get for all $x \in X, k \in \mathbb{N}$

$$c(x) = \frac{1}{2^{3k}} c(2^k x). \quad (16)$$

For the purpose of proving the uniqueness of the set valued cubic mapping c , let $c' : X \rightarrow C_{cb}(Y)$ be another cubic set valued mapping satisfying Equation (11) and Inequality (12). Then, Equation (16) is also satisfied by c' . Now, using Equation (16), we get for all $x \in X$

$$\begin{aligned} h(c(x), c'(x)) &\leq \frac{1}{2^{3k}} \left(h\left(c\left(2^k x\right), f\left(2^k x\right)\right) + h\left(f\left(2^k x\right), c'\left(2^k x\right)\right) \right) \\ &\leq \frac{1}{2^{3k}} \frac{1}{2^3(n-2)} \Phi\left(0, 2^k x, \dots, 2^k x, 0\right) \quad [\text{by (12)}] \\ &= \frac{1}{2^3(n-2)} \sum_{i=k}^{\infty} \frac{1}{2^{3i}} \Theta(2^i x) \end{aligned}$$

which tends to 0 as $k \rightarrow \infty$ by the virtue of convergence of the series.

$\sum_{k=0}^{\infty} \frac{1}{2^{3k}} \phi\left(2^k x_1, 2^k x_2, \dots, 2^k x_n\right)$ for all $x_1, x_2, \dots, x_n \in X$. That is, $c(x) = c'(x)$ for all $x \in X$. This shows that c is unique. $\square$

We note special cases of the above theorem in the following corollaries.

Corollary 5. Let X and Y be real normed spaces, of which Y is a real Banach space and $n \geq 3$ is a positive integer. Let $f : X \rightarrow C_{cb}(Y)$ be a mapping which satisfies $f(0) = \{0\}$ and the functional inequality

$$\begin{aligned} &h\left(2f\left(\sum_{i=1}^{n-1} x_i + 2x_n\right) \oplus 2f\left(\sum_{i=1}^{n-1} x_i - 2x_n\right) \oplus 2 \sum_{i=1}^{n-1} f(2x_i) \oplus 7(n-1)\right. \\ &\quad \left.[f(x_1) \oplus f(-x_1)], 4f\left(\sum_{i=1}^{n-1} x_i\right) \oplus 8 \sum_{i=1}^{n-1} [f(x_i + x_n) \oplus f(x_i - x_n)]\right) \\ &\leq \epsilon (\|x_1\|^p + \|x_2\|^p + \dots + \|x_n\|^p) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$ and $\epsilon \geq 0, 0 < p < 3$. Then, we can find a unique cubic mapping $c : X \rightarrow C_{cb}(Y)$ defined by

$$c(x) := \lim_{k \rightarrow \infty} \frac{1}{2^{3k}} f(2^k x)$$

satisfying the following inequality

$$h(f(x), c(x)) \leq \frac{\epsilon \|x\|^p}{2(2^3 - 2^p)}$$

for all $x \in X$.

Proof. Define $\phi(x_1, x_2, \dots, x_n) = \epsilon (\|x_1\|^p + \|x_2\|^p + \dots + \|x_n\|^p)$. Then, clearly, $\Phi(x_1, x_2, \dots, x_n) = \frac{\epsilon}{1-2^{p-3}} \sum_{i=1}^n \|x_i\|^p < \infty$. By the application of Theorem 3, the proof is completed. $\square$

Corollary 6. Let X be a real vector space, Y be a real Banach space and $n \geq 3$ be a positive integer. Let $f : X \rightarrow C_{cb}(Y)$ be a mapping which satisfies $f(0) = \{0\}$ and the functional inequality

$$h \left(2f \left(\sum_{i=1}^{n-1} x_i + 2x_n \right) \oplus 2f \left(\sum_{i=1}^{n-1} x_i - 2x_n \right) \oplus 2 \sum_{i=1}^{n-1} f(2x_i) \oplus 7(n-1) \right. \\ \left. [f(x_1) \oplus f(-x_1)], 4f \left(\sum_{i=1}^{n-1} x_i \right) \oplus 8 \sum_{i=1}^{n-1} [f(x_i + x_n) \oplus f(x_i - x_n)] \right) \leq \epsilon$$

for all $x_1, x_2, \dots, x_n \in X$ and $\epsilon \geq 0$. Then, we can find a unique cubic mapping $c : X \rightarrow C_{cb}(Y)$ defined by

$$c(x) := \lim_{k \rightarrow \infty} \frac{1}{2^{3k}} f(2^k x)$$

satisfying the following inequality

$$h(f(x), c(x)) \leq \frac{\epsilon}{14(n-2)}$$

for all $x \in X$.

Proof. Define $\phi(x_1, x_2, \dots, x_n) = \epsilon$. Then, clearly, $\Phi(x_1, x_2, \dots, x_n) = \frac{8\epsilon}{7} < \infty$. By the application of Theorem 3, the proof is completed. $\square$

The following is a stability result with another control function.

Theorem 4. Let X be a real vector space, Y be a real Banach space and $n \geq 3$ be a positive integer. Let $\phi : X^n \rightarrow [0, \infty)$ be such that

$\Phi(x_1, x_2, \dots, x_n) := \sum_{k=0}^{\infty} 2^{3k} \phi \left(\frac{x_1}{2^k}, \frac{x_2}{2^k}, \dots, \frac{x_n}{2^k} \right) < \infty$ for all $x_1, x_2, \dots, x_n \in X$. Suppose that $f : X \rightarrow C_{cb}(Y)$ is a mapping which satisfies $f(0) = \{0\}$ and the functional inequality

$$h \left(2f \left(\sum_{i=1}^{n-1} x_i + 2x_n \right) \oplus 2f \left(\sum_{i=1}^{n-1} x_i - 2x_n \right) \oplus 2 \sum_{i=1}^{n-1} f(2x_i) \oplus 7(n-1) \right. \\ \left. [f(x_1) \oplus f(-x_1)], 4f \left(\sum_{i=1}^{n-1} x_i \right) \oplus 8 \sum_{i=1}^{n-1} [f(x_i + x_n) \oplus f(x_i - x_n)] \right) \\ \leq \phi(x_1, x_2, \dots, x_n)$$

for all $x_1, x_2, \dots, x_n \in X$. Then, we can find a unique cubic mapping $c : X \rightarrow C_{cb}(Y)$ defined by

$$c(x) := \lim_{k \rightarrow \infty} 2^{3k} f \left(\frac{x}{2^k} \right)$$

satisfying the following inequality

$$h(f(x), c(x)) \leq \frac{1}{2(n-2)} \Phi \left(0, \frac{x}{2}, \dots, \frac{x}{2}, 0 \right)$$

for all $x \in X$.

Proof. Theorem 4 is a restatement of Theorem 3 only by making a minor change in the control function ϕ and therefore the proof remains almost the same as that of Theorem 3. The proof is omitted. $\square$

We note a special case of the above theorem in the following corollary.

Corollary 7. Let X and Y be real normed spaces, of which Y is a real Banach space and $n \geq 3$ be a positive integer. Let $f : X \rightarrow C_{cb}(Y)$ be a mapping which satisfies $f(0) = \{0\}$ and the functional inequality

$$\begin{aligned} & h \left(2f \left(\sum_{i=1}^{n-1} x_i + 2x_n \right) \oplus 2f \left(\sum_{i=1}^{n-1} x_i - 2x_n \right) \oplus 2 \sum_{i=1}^{n-1} f(2x_i) \oplus 7(n-1) \right. \\ & \left. [f(x_1) \oplus f(-x_1)], 4f \left(\sum_{i=1}^{n-1} x_i \right) \oplus 8 \sum_{i=1}^{n-1} [f(x_i + x_n) \oplus f(x_i - x_n)] \right) \\ & \leq \epsilon (\|x_1\|^p + \|x_2\|^p + \dots + \|x_n\|^p) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$ and $\epsilon \geq 0, p > 3$. Then, we can find a unique cubic mapping $c : X \rightarrow C_{cb}(Y)$ defined by

$$c(x) := \lim_{k \rightarrow \infty} 2^{3k} f \left(\frac{x}{2^k} \right)$$

satisfying the following inequality

$$h(f(x), c(x)) \leq \frac{\epsilon \|x\|^p}{2(2^p - 2^3)}$$

for all $x \in X$.

Proof. Define $\phi(x_1, x_2, \dots, x_n) = \epsilon (\|x_1\|^p + \|x_2\|^p + \dots + \|x_n\|^p)$. Then, clearly, $\Phi(x_1, x_2, \dots, x_n) = \frac{\epsilon}{1-2^{3-p}} \sum_{i=1}^n \|x_i\|^p < \infty$. By the application of Theorem 4, the proof is completed. $\square$

5. Conclusions

In this paper, our results are obtained in the general structure of Banach spaces. It is possible that similar results are valid for set valued functional equations of other kinds as well. In addition, in more specialized structures, such as Hilbert spaces, etc., the results may be true under less restricted conditions. It is also interesting to see whether the functional equations exhibit stability in the structure of topological vector spaces. These problems may be investigated in future works.

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