Article

Duality Results in Quasiinvex Variational Control Problems with Curvilinear Integral Functionals

Elena-Corina Cipu

Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania;
corina.cipu@upb.ro

Received: 1 August 2019; Accepted: 30 August 2019; Published: 2 September 2019

Abstract: In this paper, we formulate and prove weak, strong and converse duality results in variational control problems involving (ρ, b) -quasiinvex path-independent curvilinear integral cost functionals.

Keywords: duality; multiobjective fractional variational control problem; (ρ, b) -quasiinvexity; path-independent curvilinear integral functional

MSC: 90C46; 65K10; 90C29; 26B25; 49K20

1. Introduction

It is well known that convexity theory is an important mathematical tool for studying a wide class of unrelated problems in a unified and general framework. Based on the works of Hanson [1], Craven and Glover [2], Mond and Smart [3], Mond and Hanson [4] and Preda [5], many duality and sufficiency results associated with variational control problems have been formulated and investigated. Over time, the multiobjective variational programming problem with mixed constraints has also been studied by many researchers. For other contributions and numerous approaches on different aspects of vector variational control problems (such as: conditions of optimality, models of duality, fields of applicability), the reader is directed to Zhian and Qingkai [6], Zalmai [7], Gulati et al. [8], Hachimi and Aghezzaf [9], Zhang et al. [10], Soleimani and Tammer [11], Ansari et al. [12] and Tung [13].

In this paper, motivated and inspired by the aforementioned works, under (ρ, b) -quasi-invexity assumptions, a duality model of Mond–Weir type is studied for a new multiobjective fractional variational control problem governed by path-independent integral functionals (initiated by Mititelu and Treanță [14]). Taking into account the necessary efficiency conditions formulated in Mititelu and Treanță [14], in accordance with Treanță [15–20] and following Treanță and Mititelu [21], we shall formulate and prove weak, strong and converse duality results for the considered variational control problem. Due to the physical significance (mechanical work) of the functionals used, the present work also has a huge potential regarding the applicability of the obtained results.

The present paper is structured as follows. Section 2 includes some notations, working hypotheses and problem description. Section 3 introduces the main results of this paper. More precisely, three duality results are established for the considered variational control problem. Finally, Section 4 concludes the paper.

2. Preliminaries

In the following, we consider the compact domain Θ in the Euclidean real space $\mathbb{R}^m$. Denote by $t = (t^\ell)$, $\ell \in \overline{1, m}$, $x = (x^j)$, $j \in \overline{1, n}$ and $v = (v^i)$, $i \in \overline{1, k}$, the points in Θ , $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively. Also, consider $\Theta \supset \Gamma : t = t(\tau)$, $\tau \in [a, b]$, a piecewise smooth curve joining the different points $t_1 = (t_1^1, \dots, t_1^m)$, $t_2 = (t_2^1, \dots, t_2^m)$ in Θ . Denote by $\mathcal{X}$ the space of piecewise smooth *state functions*

$x : \Theta \rightarrow \mathbb{R}^n$ and by $\mathcal{U}$ the space of piecewise continuous control functions $v : \Theta \rightarrow \mathbb{R}^k$, endowed with the norm $\|\cdot\|_\infty$. Also, for any two vectors $\psi, \omega \in \mathbb{R}^p$, the following convention for equalities and inequalities is used

$$\psi = \omega \Leftrightarrow \psi_r = \omega_r, \quad \psi \leq \omega \Leftrightarrow \psi_r \leq \omega_r, \quad \psi < \omega \Leftrightarrow \psi_r < \omega_r,$$

$$\psi \preceq \omega \Leftrightarrow \psi \leq \omega, \quad \psi \neq \omega, \quad r \in \overline{1, p}$$

and, for $\mathcal{P} := \Theta \times \mathbb{R}^n \times \mathbb{R}^k$, the functions

$$f = (f^r)_{r \in \overline{1, p}} : \mathcal{P} \rightarrow \mathbb{R}^p, \quad g = (g^r)_{r \in \overline{1, p}} : \mathcal{P} \rightarrow \mathbb{R}^p$$

are considered closed Lagrange one-form densities, leading to the following path-independent curvilinear integral functionals

$$F^r(x, v) = \int_{\Gamma} f_{\ell}^r(t, x(t), v(t)) dt^{\ell}, \quad G^r(x, v) = \int_{\Gamma} g_{\ell}^r(t, x(t), v(t)) dt^{\ell}, \quad r \in \overline{1, p}.$$

Also, assume that the Lagrange matrix densities $\mathbf{X} = (X_{\ell}^j)_{j \in \overline{1, n}, \ell \in \overline{1, m}} : \mathcal{P} \rightarrow \mathbb{R}^{nm}$ define the following first-order partial differential equations (PDEs)

$$X_{\ell}^j(t, x(t), v(t)) = x_{\ell}^j(t) = \frac{\partial x^j}{\partial t^{\ell}}(t),$$

satisfying the closeness conditions $D_{\ell'} X_{\ell}^j = D_{\ell} X_{\ell'}^j$, $\ell \neq \ell'$, where D_{ℓ} is the total derivative operator, and

$$Y = (Y_s)_{s \in \overline{1, q}} : \mathcal{P} \rightarrow \mathbb{R}^q$$

are continuously differentiable functions.

The problem to be studied in the present paper is the following multiobjective fractional variational control problem

$$(FP) \quad \min_{(x, v)} \left\{ K(x, v) = \left(\frac{F^1(x, v)}{G^1(x, v)}, \dots, \frac{F^p(x, v)}{G^p(x, v)} \right) \right\}$$

subject to

$$\frac{\partial x^j}{\partial t^{\ell}}(t) - X_{\ell}^j(t, x(t), v(t)) = 0, \quad j \in \overline{1, n}, \ell \in \overline{1, m}, t \in \Theta, \quad (1)$$

$$Y(t, x(t), v(t)) \leq 0, \quad t \in \Theta, \quad (2)$$

$$x(t_1) = x_1 = \text{given}, \quad x(t_2) = x_2 = \text{given}. \quad (3)$$

The set of all feasible solutions (domain) in (FP) is defined by

$$D := \{(x, v) \mid x \in \mathcal{X}, v \in \mathcal{U} \text{ satisfying (1), (2), (3)}\}.$$

Further, in order to introduce the main results of this paper, in accordance with Treanță [20], Mititelu and Treanță [14], Treanță [15–17] and following Treanță and Mititelu [21], we formulate some auxiliary mathematical tools.

Definition 1 (Treanță and Mititelu [21]). A feasible solution $(x^0, v^0) \in D$ in (FP) is called efficient solution if there is no other $(x, v) \in D$ such that $K(x, v) \preceq K(x^0, v^0)$.

Consider $\rho \in \mathbb{R}^p$ and the following vector curvilinear integral functional

$$F : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^p, \quad F(x, v) = \int_{\Gamma} f_{\ell}(t, x(t), x_v(t), v(t)) dt^{\ell},$$

where $f_{\ell} = (f_{\ell}^r)$, $r \in \overline{1, p}$, $\ell \in \overline{1, m}$. Also, we consider the following functional $b : (\mathcal{X} \times \mathcal{U})^2 \rightarrow [0, \infty)$ and the distance function $d((x, v), (x^0, v^0))$ on $\mathcal{X} \times \mathcal{U}$.

Definition 2 (Treanță and Mititelu [21]). (i) If there exist

$$\eta : \Theta \times (\mathbb{R}^n \times \mathbb{R}^k)^2 \rightarrow \mathbb{R}^n, \quad \eta = \eta(t, x(t), v(t), x^0(t), v^0(t))$$

of C^1 -class with $\eta(t, x^0(t), v^0(t), x^0(t), v^0(t)) = 0$, $\forall t \in \Theta$, $\eta(t_1) = \eta(t_2) = 0$, and

$$\xi : \Theta \times (\mathbb{R}^n \times \mathbb{R}^k)^2 \rightarrow \mathbb{R}^k, \quad \xi = \xi(t, x(t), v(t), x^0(t), v^0(t))$$

of C^0 -class with $\xi(t, x^0(t), v^0(t), x^0(t), v^0(t)) = 0$, $\forall t \in \Theta$, $\xi(t_1) = \xi(t_2) = 0$, such that for any $(x, v) \in \mathcal{X} \times \mathcal{U}$:

$$\begin{aligned} F(x, v) \leq F(x^0, v^0) \implies \\ b(x, v, x^0, v^0) \int_{\Gamma} \left[\frac{\partial f_{\ell}}{\partial x} (t, x^0(t), x_v^0(t), v^0(t)) \eta + \frac{\partial f_{\ell}}{\partial x_v} (t, x^0(t), x_v^0(t), v^0(t)) D_v \eta \right] dt^{\ell} \\ + b(x, v, x^0, v^0) \int_{\Gamma} \frac{\partial f_{\ell}}{\partial v} (t, x^0(t), x_v^0(t), v^0(t)) \xi dt^{\ell} \\ \leq -\rho b(x, v, x^0, v^0) d^2((x, v), (x^0, v^0)), \end{aligned}$$

then F is called (ρ, b) -quasiinvex at (x^0, v^0) with respect to η , ξ and d ;

(ii) If, in the same hypotheses as above, with $(x, v) \neq (x^0, v^0)$, we have

$$\begin{aligned} F(x, v) \leq F(x^0, v^0) \implies \\ b(x, v, x^0, v^0) \int_{\Gamma} \left[\frac{\partial f_{\ell}}{\partial x} (t, x^0(t), x_v^0(t), v^0(t)) \eta + \frac{\partial f_{\ell}}{\partial x_v} (t, x^0(t), x_v^0(t), v^0(t)) D_v \eta \right] dt^{\ell} \\ + b(x, v, x^0, v^0) \int_{\Gamma} \frac{\partial f_{\ell}}{\partial v} (t, x^0(t), x_v^0(t), v^0(t)) \xi dt^{\ell} \\ < -\rho b(x, v, x^0, v^0) d^2((x, v), (x^0, v^0)), \end{aligned}$$

then F is called strictly (ρ, b) -quasiinvex at (x^0, v^0) with respect to η , ξ and d .

According to Mititelu and Treanță [14] and Treanță [20], if $(x^0, v^0) \in D$ is an efficient solution of the problem (FP), there exist $\theta = (\theta^r)$, $\mu(t)$ and $\lambda(t)$, with $\mu(t) = (\mu^s(t))$, $\lambda(t) = (\lambda_j^{\ell}(t))$ piecewise smooth functions, fulfilling

$$\begin{aligned} \theta^r \left[G^r(x^0, v^0) \frac{\partial f_{\ell}^r}{\partial x^j} (t, x^0(t), v^0(t)) - F^r(x^0, v^0) \frac{\partial g_{\ell}^r}{\partial x^j} (t, x^0(t), v^0(t)) \right] \\ + \lambda_j^{\ell}(t) \frac{\partial X_{\ell}^j}{\partial x^j} (t, x^0(t), v^0(t)) + \mu^s(t) \frac{\partial Y_s}{\partial x^j} (t, x^0(t), v^0(t)) + \frac{\partial \lambda_j^{\ell}}{\partial t^{\ell}}(t) = 0, \quad j = \overline{1, n}, \ell = \overline{1, m}, \end{aligned} \quad (4)$$

$$\theta^r \left[G^r(x^0, v^0) \frac{\partial f_\ell^r}{\partial v^i} (t, x^0(t), v^0(t)) - F^r(x^0, v^0) \frac{\partial g_\ell^r}{\partial v^i} (t, x^0(t), v^0(t)) \right] \quad (5)$$

$$+ \lambda_j^\ell(t) \frac{\partial X_\ell^j}{\partial v^i} (t, x^0(t), v^0(t)) + \mu^s(t) \frac{\partial Y_s}{\partial v^i} (t, x^0(t), v^0(t)) = 0, \quad i = \overline{1, k}, \ell = \overline{1, m},$$

$$\mu^s(t) Y_s(t, x^0(t), v^0(t)) = 0 \quad (\text{no summation}), \quad (\theta, \mu(t)) \succeq 0, \quad (6)$$

for all $t \in \Theta$, except at discontinuities.

Definition 3 (Treanță and Mititelu [21]). The feasible solution $(x^0, v^0) \in D$ is a normal efficient solution for (FP) if the necessary efficiency conditions formulated in (4)–(6) hold for $\theta \succeq 0$ and $e^t \theta = 1$, $e^t = (1, \dots, 1) \in \mathbb{R}^p$.

3. Main Results

In this section, on the line of Treanță and Mititelu [21], let $\{Q_1, Q_2, \dots, Q_\zeta\}$ be a partition of the set $Q = \{1, 2, \dots, q\}$ (with $\zeta < q$). For $(z, u) \in \mathcal{X} \times \mathcal{U}$, we associate to (FP) the following multiobjective fractional variational control problem

$$(DFP) \quad \max_{(z, u)} \left\{ K(z, u) = \left(\frac{\int_\Gamma f_\ell^1(t, z(t), u(t)) dt^\ell}{\int_\Gamma g_\ell^1(t, z(t), u(t)) dt^\ell}, \dots, \frac{\int_\Gamma f_\ell^p(t, z(t), u(t)) dt^\ell}{\int_\Gamma g_\ell^p(t, z(t), u(t)) dt^\ell} \right) \right\}$$

subject to

$$\theta^r \left[G^r(z, u) \frac{\partial f_\ell^r}{\partial z^j} (t, z(t), u(t)) - F^r(z, u) \frac{\partial g_\ell^r}{\partial z^j} (t, z(t), u(t)) \right] \quad (7)$$

$$+ \lambda_j^\ell(t) \frac{\partial X_\ell^j}{\partial z^j} (t, z(t), u(t)) + \mu^s(t) \frac{\partial Y_s}{\partial z^j} (t, z(t), u(t)) + \frac{\partial \lambda_j^\ell}{\partial t^\ell} (t) = 0, \quad j = \overline{1, n}, \ell = \overline{1, m},$$

$$\theta^r \left[G^r(z, u) \frac{\partial f_\ell^r}{\partial u^i} (t, z(t), u(t)) - F^r(z, u) \frac{\partial g_\ell^r}{\partial u^i} (t, z(t), u(t)) \right] \quad (8)$$

$$+ \lambda_j^\ell(t) \frac{\partial X_\ell^j}{\partial u^i} (t, z(t), u(t)) + \mu^s(t) \frac{\partial Y_s}{\partial u^i} (t, z(t), u(t)) = 0, \quad i = \overline{1, k}, \ell = \overline{1, m},$$

$$\lambda_j^\ell(t) \left[X_\ell^j(t, z(t), u(t)) - \frac{\partial z^j}{\partial t^\ell} (t) \right] \geq 0, \quad (9)$$

$$\mu^{Q_\theta}(t) Y_{Q_\theta}(t, z(t), u(t)) \geq 0, \quad \theta = \overline{1, \zeta}, \quad (10)$$

$$\theta = (\theta^r) \succeq 0, \mu(t) = (\mu^s(t)) \geq 0, y(t_1) = x_1 = \text{given}, y(t_2) = x_2 = \text{given}. \quad (11)$$

Because of the inequality constraints (involving the partition $\{Q_1, Q_2, \dots, Q_\zeta\}$ of the set $Q = \{1, 2, \dots, q\}$), the dual problem (DFP) is of Mond-Weir type (see Mond and Weir [22]) since it has the same objective vector as the primal problem (FP).

The aforementioned expression $\mu^{Q_\theta}(t) Y_{Q_\theta}(t, z(t), u(t))$ has the following meaning

$$\mu^{Q_\theta}(t) Y_{Q_\theta}(t, z(t), u(t)) = \sum_{s \in Q_\theta} \mu^s(t) Y_s(t, z(t), u(t)), \quad \theta = \overline{1, \zeta}.$$

The concept of efficient solution associated with the problem (DFP) is similar to that formulated in Definition 1.

In this section, we prove that the multiobjective fractional variational control problems (FP) and (DFP) are a dual pair subject to (ρ, b) -quasiinvexity assumptions on the modified objective and constraints. Further, denote by $\mathbb{D}$ the domain associated with (DFP).

Theorem 1 (Weak Duality). Consider $(x, v) \in D$ and $(z, u, \theta, \lambda, \mu) \in \mathbb{D}$ feasible solutions for (FP) and (DFP), respectively. Also, assume that the following conditions are fulfilled:

(a) Each functional

$$\mathcal{F}_r^{z,u}(x, v) = \int_{\Gamma} [G^r(z, u) f_{\ell}^r(t, x(t), v(t)) - F^r(z, u) g_{\ell}^r(t, x(t), v(t))] dt^{\ell}, \quad r = \overline{1, p}$$

is (ρ_r^1, b) -quasiinvex at (z, u) with respect to η , ξ and d ;

(b) The functional $X(x, v) = \int_{\Gamma} \lambda_{\ell}^{\ell}(t) \left[X_{\ell}^j(t, x(t), v(t)) - \frac{\partial x^j}{\partial t^{\ell}}(t) \right] dt^{\ell}$ is (ρ^2, b) -quasiinvex at (z, u) with respect to η , ξ and d ;

(c) Each functional

$$Y_{\theta}(x, v) = \int_{\Gamma} \mu^{Q_{\theta}}(t) Y_{Q_{\theta}}(t, x(t), v(t)) dt^{\ell}, \quad \theta = \overline{1, \varsigma}$$

is (ρ_{θ}^3, b) -quasiinvex at (z, u) with respect to η , ξ and d ;

(d) At least one of the functionals given in (a)–(c) is strictly (ρ, b) -quasiinvex at (z, u) with respect to η , ξ and d , where $\rho = \rho_r^1, \rho^2$ or ρ_{θ}^3 ;

$$(e) \theta^r \rho_r^1 + \rho^2 + \sum_{\theta=1}^{\varsigma} \rho_{\theta}^3 \geq 0 \quad (\rho_r^1, \rho^2, \rho_{\theta}^3 \in \mathbb{R}).$$

Then the infimum of (FP) is greater than or equal to the supremum of (DFP).

Proof. Denote by $\alpha(x, v)$ and $\beta(z, u, \theta, \lambda, \mu)$ the value of problem (FP) at $(x, v) \in D$ and the value of problem (DFP) at $(z, u, \theta, \lambda, \mu) \in \mathbb{D}$, respectively. By reductio ad absurdum, suppose that $\alpha(x, v) \preceq \beta(z, u, \theta, \lambda, \mu)$. Further, for $r = \overline{1, p}$ and $\theta = \overline{1, \varsigma}$, consider the following non-empty set

$$S = \{(x, v) \in \mathcal{X} \times \mathcal{U} \mid \mathcal{F}_r^{z,u}(x, v) \leq \mathcal{F}_r^{z,u}(z, u), X(x, v) \leq X(z, u), Y_{\theta}(x, v) \leq Y_{\theta}(z, u)\}.$$

Using (a), for $(x, v) \in S$ and $r = \overline{1, p}$, we get

$$\begin{aligned} \mathcal{F}_r^{z,u}(x, v) &\leq \mathcal{F}_r^{z,u}(z, u) \implies \\ b(x, v, z, u) &\int_{\Gamma} [G^r(z, u)(f_{\ell}^r)_z(t, z(t), u(t)) - F^r(z, u)(g_{\ell}^r)_z(t, z(t), u(t))] \eta dt^{\ell} \\ &+ b(x, v, z, u) \int_{\Gamma} [G^r(z, u)(f_{\ell}^r)_u(t, z(t), u(t)) - F^r(z, u)(g_{\ell}^r)_u(t, z(t), u(t))] \xi dt^{\ell} \\ &\leq -\rho_r^1 b(x, v, z, u) d^2((x, v), (z, u)). \end{aligned}$$

Multiplying by $\theta = (\theta^r) \succeq 0$ and taking summation over $r = \overline{1, p}$, we find

$$\begin{aligned} b(x, v, z, u) &\int_{\Gamma} \theta^r [G^r(z, u)(f_{\ell}^r)_z(t, z(t), u(t)) - F^r(z, u)(g_{\ell}^r)_z(t, z(t), u(t))] \eta dt^{\ell} \\ &+ b(x, v, z, u) \int_{\Gamma} \theta^r [G^r(z, u)(f_{\ell}^r)_u(t, z(t), u(t)) - F^r(z, u)(g_{\ell}^r)_u(t, z(t), u(t))] \xi dt^{\ell} \\ &\leq -\theta^r \rho_r^1 b(x, v, z, u) d^2((x, v), (z, u)). \end{aligned} \quad (12)$$

For $(x, v) \in S$, the inequality $X(x, v) \leq X(z, u)$ holds and, according to (b), it follows

$$b(x, v, z, u) \int_{\Gamma} \left[\lambda_j^\ell(t) (X_\ell^j)_z(t, z(t), u(t)) \eta - \lambda^\ell(t) D_\ell \eta + \lambda_j^\ell(t) (X_\ell^j)_u(t, z(t), u(t)) \xi \right] dt^\ell \quad (13)$$

$$\leq -\rho^2 b(x, v, z, u) d^2((x, v), (z, u)).$$

Also, for $(x, v) \in S$, the inequality $Y_\vartheta(x, v) \leq Y_\vartheta(z, u)$, $\vartheta = \overline{1, \zeta}$, gives (see (c))

$$b(x, v, z, u) \int_{\Gamma} \left[\mu^{\vartheta}(t) (Y_{Q_\vartheta})_z(t, z(t), u(t)) \eta + \mu^{\vartheta}(t) (Y_{Q_\vartheta})_u(t, z(t), u(t)) \xi \right] dt^\ell$$

$$\leq -\rho_\vartheta^3 b(x, v, z, u) d^2((x, v), (z, u))$$

and making summation over $\vartheta = \overline{1, \zeta}$ in the previous inequality, it results

$$b(x, v, z, u) \int_{\Gamma} \left[\mu^s(t) (Y_s)_z(t, z(t), u(t)) \eta + \mu^s(t) (Y_s)_u(t, z(t), u(t)) \xi \right] dt^\ell \quad (14)$$

$$\leq -\sum_{\vartheta=1}^{\zeta} \rho_\vartheta^3 b(x, v, z, u) d^2((x, v), (z, u)).$$

Computing (12) + (13) + (14) and using (d), we obtain

$$b(x, v, z, u) \int_{\Gamma} \theta^r [G^r(z, u) (f_\ell^r)_z(t, z(t), u(t)) - F^r(z, u) (g_\ell^r)_z(t, z(t), u(t))] \eta dt^\ell$$

$$+ b(x, v, z, u) \int_{\Gamma} \left[\lambda_j^\ell(t) (X_\ell^j)_z(t, z(t), u(t)) + \mu^s(t) (Y_s)_z(t, z(t), u(t)) \right] \eta dt^\ell$$

$$+ b(x, v, z, u) \int_{\Gamma} \theta^r [G^r(z, u) (f_\ell^r)_u(t, z(t), u(t)) - F^r(z, u) (g_\ell^r)_u(t, z(t), u(t))] \xi dt^\ell$$

$$+ b(x, v, z, u) \int_{\Gamma} \left[\lambda_j^\ell(t) (X_\ell^j)_u(t, z(t), u(t)) + \mu^s(t) (Y_s)_u(t, z(t), u(t)) \right] \xi dt^\ell$$

$$- b(x, v, z, u) \int_{\Gamma} \left[\lambda^\ell(t) D_\ell \eta \right] dt^\ell < -(\theta^r \rho_r^1 + \rho^2 + \sum_{\vartheta=1}^{\zeta} \rho_\vartheta^3) b(x, v, z, u) d^2((x, v), (z, u)).$$

The previous inequality implicates $b(x, v, z, u) > 0$ and, as a consequence, we can rewrite it as

$$\int_{\Gamma} \theta^r [G^r(z, u) (f_\ell^r)_z(t, z(t), u(t)) - F^r(z, u) (g_\ell^r)_z(t, z(t), u(t))] \eta dt^\ell$$

$$+ \int_{\Gamma} \left[\lambda_j^\ell(t) (X_\ell^j)_z(t, z(t), u(t)) + \mu^s(t) (Y_s)_z(t, z(t), u(t)) \right] \eta dt^\ell$$

$$+ \int_{\Gamma} \theta^r [G^r(z, u) (f_\ell^r)_u(t, z(t), u(t)) - F^r(z, u) (g_\ell^r)_u(t, z(t), u(t))] \xi dt^\ell$$

$$+ \int_{\Gamma} \left[\lambda_j^\ell(t) (X_\ell^j)_u(t, z(t), u(t)) + \mu^s(t) (Y_s)_u(t, z(t), u(t)) \right] \xi dt^\ell$$

$$- \int_{\Gamma} \left[\lambda^\ell(t) D_\ell \eta \right] dt^\ell < -(\theta^r \rho_r^1 + \rho^2 + \sum_{\vartheta=1}^{\zeta} \rho_\vartheta^3) d^2((x, v), (z, u)).$$

Now, considering the constraints (7), (8) of (DFP), we obtain

$$- \int_{\Gamma} \eta D_\ell \lambda^\ell(t) dt^\ell - \int_{\Gamma} \left[\lambda^\ell(t) D_\ell \eta \right] dt^\ell < -(\theta^r \rho_r^1 + \rho^2 + \sum_{\vartheta=1}^{\zeta} \rho_\vartheta^3) d^2((x, v), (z, u)).$$

By direct computation, we get

$$D_\ell [\eta \lambda^\ell(t)] = \lambda^\ell(t) D_\ell \eta + \eta D_\ell \lambda^\ell(t),$$

$$\int_\Gamma \eta D_\ell \lambda^\ell(t) dt^\ell = \int_\Gamma D_\ell [\eta \lambda^\ell(t)] dt^\ell - \int_\Gamma [\lambda^\ell(t) D_\ell \eta] dt^\ell$$

but, applying the condition $\eta(t_1) = \eta(t_2) = 0$ and the result “A total divergence is equal to a total derivative”, we get

$$\int_\Gamma D_\ell [\eta \lambda^\ell(t)] dt^\ell = 0.$$

It results that

$$-\int_\Gamma \eta D_\ell \lambda^\ell(t) dt^\ell - \int_\Gamma [\lambda^\ell(t) D_\ell \eta] dt^\ell = 0.$$

Consequently,

$$0 < -(\theta^r \rho_r^1 + \rho^2 + \sum_{\theta=1}^{\xi} \rho_\theta^3) d^2((x, v), (z, u))$$

and applying the hypothesis (e) and $d((x, v), (z, u)) \geq 0$, we get a contradiction. The proof is complete. $\square$

Theorem 2 (Strong Duality). Under the assumptions formulated in Theorem 1, if $(x^0, v^0) \in D$ is a normal efficient solution of the primal problem (FP), then there exist $\theta^0 = (\theta^0)^r$, $\mu^0(t)$ and $\lambda^0(t)$ [with $\mu^0(t) = (\mu^0)^s(t)$, $\lambda^0(t) = (\lambda^0)_j^\ell(t)$ piecewise smooth functions], such that $(x^0, v^0, \theta^0, \lambda^0, \mu^0) \in \mathbb{D}$ is efficient solution for (DFP). Moreover, the corresponding objective values are equal.

Proof. Taking into account that $(x^0, v^0) \in D$ is a normal efficient solution of (FP), the necessary efficiency conditions, formulated in (4)–(6), imply that there exist $\theta^0 = (\theta^0)^r$, $\mu^0(t)$ and $\lambda^0(t)$, with $\mu^0(t) = (\mu^0)^s(t)$, $\lambda^0(t) = (\lambda^0)_j^\ell(t)$ piecewise smooth functions, such that $(x^0, v^0, \theta^0, \lambda^0, \mu^0)$ is feasible solution for (DFP). Since

$$\frac{\partial x^{0j}}{\partial t^\ell}(t) = X_\ell^j(t, x^0(t), v^0(t)), \quad j = \overline{1, n}, \ell = \overline{1, m}, t \in \Theta$$

and (by (6))

$$\mu^s(t) Y_s(t, x^0(t), v^0(t)) = 0, \quad (\text{summation over } s), \quad t \in \Theta,$$

it results in the dual objective having the same value as the primal objective. Also, by Theorem 1, $(x^0, v^0, \theta^0, \lambda^0, \mu^0) \in \mathbb{D}$ is efficient solution in (DFP). $\square$

Theorem 3 (Converse Duality). Consider $(x^0, v^0, \theta^0, \lambda^0, \mu^0) \in \mathbb{D}$ is an efficient solution for (DFP) and the following conditions are satisfied:

- (a) $(\bar{x}, \bar{u}) \in D$ is a normal efficient solution for (FP);
- (b) the conditions provided by Theorem 1 hold for $(x^0, v^0, \theta^0, \lambda^0, \mu^0)$.

Then $(\bar{x}, \bar{u}) = (x^0, v^0)$ and the corresponding objective values are equal.

Proof. By contrary, let us consider that (x^0, v^0) is not a normal efficient solution in (FP). In consequence, $(\bar{x}, \bar{u}) \neq (x^0, v^0)$. Since $(\bar{x}, \bar{u}) \in D$ is a normal efficient solution for (FP), in accordance with Mititelu and Treanță [14], there exist $\bar{\theta} = (\bar{\theta})^r$, $\bar{\mu}(t)$ and $\bar{\lambda}(t)$, with $\bar{\mu}(t) = (\bar{\mu})^s(t)$, $\bar{\lambda}(t) = (\bar{\lambda})_j^\ell(t)$ piecewise smooth functions, fulfilling (4)–(6) and Definition 3. It results in

$$\bar{\lambda}_j^\ell(t) \left[X_\ell^j(t, \bar{x}(t), \bar{u}(t)) - \frac{\partial \bar{x}^j}{\partial t^\ell}(t) \right] \geq 0,$$

$$\bar{\mu}^{Q_\vartheta}(t) Y_{Q_\vartheta}(t, \bar{x}(t), \bar{u}(t)) \geq 0, \quad \vartheta = \overline{1, \zeta}$$

and, therefore, $(\bar{x}, \bar{u}, \bar{\theta}, \bar{\lambda}, \bar{\mu}) \in \mathbb{D}$. Moreover, we have $\alpha(\bar{x}, \bar{u}) = \beta(\bar{x}, \bar{u}, \bar{\theta}, \bar{\lambda}, \bar{\mu})$. In accordance to Theorem 1, we obtain that $\alpha(\bar{x}, \bar{u}) \preceq \beta(x^0, v^0, \theta^0, \lambda^0, \mu^0)$, or, equivalently, $\beta(\bar{x}, \bar{u}, \bar{\theta}, \bar{\lambda}, \bar{\mu}) \preceq \beta(x^0, v^0, \theta^0, \lambda^0, \mu^0)$ is not true. This contradicts the maximal efficiency of the point $(x^0, v^0, \theta^0, \lambda^0, \mu^0)$. Hence, $(\bar{x}, \bar{u}) = (x^0, v^0)$ and the corresponding objective values are equal. $\square$

4. Conclusions

In this paper, a dual of Mond–Weir type has been investigated for a new class of multiobjective fractional variational control problems (introduced by Mititelu and Treanță [14]). Under (ρ, b) -quasiinvexity hypotheses, weak, strong and converse duality results have been formulated and proved. The present work can be extended for other classes of optimization problems such as uncertain variational control problems.

Funding: The APC was funded by University Politehnica of Bucharest, “PubArt” program.

Conflicts of Interest: The author declares no conflict of interest.

References

- Hanson, M.A. On sufficiency of Kuhn-Tucker conditions. *J. Math. Anal. Appl.* **1981**, *80*, 545–550. [\[CrossRef\]](#)
- Craven, B.D.; Glover, B.M. Invex functions and duality. *J. Aust. Math. Soc. (Ser. A)* **1985**, *39*, 1–20. [\[CrossRef\]](#)
- Mond, B.; Smart, I. Duality and sufficiency in control problems with invexity. *J. Math. Anal. Appl.* **1988**, *136*, 325–333. [\[CrossRef\]](#)
- Mond, B.; Hanson, M.A. Duality for control problems. *SIAM J. Control* **1968**, *6*, 114–120. [\[CrossRef\]](#)
- Preda, V. On duality and sufficiency in control problems with general invexity. *Bull. Math. Soc. Sci. Math. Roum.* **1991**, *35*, 271–280.
- Zhian, L.; Qingkai, Y. Duality for a class of multiobjective control problems with generalized invexity. *J. Math. Anal. Appl.* **2001**, *256*, 446–461. [\[CrossRef\]](#)
- Zalmay, G.J. Generalized $(\mathcal{F}, b, \phi, \rho, \theta)$ -univex n -set functions and semiparametric duality models in multiobjective fractional subset programming. *Int. J. Math. Math. Sci.* **2005**, *2005*, 949–973. [\[CrossRef\]](#)
- Gulati, T.R.; Husain, I.; Ahmed, A. Optimality conditions and duality for multiobjective control problems. *J. Appl. Anal.* **2005**, *11*, 225–245. [\[CrossRef\]](#)
- Hachimi, M.; Aghezzaf, B. Sufficiency and duality in multiobjective variational problems with generalized type I functions. *J. Glob. Optim.* **2006**, *34*, 191–218. [\[CrossRef\]](#)
- Zhang, J.; Liu, S.; Li, L.; Feng, Q. Sufficiency and duality for multiobjective variational control problems with G -invexity. *Comput. Math. Appl.* **2012**, *63*, 838–850. [\[CrossRef\]](#)
- Soleimani, B.; Tammer, C. A vector-valued Ekeland’s variational principle in vector optimization with variable ordering structures. *J. Nonlinear Var. Anal.* **2017**, *1*, 89–110.
- Ansari, Q.H.; Köbis, E.; Yao, J.C. Vector variational inequalities and vector optimization. *Theory and Applications. Vector Optimization*; Springer: Cham, Switzerland, 2018; p. xiii+509.
- Tung, L.T. Karush-Kuhn-Tucker optimality conditions for nonsmooth multiobjective semidefinite and semi-infinite programming. *J. Appl. Numer. Optim.* **2019**, *1*, 63–75.
- Mititelu, Ș.; Treanță, S. Efficiency conditions in vector control problems governed by multiple integrals. *J. Appl. Math. Comput.* **2018**, *57*, 647–665. [\[CrossRef\]](#)
- Treanță, S. Multiobjective fractional variational problem on higher-order jet bundles. *Commun. Math. Stat.* **2016**, *4*, 323–340. [\[CrossRef\]](#)
- Treanță, S. Sufficient efficiency conditions associated with a multidimensional multiobjective fractional variational problem. *J. Multidiscip. Model. Optim.* **2018**, *1*, 1–13.
- Treanță, S. PDE-constrained vector variational problems governed by curvilinear integral functionals. *Appl. Anal. Optim.* **2019**, *3*, 83–101.

18. Treanță, S. On Controlled Variational Inequalities Involving Convex Functionals. In *Optimization of Complex Systems: Theory, Models, Algorithms and Applications*. WCGO 2019. *Advances in Intelligent Systems and Computing*; Le Thi, H., Le, H., Pham Dinh, T., Eds.; Springer: Cham, Switzerland, 2019; Volume 991, pp. 164–174.
19. Treanță, S. Constrained variational problems governed by second-order Lagrangians. *Appl. Anal.* **2018**, 1–18. [[CrossRef](#)]
20. Treanță, S. On a modified optimal control problem with first-order PDE constraints and the associated saddle-point optimality criterion. *Eur. J. Control* **2019**. [[CrossRef](#)]
21. Treanță, S.; Mititelu, Ș. Duality with (ρ, b) -quasiinvexity for multidimensional vector fractional control problems. *J. Inform. Optim. Sci.* **2019**, 1–17. [[CrossRef](#)]
22. Mond, B.; Weir, T. Generalized concavity and duality. In *Generalized Concavity in Optimization and Economics*; Schaible, S., Ziemba, W.T., Eds.; Academic Press: New York, NY, USA, 1981; pp. 263–279.

© 2019 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).