Article

Mysterious Circle Numbers. Does $\pi_{p,q}$ Approach π_p When q Is Tending to p ?

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Abstract: This paper aims to introduce a mathematical-philosophical type of question from the fascinating world of generalized circle numbers to the widest possible readership. We start with recalling well-known (in part from school) properties of the polygonal approximation of the common circle when approximating the famous circle number π by convergent sequences of upper and lower bounds being based upon the lengths of polygons. Next, we shortly refer to some results from the literature where suitably defined generalized circle numbers of l_p - and $l_{p,q}$ -circles, π_p and $\pi_{p,q}$, respectively, are considered and turn afterwards over to the approximation of an l_p -circle by a family of $l_{p,q}$ -circles with q converging to p , $q \rightarrow p$. Then we ask whether or not there holds the continuity property $\pi_{p,q} \rightarrow \pi_p$ as $q \rightarrow p$. The answer to this question leads us to the answer of the question stated in the paper's title. Presenting here for illustration true paintings instead of strong technical or mathematical drawings intends both to stimulate opening heart and senses of the reader for recognizing generalized circles in his real life and to suggest the philosophical challenge of the consequences coming out from the demonstrated non-continuity property.

Keywords: $l_{p,q}$ -circle approximation of l_p -circle; generalized circumference; l_p -circle number; $l_{p,q}$ -circle number; non-continuity property

1. Introduction

The development of sciences is closely connected with the identification of new structures in reality and the construction of corresponding new mathematical models. The type of mathematical models dealt with here is that of generalized circles changing shape along with changing a generalized radius variable. Generally spoken, the theory of circle numbers π_q and $\pi_{p,q}$ belongs to the research area of so-called non-Euclidean geometry. More specifically it may be identified being part of Minkowski geometry which should, however, carefully be distinguished from Minkowskian geometry playing itself a basic role in physics of the four-dimensional space-time model.

The circumference-to-diameter ratio of circles in a two-dimensional norm space was studied for the particular class of l_p -norms in [1] and for the general case in [2]. The closely related ratio of the length of the unit circle to the area of the circle disc on Minkowski planes was studied in [3]. For a basic introduction to Minkowski geometry we refer to [4]. More details on various approaches to generalizing the famous circle number π , as well as extensions to dimension three, can be found in the recent paper [5] and the references given there.

While there is a certain preference in systematic Minkowski geometry to measure geometric objects on the basis of the space's own norm, the present approach is based upon using another norm being called within a more general framework a dual norm.

It is well known that, for all n , the perimeter of a unit circle can be estimated above and below by the lengths of the smallest outer and biggest inner regular polygons having n vertices, respectively. Both sequences are converging toward two times the famous circle number π as n is approaching infinity.

Similarly, for $n \geq 3$, let a regular polygon P_n be defined by its vertices $I_{n,i}^T = (\cos(2\pi(i-1)/n), \sin(2\pi(i-1)/n))$, $i = 1, \dots, n$ and denote the circumscribed polygonal disc K_n . One may consider P_n to be a generalized circle. The functional $h_{K_n} : \mathbb{R}^2 \rightarrow [0, \infty)$ defined by $h_{K_n}(x) = \inf\{\lambda > 0 : x \in \lambda K_n\}$ is called the Minkowski functional of K_n . Because $h_{K_n}(x) = r, \forall x \in rP_n$ we call r the polygonal radius of $P_n(r) = rP_n$. With a suitable definition of the K_n -generalized arc-length of $P_n(r)$, $\mathcal{U}_n(P_n(r))$, it is proved in [6] that the ratios $\mathcal{U}_n(P_n(r))/(2r)$ and $A_n^*(r)/r^2$ do not depend on $r > 0$ and are equal where $A_n^*(r)$ is the area content of $K_n(r) = rK_n$. The number

$$\frac{\mathcal{U}_n(P_n(1))}{2} = A_n^*(1) = \pi_n^* = \pi \frac{\sin(2\pi/n)}{2\pi/n}$$

is called the polygonal circle number. Clearly, $\pi_n^* \rightarrow \pi$ as $n \rightarrow \infty$.

It may sound surprising that circle numbers are around us, but people know from their maths at least that circles $C(r)$ with radius r and diameter d have circumference $U = \pi \cdot d$ and area content $A = \pi \cdot r^2$. What most people are not aware of is that beyond Euclidean and polygonal circles there are really many different types of circles each of them having an own circle number. Let us call, e.g., the set of points (x, y) from the plane which satisfy the equation

$$|x|^p + |y|^p = r^p$$

the p -circle with p -radius r , and denote it $C_p(r)$. For $p = 1/2$ see Figure 1, for other values of p see Figure 2. Commonly, the function $f(x, y) = (|x|^p + |y|^p)^{1/p}$ is called the l_p -norm if $p \geq 1$. The reader who remembers the definition of what is called a circle at school might ask why we can call $P_n(r)$ and $C_p(r)$ a circle, here. The answer to this question is that in mathematics it can reasonably be explained by introducing the Minkowski functional in which sense all points from $P_n(r)$ or $C_p(r)$ have the same (generalized) distance from the origin.

Clearly, $C_2(r)$ is the common and well-known Euclidean circle of radius r , and $C_p(1)$ is called the unit l_p -circle. The p -circumference $U_p(r)$ of $C_p(r)$, with $U_2(r)$ being the common notion of circumference, and the area content $A_p(r)$ of the l_p -circle disc $K_p(r)$ circumscribed by $C_p(r)$ can be proved to have for all $r > 0$ the properties

$$\frac{A_p(r)}{r^2} = \pi_p = \frac{U_p(r)}{2r}, \quad (1)$$

π_p is thus called the l_p -circle number.

To precisely explain the way of defining the p -circumference of $C_p(r)$ we would need to slightly modify that limiting process with the help of which our teachers at school approximated a circle by a sequence of polygons having more and more corners located on the circle or only a little bit outside or inside of it, whereby we would have to measure the length of a each polygonal segment in a suitable way. Differently from what is done in systematic Minkowski geometry where the space's own norm (more generally: Minkowski functional) is used we are using a dual norm (Minkowski functional) for measuring the arc-length of a norm (Minkowski) circle. But, instead of going into those details here, we refer to [7] and the extensive work of many authors mentioned there and in its references.

Fenster

Figure 1. Unit l_p -circle with $p = 1/2$ (*Fenster*, Oil and pencil on paper, 42×30).

Infinity Pools

Figure 2. Unit circles with $p = 1, 2$ and a big positive number from left to right (*Infinity pools*, Oil on paper, 30×42).

Now, notice that ellipses can also be considered from an abstract point of view as (generalized) circles, thus there exist also ellipses numbers. To explain this in more detail, let us imagine we are given the ellipse $E_{(a,b)}(r)$ of (a, b) -ellipse radius r being the set of points in the plane satisfying the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2,$$

see Figure 5 where $(a, b) = (2, 1)$ and $(1, 2)$.

Completing a similar limiting process as in case of l_p -circles, and denoting the (a, b) -circumference of $E_{(a,b)}(r)$ by $U_{(a,b)}(r)$ and the area content of the (a, b) -ellipse disc $K_{(a,b)}(r)$ circumscribed by $E_{(a,b)}(r)$ by $A_{(a,b)}(r)$, we see that similarly to Equation (1) the following equations hold for all $r > 0$

$$\frac{A_{(a,b)}(r)}{r^2} = \pi_{(a,b)} = \frac{U_{(a,b)}(r)}{2r}. \quad (2)$$

The circle number $\pi_{(a,b)}$ is also called (a, b) -ellipse number. Note that an (a, b) -ellipse of radius r becomes a common l_2 -circle of radius ar if $a = b$.

Without going into the technical details, we recognized so far that different methods of measuring the length of a curve (the circumference of a generalized circle) may result in showing us that different mathematical structures reflect similar properties like the well-known circle number $\pi = \pi_2 = 3.14159 \dots$ does. Let us further remark here that the p -circle and (a, b) -ellipse discs $K_p(r)$ and $K_{(a,b)}(r)$, respectively, have the following property of positive homogeneity in common:

$$K_p(r) = r \cdot K_p(1), \quad K_{(a,b)}(r) = r \cdot K_{(a,b)}(1).$$

Here, $r \cdot K$ is the set of all points (rx, ry) from the plane where (x, y) belongs to K .

We present here for illustration true paintings instead of strong technical or mathematical drawings intending both to stimulate opening heart and senses of the reader for recognizing generalized circles in his real life and to suggest the philosophical challenge of the consequences coming out from the demonstrated non-continuity property. For paintings of Vincent Wenzel see [8,9].

2. Materials and Methods

If things of real life or from sciences are changing then often they are doing this continuously in time or in dependence of another parameter, but by far not always. Sometimes there is a qualitative change of property in the limit. The basic object of investigation in this section is such a so-called discontinuity problem.

There are many types of (generalized) circles allowing to define circle numbers for although they do not possess the homogeneity property mentioned in the last section. In high risk theory, where the work in [10] got much of its motivation from, one may be confronted with the (p, q) -circle $C^{(p,q)}(r)$ of (p, q) -radius parameter r consisting of all points in the plane satisfying

$$\frac{|x|^p}{p} + \frac{|y|^q}{q} = r.$$

In Figures 3 and 4, the (p, q) -circle discs $K^{(p,q)}(r)$ circumscribed by the (p, q) -circles $C^{(p,q)}(r)$ are convex or radially concave, respectively. Varying shape of $C^{(p,q)}(r)$ for varying values of r is caused by absence of homogeneity in these figures.

We note that a (p, q) -circle becomes a p -circle if $p = q$. However, as already mentioned, the (p, q) -circle discs do not have the property of positive homogeneity. Instead, one can prove that $K^{(p,q)}(r)$ consists of all points $(r^{1/p}x, r^{1/q}y)$ from the plane such that (x, y) belongs to $K^{(p,q)}(1)$.

Figure 3. (p, q) -circles for $(p, q) = (2, 0.6)$ and several values of r (*Humboldt Hain*, Oil and pencil on prepared paper, 42×30).

Figure 4. (p, q) -circles for $(p, q) = (0.2, 3/4)$ and several values of r (*Berliner Puppen-Zimmer*, Oil on paper, 30×42).

It may be astonishing that there exist circle numbers of such sets $K^{(p,q)}(r)$, too. If one defines again in a suitable (non-Euclidean) way how to measure length of $C^{(p,q)}(r)$ one can prove that

$$\frac{A^{(p,q)}(r)}{r^{\frac{1}{p} + \frac{1}{q}}} = \pi_{p,q} = \frac{U^{(p,q)}(r)}{(\frac{1}{p} + \frac{1}{q})r^{\frac{1}{p} + \frac{1}{q} - 1}} \quad (3)$$

holds for certain well defined positive real numbers $\pi_{p,q}$ with $p \neq q$ and all $r > 0$. Here, $U^{(p,q)}(r)$ and $A^{(p,q)}(r)$ denote (p,q) -circumference and area content of $C^{(p,q)}(r)$ and $K^{(p,q)}(r)$, respectively.

3. Results

The answer to the question asked in the title of this note is generally ‘no’, meaning, from a more philosophical point of view, that ‘accumulating quantity’ in the limiting process where $|p - q|$ is approaching zero results in a ‘jump of quality’. But, there is an exceptional case where the answer to the question in the title is ‘yes’, namely

$$\pi^{(1,1)} = \pi_1.$$

The proof follows by combining earlier results as in [7] with those in [10]. To be specific, we note that one can derive from Equation (2) in [7] that

$$\pi_p = \frac{1}{\Gamma(\frac{2}{p} + 1)} \left(\int_{-\infty}^{\infty} e^{-|s|^p} ds \right)^2.$$

Similarly, Formula (11) in [10] can be reformulated as

$$\pi_{p,q} = \frac{p^{\frac{1}{p}} q^{\frac{1}{q}}}{\Gamma(\frac{1}{p} + \frac{1}{q} + 1)} \int_{-\infty}^{\infty} e^{-|s|^p} ds \int_{-\infty}^{\infty} e^{-|s|^q} ds.$$

Thus, $\pi_{p,q} \rightarrow \pi_p$ as $q \rightarrow p$ if and only if $p = 1$.

Having this result in mind one may ask how useful it is for the applications.

Application To give a very first answer to the question just asked let $g : [0, \infty) \rightarrow [0, \infty)$ be a so-called probability density generating function satisfying the assumption

$$0 < I(g; p, q) = \int_0^{\infty} r^{\frac{1}{p} + \frac{1}{q} - 1} g(r) dr < \infty.$$

If

$$\varphi_{g;p,q}(x, y) = C(g; p, q) g\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right), \quad (x, y)^T \in \mathbb{R}^2$$

denotes the corresponding (p, q) -spherical probability density then it is known that the constant $C(g; p, q)$ allows the representation

$$C(g; p, q) = \frac{1}{2I(g; p, q)\pi_{p,q}}.$$

To be more specific, let

$$g_{N,\kappa}(r) = \begin{cases} r^{N-1} e^{-\kappa r} & \text{if } r > 0 \\ g(r) = 0 & \text{elsewhere} \end{cases}$$

denote a Gamma type density generating function. Then

$$C(g_{N,\kappa}; p, q) = \frac{\kappa^{N-1+\frac{1}{p}+\frac{1}{q}}}{\Gamma(N-1+\frac{1}{p}+\frac{1}{q}) 4B(\frac{1}{p}, \frac{1}{q}) p^{\frac{1}{p}-1} q^{\frac{1}{q}-1}}$$

and the resulting density of a corresponding two-dimensional random vector is

$$\varphi_{g_{N,\kappa;p,q}}(x,y) = C(g_{N,\kappa;p,q}) \left(\frac{|x|^p}{p} + \frac{|y|^q}{q} \right)^{N-1} e^{-\kappa \left(\frac{|x|^p}{p} + \frac{|y|^q}{q} \right)}, (x,y)^T \in \mathbb{R}^2.$$

In statistical data analysis one is interested in estimating the complete probabilistic model or at least some of its parameters. The accuracy of such estimates depends strongly on the underlying sample size. If the latter appears to be sufficient for detecting parameter deviations of a certain stochastic order of magnitude ε , say, but $|p - q|$ is much smaller than ε then it becomes impossible to statistically distinguish between the models where $p \neq q$ or $p = q$, respectively, and the constant $\pi_{p,q}$ may be quite different from π_p . Suitable efforts are needed to be made to overcome this statistical problem. Clearly, other statistical decisions are affected in a similar way.

The reader is invited to jointly with the grouch in Figure 5 have his own philosophical reflections on the mathematical circumstances presented in this section.

Grantler mit Blume

Figure 5. Philosophical reflection of a mathematical phenomenon (*Grantler mit Blume*, Charcoal on paper, 42 × 30).

4. Discussion

Generalized circles may be detected everywhere, no matter we do recognize them or not. Circles of different type are present in science and technique. When dealing with circles of unusual shape we might try to use advanced approximation techniques or we can just deal with them as they are. In the latter case we are challenged to generalize the notions of radius and arc-length what might be really hard for one or another reader because many teaching systems do not prepare for such intervention in an established field like geometry. All points of a norm circle are defined here to have same distance from a center, just in opposite to what we learned at school, and arc-length is measured then using

a circle with respect to another norm, for example the dual one as in the present approach. It may even happen that circles change their shape if the radius changes its magnitude along with the change of a certain parameter. In this case the new question arises which is formulated in the title of this paper. As we have seen, this type of dynamics is occasionally accompanied by jumps in the behavior of our objects of interest: the circle numbers $\pi_{p,q}$ are not converging to the circle number π_p if q is approaching p , in general. In practice, two values of p and q may differ not very much from each other. If one is interested, e.g., in knowing the parameters of the density $\varphi_{g;p,q}$ this circumstance may lead to consequences for developing a specific statistical method which, however, is not in the scope of the present paper.

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