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Existence and Uniqueness of Zeros for Vector-Valued Functions with *K*-Adjustability Convexity and Their Applications

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Abstract: In this paper, we introduce the new concepts of K-adjustability convexity and strictly K-adjustability convexity which respectively generalize and extend the concepts of K-convexity and strictly K-convexity. We establish some new existence and uniqueness theorems of zeros for vector-valued functions with K-adjustability convexity. As their applications, we obtain existence theorems for the minimization problem and fixed point problem which are original and quite different from the known results in the existing literature.

Keywords: K-convexity; strictly K-convexity; K-adjustability convexity; strictly K-adjustability convexity; nonlinear scalarization function; (e, K)-lower semicontinuous; zero for a vector-valued function; minimization problem; fixed point problem

MSC: 26B25; 47H10; 65K10

1. Introduction and Preliminaries

It is well known that convex analysis has played an important role in almost all branches of mathematics, physics, economics, and engineering. Convexity is an ancient and natural notion and the theory of convex functions is an essential part of the general subject of convexity.

Let V be a vector space. A nonempty subset A of V is called *convex* if for any $x, y \in A$, $\lambda x + (1 - \lambda)y \in A$ for all $\lambda \in [0, 1]$. Let X be a nonempty convex subset of V. A real-valued function $f : X \to \mathbb{R}$ is called *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

for all $x, y \in X$ and $t \in [0, 1]$. If the above inequality (1) is strict whenever $x \neq y$ and $0 < t < 1$, then f is called *strictly convex*. A function $f : X \to \mathbb{R}$ is called *concave* (resp. *strictly concave*) if $-f$ is convex (resp. strictly convex). A large amount of new notions of generalized convexity and concavity have been investigated by several authors; see, for example, ref. [1–15] and references therein.

The general vector optimization problem (VOP) for a vector-valued function $f : X \to V_2$ can be formalized as follows:

$$(VOP) \quad \begin{cases} \text{Optimize } f(x), \\ \text{subject to } x \in X \end{cases},$$

where V_1 and V_2 be vector spaces and X is a nonempty subset of V_1. Vector optimization problems have been intensively investigated, and various feasible methods have been proposed over a century and has made more important contributions to improve our understanding of the real world around us in various fields. Convex analysis and vector optimization has wide and significant applications in many areas of mathematics, including nonlinear analysis, finance mathematics, vector differential equations and inclusions, dynamic system theory, control theory, economics, game theory, machine learning, multiobjective programming, multi-criteria decision making, game theory, signal processing, and so forth. For more details, see, e.g., ref. [1,7–10,16] and references therein.

In reality, we often encounter non-convex functions or non-concave functions when solving problems in the real world, so these known results for convex functions or concave functions are not easily applicable to work. Motivated by that reason, in this paper, we study and introduce the new concepts of K-adjustability convexity and strictly K-adjustability convexity (see Definition 1 below). A nontrivial example is given to illustrate that the concept of K-adjustability convexity is a real generalization of the concept of K-convexity. In Section 3, we establish some new existence and uniqueness theorems of zeros for vector-valued functions with K-adjustability convexity. As their applications, we obtain existence theorems for minimization problem and fixed point problem which are original and quite different from the known results in the literature.

2. New Concepts of K-Adjustability Convexity and Strictly K-Adjustability Convexity

Let V be a topological vector space (t.v.s., for short) with its zero vector θ_V. Let A be a nonempty subset of V. We use the notations $\overline{A}$, $co(A)$ and $\overline{co}(A)$ to denote the closure, convex hull and closed convex hull (i.e., the closure of the convex hull) of A, respectively. A nonempty subset K of V is called a convex cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for $\lambda \geq 0$. A cone K is pointed if $K \cap (-K) = \{\theta_V\}$. For a given cone $K \subseteq V$, we can define a partial ordering $\precsim_K$ with respect to K by

$$x \precsim_K y \Longleftrightarrow y - x \in K.$$

$x \prec_K y$ will stand for $x \precsim_K y$ and $x \neq y$, while $x \ll_K y$ will stand for $y - x \in intK$, where $intK$ denotes the interior of K. A function $\varphi : V \to V$ is called to be $\precsim_K$*-nondecreasing* if $x, y \in V$ with $x \precsim_K y$ implies $\varphi(x) \precsim_K \varphi(y)$.

Let X be a topological space. A real-valued function $h : X \to \mathbb{R}$ is *lower semicontinuous* (in short *lsc*) (resp. *upper semicontinuous*, in short *usc*) if $\{x \in X : h(x) \leq r\}$ (resp. $\{x \in X : h(x) \geq r\}$) is *closed* for each $r \in \mathbb{R}$.

Let Y be a t.v.s. with its zero vector θ, K be a proper (i.e., $K \neq Y$), closed and convex pointed cone in Y with $intK \neq \emptyset$, $e \in intK$, and $\precsim_K$ be a partial ordering with respect to K. A vector-valued function $f : X \to Y$ is said to be (e, K)*-lower semicontinuous* [9,17] if for each $r \in \mathbb{R}$, the set $\{x \in X : f(x) \in re - K\}$ is closed.

In this paper, we introduce the concepts of K-adjustability convexity and strictly K-adjustability convexity.

Definition 1. *Let V_1 and V_2 be vector spaces, X be a nonempty convex set in V_1, K be a given convex cone in V_2 and $\mu : V_2 \to V_2$ be a mapping. A vector-valued function $f : X \to V_2$ is called*

(i) K-adjustability convex with respect to μ (abbreviated as (K, μ)-adjconvex) if

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) \in K \tag{2}$$

for all $x, y \in X$ and $t \in [0, 1]$. In particular, f is called K-convex if μ is an identity mapping on V_2 and (2) becomes

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in K$$

for all $x, y \in X$ and $t \in [0, 1]$.

(ii) strictly K-adjustability convex with respect to μ (abbreviated as strictly (K, μ)-adjconvex) if

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) \in intK \tag{3}$$

for all $x, y \in X$ with $x \neq y$ and $t \in (0,1)$. In particular, f is called strictly K-convex if μ is an identity mapping on V_2 and (3) becomes

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in intK$$

for all $x, y \in X$ with $x \neq y$ and $t \in (0,1)$.

Here, we give an example where f is K-adjconvex but not K-convex.

Example 1. Let $V_1 = \mathbb{R}$, $V_2 = \mathbb{R}^2$, $X = [-1,1]$ and $K = \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_i \geq 0, i = 1,2\}$. Then X is a nonempty convex subset of V_1 and K is a convex cone in V_2. Let $f : X \to V_2$ be defined by

$$f(x) = \begin{cases} (-x, 0), & x \in [0,1], \\ (0, x), & x \in [-1, 0). \end{cases}$$

Take $\hat{x} = \frac{1}{2}$ and $\hat{y} = -\frac{1}{2}$. Thus, we get

$$\frac{1}{2}f(\hat{x}) + \frac{1}{2}f(\hat{y}) - f\left(\frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}\right) = \left(-\frac{1}{4}, -\frac{1}{4}\right) - (0,0) = \left(-\frac{1}{4}, -\frac{1}{4}\right) \notin K,$$

which show that f is not K-convex. Now, let $\mu : V_2 \to V_2$ be defined by

$$\mu(x,y) = (\max\{|x|, |y|\}, 0) \quad \text{for } (x,y) \in V_2.$$

We claim that f is (K,μ)-adjconvex. Let $x, y \in X$ and $t \in [0,1]$ be given. We consider the following four possible cases:

Case 1. If $x, y \in [0,1]$, then $tf(x) + (1-t)f(y) = (a, 0)$ for some $a \leq 0$ and $f(tx + (1-t)y) = (b, 0)$ for some $b \leq 0$. Since $\max\{|a|, |0|\} - b \geq 0$, we obtain

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) = (\max\{|a|, |0|\} - b, 0) \in K.$$

Case 2. If $x, y \in [-1, 0)$, then $tf(x) + (1-t)f(y) = (0, c)$ for some $c \leq 0$ and $f(tx + (1-t)y) = (0, d)$ for some $d < 0$. So $\mu(tf(x) + (1-t)f(y)) = (\max\{|0|, |c|\}, 0)$ and we get

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) \in K.$$

Case 3. Assume that $x \in [0,1]$ and $y \in [-1, 0)$. Then $tf(x) + (1-t)f(y) = (m, n)$ for some $m, n \leq 0$.

- If $tx + (1-t)y \in [0,1]$, then $f(tx + (1-t)y) = (\lambda, 0)$ for some $\lambda \leq 0$. Hence, we have

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) = (\max\{|m|, |n|\} - \lambda, 0) \in K.$$

- If $tx + (1-t)y \in [-1, 0)$, then $f(tx + (1-t)y) = (0, s)$ for some $s < 0$. Therefore, we get

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) = (\max\{|m|, |n|\}, -s) \in K.$$

Case 4. Assume that $x \in [-1, 0)$ and $y \in [0,1]$. Following the same argument as Case 3, we can verify

$$\mu(tf(x) + (1-t)f(y)) - f(tx + (1-t)y) \in K.$$

Therefore, by above cases, we prove that f is (K,μ)-adjconvex.

In Definition 1, if we take $V = V_1$, $V_2 = \mathbb{R}$, $K = [0, +\infty) \subset \mathbb{R}$, then we obtain the following concepts.

Definition 2. *Let X be a nonempty convex subset of a vector space V and $\mu : \mathbb{R} \to \mathbb{R}$ be a function. A real-valued function $f : X \to \mathbb{R}$ is called*

(i) adjustability convex with respect to μ (abbreviated as (μ)-adjconvex) if

$$f(tx + (1-t)y) \leq \mu(tf(x) + (1-t)f(y))$$

for all $x, y \in X$ and $t \in [0,1]$. In particular, if μ is an identity mapping on $\mathbb{R}$, then f is called convex.

(ii) strictly adjustability convex with respect to μ (abbreviated as strictly (μ)-adjconvex) if

$$f(tx + (1-t)y) < \mu(tf(x) + (1-t)f(y))$$

for all $x, y \in X$ with $x \neq y$ and $t \in (0,1)$. In particular, if μ is an identity mapping on $\mathbb{R}$, then f is called strictly convex.

In the following, unless otherwise specified, we always suppose that Y is a locally convex Hausdorff t.v.s. with its zero vector θ, K be a proper, closed and convex pointed cone in Y with $intK \neq \emptyset$, $e \in intK$, and $\precsim_K$ be a partial ordering with respect to K. Recall that the nonlinear scalarization function $\xi_e : Y \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}, \text{ for all } y \in Y.$$

Obviously, $\xi_e(\theta) = 0$.

The following known result is very crucial in our proofs.

Lemma 1. *(see [1,5,16,18–23]). For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied:*

(i) $\xi_e(y) \leq r \iff y \in re - K$;
(ii) $\xi_e(y) > r \iff y \notin re - K$;
(iii) $\xi_e(y) \geq r \iff y \notin re - intK$;
(iv) $\xi_e(y) < r \iff y \in re - intK$;
(v) $\xi_e(\cdot)$ is positively homogeneous and continuous on Y;
(vi) if $y_1 \in y_2 + K$ (i.e., $y_2 \precsim_K y_1$), then $\xi_e(y_2) \leq \xi_e(y_1)$;
(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ for all $y_1, y_2 \in Y$.

By Applying (i) of Lemma 1, one can easily verify the following result; see also [19,24].

Lemma 2. *Let X be a topological space and $f : X \to Y$ be a vector-valued function. Then f is (e,K)-lower semicontinuous if and only if $\xi_e \circ f$ is lower semicontinuous.*

3. New Existence Results and Their Applications to Minimization Problem and Fixed Point Problem

The following lemma is very important and will be used for proving our main results.

Lemma 3. *Let $\mu : Y \to Y$ be a vector-valued function satisfying the following condition:*

(A) For any $\epsilon > 0$, there exists $\gamma > 0$ such that

$$x \notin -K \text{ and } x \ll_K \gamma e \quad \text{implies} \quad \mu(x) \ll_K \epsilon e.$$

Then there exists a strictly decreasing sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ of positive real numbers such that $\mu(\lambda_{n+1}e) \ll_K \lambda_n e$ for all $n \in \mathbb{N}$ and $\lambda_n \downarrow 0$ as $n \to \infty$.

Proof. Given $\lambda_1 > 0$. Then, by (A), there exists $\delta_1 > 0$ such that

$$x \notin -K \text{ and } x \ll_K \delta_1 e \quad \text{implies} \quad \mu(x) \ll_K \lambda_1 e. \tag{4}$$

Let $\lambda_2 = \min\left\{\frac{\delta_1}{2}, \frac{\lambda_1}{2}\right\}$ and take $w_1 = \lambda_2 e \in intK$. Then we have the following:

- $w_1 \notin -K$;
- $w_1 \ll_K \delta_1 e$;
- $\lambda_2 < \lambda_1$.

So we have from (4) that

$$\mu(w_1) \ll_K \lambda_1 e.$$

For λ_2, it must exist $\delta_2 > 0$ such that

$$x \notin -K \text{ and } x \ll_K \delta_2 e \quad \text{implies} \quad \mu(x) \ll_K \lambda_2 e. \tag{5}$$

Put $\lambda_3 = \min\left\{\frac{\delta_2}{2}, \frac{\lambda_2}{2}\right\}$ and $w_2 = \lambda_3 e \in intK$. Thus $\lambda_3 < \lambda_2$ and, by (5), we obtain

$$\mu(w_2) \ll_K \lambda_2 e.$$

Continuing this process, for λ_j, $j \in \mathbb{N}$ with $j \geq 2$, it must exist $\delta_j > 0$ such that

$$x \notin -K \text{ and } x \ll_K \delta_j e \quad \text{implies} \quad \mu(x) \ll_K \lambda_j e. \tag{6}$$

Take

$$\lambda_{j+1} = \min\left\{\frac{\delta_j}{2}, \frac{\lambda_j}{2}\right\} \tag{7}$$

and

$$w_j = \lambda_{j+1} e \in Y.$$

Then we get from (6) and (7) that $\lambda_{j+1} < \lambda_j$ and $\mu(w_j) \ll_K \lambda_j e$. Therefore, we can construct a strictly decreasing sequences $\{\lambda_n\}$ of positive real numbers such that

$$\mu(\lambda_{n+1} e) \ll_K \lambda_n e \quad \text{for all } n \in \mathbb{N}.$$

By (7), we have $0 < \lambda_{n+1} \leq \frac{\lambda_1}{2^n}$ for $n \in \mathbb{N}$, which yields $\lambda_n \downarrow 0$ as $n \to \infty$. The proof is completed. □

The following result is immediate from Lemma 3 if we take $Y = \mathbb{R}$, $K = [0, +\infty) \subset \mathbb{R}$ and $e = 1$.

Corollary 1. *Let $\mu : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following condition:*

($A_{\mathbb{R}}$) For any $\epsilon > 0$, there exists $c > 0$ such that

$$0 < x < c \quad \textit{implies} \quad \mu(x) < \epsilon.$$

Then there exists a strictly decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\mu(\lambda_{n+1}) < \lambda_n$ for all $n \in \mathbb{N}$ and $\lambda_n \downarrow 0$ as $n \to \infty$.

Corollary 2. *Let $\mu : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\lim\limits_{x\to 0^+} \mu(x) = 0$. Then there exists a strictly decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\mu(\lambda_{n+1}) < \lambda_n$ for all $n \in \mathbb{N}$ and $\lambda_n \downarrow 0$ as $n \to \infty$.*

Proof. For any $\epsilon > 0$, since $\lim\limits_{x\to 0^+} \mu(x) = 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$0 < x < \delta \quad \text{implies} \quad \mu(x) < \epsilon.$$

Therefore, the conclusion is immediate from Corollary 1. □

We now establish the following crucial and useful existence result which is one of the main results of this paper and will be applied to minimization problem and fixed point problem.

Theorem 1. *Let $(E, \|\cdot\|)$ be a normed linear space, Y be a locally convex Hausdorff t.v.s. with its zero vector θ, K be a proper, closed and convex pointed cone in Y with $intK \neq \emptyset$, and let $e \in intK$ be fixed. Let W be a nonempty weakly compact and convex subset of E, $\mu : Y \to Y$ be a $\precsim_K$-nondecreasing vector-valued function satisfying the condition (A) as in Lemma 3 and $f : W \to Y$ be a vector-valued function. Assume that*

(H1) for any positive real number γ, $\{x \in W : f(x) \in \gamma e - K\}$ is a nonempty closed subset of W,
(H2) f is (K, μ)-adjconvex.

Then there exists $v \in W$ such that $f(v) \in -K$.

Proof. By applying Lemma 3, there exists a strictly decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that

$$\mu(\lambda_{n+1}e) \ll_K \lambda_n e \quad \text{for all } n \in \mathbb{N}, \tag{8}$$

and $\lambda_n \downarrow 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, let

$$C_n = \{x \in W : f(x) \in \lambda_n e - K\}.$$

Define $F : W \to \mathbb{R}$ by

$$F(x) = \xi_e \circ f(x) \quad \text{for } x \in W.$$

Applying Lemma 1, we have

$$C_n = \{x \in W : F(x) \leq \lambda_n\}.$$

Thus, by (H1), C_n is a nonempty closed subset of W. Clearly, $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$. We choose an arbitrary point z_n from C_n for all $n \in \mathbb{N}$. For any $m, n \in \mathbb{N}$ with $m \geq n$, let

$$D_{m,n} = \{z_i : n+1 \leq i \leq m+1\}.$$

We verify that

$$co(D_{m,n}) \subseteq C_n \quad \text{for all } m, n \in \mathbb{N} \text{ with } m \geq n. \tag{9}$$

Indeed, let $m, n \in \mathbb{N}$ with $m \geq n$. If $m = n$, then

$$co(D_{n,n}) = \{z_{n+1}\} \subseteq C_{n+1} \subseteq C_n$$

and (9) is true. For $m \geq 2$ and $n = m-1$, $co(D_{m,m-1}) = co(\{z_m, z_{m+1}\})$. If $x \in co(D_{m,m-1})$, then there exists $t \in [0, 1]$ such that

$$x = tz_m + (1-t)z_{m+1}. \tag{10}$$

Since $z_m, z_{m+1} \in C_m$, $f(z_m), f(z_{m+1}) \in \lambda_m e - K$. Since K is a convex cone, we get

$$tf(z_m) + (1-t)f(z_{m+1}) \in \lambda_m e - K.$$

Thus, there exists $\zeta \in K$ such that $tf(z_m) + (1-t)f(z_{m+1}) = \lambda_m e - \zeta$. Since $\lambda_m e - \zeta \precsim_K \lambda_m e$ and μ is $\precsim_K$-nondecreasing, we obtain

$$\mu\left(tf(z_m) + (1-t)f(z_{m+1})\right) = \mu\left(\lambda_m e - \zeta\right) \precsim_K \mu\left(\lambda_m e\right) \tag{11}$$

Taking into account (H2), (8) and (11), we get

$$f(x) \precsim_K \mu\left(tf(z_m) + (1-t)f(z_{m+1})\right) \precsim_K \mu\left(\lambda_m e\right) \ll_K \lambda_{m-1} e$$

$$\iff \quad F(x) = \xi_e \circ f(x) < \lambda_{m-1} \quad \text{(by Lemma 1)}$$

which means that $x \in C_{m-1}$. Hence $co(D_{m,m-1}) \subseteq C_{m-1}$ and (9) is true for $m \geq 2$ and $n = m - 1 < m$. Assume that (9) is valid for $n = k < m$. Note first that

$$\begin{aligned} co(D_{m,k-1}) &= co(\{z_i : k \leq i \leq m+1\}) \\ &= co(\{z_k\} \cup \{z_{k+1}, \cdots, z_{m+1}\}) \\ &= co(\{z_k\} \cup D_{m,k}). \end{aligned}$$

Let $p \in co(D_{m,k-1})$ be given. If $p = z_i$ for some $i_0 \in \{k, k+1, \cdots, m+1\}$, then $p \in C_{i_0} \subseteq C_{k-1}$. Suppose $p \neq z_i$ for all $i \in \{k, k+1, \cdots, m+1\}$. Thus, there exist $\gamma_k, \gamma_{k+1}, \cdots, \gamma_{m+1} \in [0,1)$ with $\sum\limits_{i=k}^{m+1} \gamma_i = 1$, such that $p = \sum\limits_{i=k}^{m+1} \gamma_i z_i$. Let

$$w = \sum_{i=k+1}^{m+1} \frac{\gamma_i}{1-\gamma_k} z_i.$$

Due to $\sum\limits_{i=k+1}^{m+1} \frac{\gamma_i}{1-\gamma_k} = 1$ and applying the induction hypothesis, we know

$$w \in co(D_{m,k}) \subseteq C_k$$

and

$$p = \sum_{i=k}^{m+1} \gamma_i z_i = \gamma_k z_k + (1-\gamma_k) w.$$

Since $z_k, w \in C_k$, we have $f(z_k), f(w) \in \lambda_k e - K$ and hence

$$\gamma_k f(z_k) + (1-\gamma_k) f(w) \in \lambda_k e - K.$$

So $\gamma_k f(z_k) + (1-\gamma_k) f(w) = \lambda_k e - \beta$ for some $\beta \in K$. Since μ is $\precsim_K$-nondecreasing, by (H2), we obtain

$$f(p) \precsim_K \mu\left(\gamma_k f(z_k) + (1-\gamma_k) f(w)\right) = \mu\left(\lambda_k e - \beta\right) \precsim_K \mu\left(\lambda_k e\right) \ll_K \lambda_{k-1} e$$

which implies $p \in C_{k-1}$. Hence $co(D_{m,k-1}) \subseteq C_{k-1}$. Therefore, (9) is true by mathematic induction. For any $n \in \mathbb{N}$, let

$$U_n = \{x_i : i \geq n+1\}.$$

Then $co(U_n) \subseteq C_n$ for all $n \in \mathbb{N}$. Indeed, assume on the contrary that $co(U_{j^*}) \nsubseteq C_{j^*}$ for some $j^* \in \mathbb{N}$. So, there exist $z_{k_1}, z_{k_2}, \cdots, z_{k_s} \in U_{j^*}$ and $\alpha_1, \alpha_2, \cdots, \alpha_s \geq 0$ with $\sum\limits_{i=1}^{s} \alpha_i = 1$, such that $\sum\limits_{i=1}^{s} \alpha_i z_{k_i} \in co\left(D_{k_s-1,k_1-1}\right)$ and $\sum\limits_{i=1}^{s} \alpha_i z_{k_i} \notin C_{j^*}$. On the other hand, since $k_i \geq j^*+1$ for all $1 \leq i \leq s$, we have

$$D_{k_s-1,k_1-1} \subseteq D_{k_s-1,j^*}$$

and hence deduces from (9) that

$$co\left(D_{k_s-1,k_1-1}\right) \subseteq co\left(D_{k_s-1,j^*}\right) \subseteq C_{j^*},$$

which leads to a contradiction. Hence $co(U_n) \subseteq C_n$ for all $n \in \mathbb{N}$. By the closedness of C_n, we get

$$\overline{co}(U_n) \subseteq C_n \quad \text{for all } n \in \mathbb{N}.$$

Since $\overline{co}(U_{n+1}) \subseteq \overline{co}(U_n)$ and $\overline{co}(U_n)$ is weakly compact for all $n \in \mathbb{N}$, $\{\overline{co}(U_n) : n \in \mathbb{N}\}$ is a family of closed subsets of the weakly compact set $\overline{co}(U_1)$ which has the finite intersection property. Therefore we deduce

$$\emptyset \neq \bigcap_{n\in\mathbb{N}} \overline{co}(U_n) \subseteq \bigcap_{n\in\mathbb{N}} C_n$$

and hence we can take $v \in \bigcap\limits_{n\in\mathbb{N}} C_n \subseteq W$. So $F(v) \leq \lambda_n$ for all $n \in \mathbb{N}$. Since $\lambda_n \downarrow 0$ as $n \to \infty$, we get

$$F(v) \leq 0 \quad \Longleftrightarrow \quad f(v) \in -K.$$

The proof is completed. □

Corollary 3. *Let W be a nonempty weakly compact and convex subset of a normed linear space $(E, \|\cdot\|)$ with origin θ, $\tau : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function satisfying $\lim\limits_{x\to 0^+} \tau(x) = 0$ and $h : W \to \mathbb{R}$ be a function. Suppose that*

(a) for any positive real number γ, $\{x \in W : h(x) \leq \gamma\}$ is a nonempty closed subset of W,
(b) h is (τ)-adjconvex.

Then there exists $v \in W$ such that $h(v) \leq 0$.

Proof. Take $Y = \mathbb{R}$, $K = [0, +\infty) \subset \mathbb{R}$ and $e = 1$. Then Y is a locally convex Hausdorff t.v.s. with its zero vector $\theta = 0$, K is a proper, closed and convex pointed cone in Y with $intK = (0, +\infty) \neq \emptyset$, and $1 \in intK$. Define a partial ordering $\precsim_K$ with respect to K by

$$x \precsim_K y \Longleftrightarrow y - x \in K.$$

Then h is a mapping from W into Y and $\tau : Y \to Y$ is a $\precsim_K$-nondecreasing function satisfying the condition (A) as in Lemma 3. Clearly, conditions (a) and (b) respectively imply conditions (H1) and (H2) as in Theorem 1. Hence all the assumptions of Theorem 1 are satisfied and therefore the desired conclusion follows immediately from Theorem 1. □

As a direct consequence of Theorem 1, we obtain the following existence result.

Theorem 2. *In Theorem 1, if the condition (H1) is replaced with conditions (h1) and (h2), where*

(h1) f is (e, K)-lower semicontinuous;
(h2) for any positive real number γ, there exists $x \in W$ such that $f(x) \in \gamma e - K$.

Then there exists $v \in W$ such that $f(v) \in -K$.

Proof. For any positive real number γ, by (h1), (h2) and Lemma 2, the set

$$\{x \in W : f(x) \in \gamma e - K\}$$

is a nonempty closed subset of W. Therefore, the condition (H1) as in Theorem 1 holds. Applying Theorem 1, we can immediately obtain the conclusion. □

Corollary 4. *In Corollary 3, if the condition (a) is replaced with conditions (a1) and (a2), where*

(a1) h is lower semicontinuous;
(a2) for any positive real number γ, there exists $x \in W$ such that $h(x) \leq \gamma$.

Then there exists $v \in W$ such that $h(v) \leq 0$.

Applying Theorem 1, we can establish an existence theorem of zeros for vector-valued functions with K-adjustability convexity under an additional assumption.

Theorem 3. *In Theorem 1, if we further assume that $f(x) \in K$ for all $x \in W$, then the equation $f(x) = \theta$ has at least one root in W.*

Proof. By Theorem 1, there exists $v \in W$ such that $f(v) \in -K$. Therefore, by our hypothesis, we get

$$f(v) \in K \cap (-K) = \{\theta\},$$

which deduces $f(v) = \theta$. Hence v is a root of $f(x) = \theta$. The proof is completed. □

As an immediate consequence of Theorem 3, we obtain the following new existence theorem.

Corollary 5. *In Corollary 3 (or Corollary 4), if we further assume that $h(x) \geq 0$ for all $x \in W$, then the equation $h(x) = 0$ has at least one root in W.*

The following new existence and uniqueness theorem of zeros for vector-valued functions with strictly (K, μ)-adjconvexity is established by applying Theorem 3.

Theorem 4. *In Theorem 1, if we further assume $\mu(\theta) = \theta$, $f(x) \in K$ for all $x \in W$ and the condition (H2) is replaced with (H3), where*

(H3) f is strictly (K, μ)-adjconvex,

then the equation $f(x) = \theta$ has a unique root in W.

Proof. Applying Theorem 3, the equation $f(x) = \theta$ has at least one root in W. Assume that $u, v \in W$ are two distinct roots of $f(x) = \theta$. Since W is convex and $\mu(\theta) = \theta$, we have $\frac{1}{2}u + \frac{1}{2}v \in W$ and $\mu\left(\frac{1}{2}f(u) + \frac{1}{2}f(v)\right) = \theta$. By (H3), we get

$$\mu\left(\frac{1}{2}f(u) + \frac{1}{2}f(v)\right) - f\left(\frac{1}{2}u + \frac{1}{2}v\right) \in intK$$

which implies

$$f\left(\frac{1}{2}u + \frac{1}{2}v\right) \in -intK \cap K = \emptyset,$$

a contradiction. Therefore, the equation $f(x) = \theta$ has a unique root in W. The proof is completed. □

Corollary 6. *In Corollary 3, if we further assume that $\tau(0) = 0$, $h(x) \geq 0$ for all $x \in W$ and the condition (b) is replaced with*

(b1) h is strictly (τ)-adjconvex,

then the equation $h(x) = 0$ has a unique root in W.

As an interesting application of Corollary 5, we prove the following minimization theorem.

Theorem 5. *Let W be a nonempty weakly compact and convex subset of a normed linear space $(E, \|\cdot\|)$ with origin θ and $g : W \to \mathbb{R}$ be a convex, lower semicontinuous and bounded below function. Then*

$$\arg\min_{x \in W} g(x) := \left\{ y \in W : g(y) = \inf_{z \in W} g(z) \right\} \neq \emptyset.$$

Moreover, if g is strictly convex, then $\arg\min_{x \in W} g(x)$ is a singleton set.

Proof. Since g is bounded below, $\inf_{z \in W} g(z)$ exists. Let $h : W \to \mathbb{R}$ be defined by

$$h(x) = g(x) - \inf_{z \in W} g(z) \quad \text{for } x \in W.$$

Clearly, the following hold:

- $h(x) \geq 0$ for all $x \in W$,
- h is convex and lower semicontinuous.

Notice that for any $\gamma > 0$, there exists $x_\gamma \in W$ such that $g(x_\gamma) < \inf_{z \in W} g(z) + \gamma$. Thus, we have

- *For any positive real number γ, there exists $x \in W$ such that $h(x) \leq \gamma$.*

Applying Corollary 5, there exists $v \in W$ such that $h(v) = 0$, or equivalence, $g(v) = \inf_{z \in W} g(z)$. Hence $\arg\min_{x \in W} g(x) \neq \emptyset$. Assume that there exist $u, v \in \arg\min_{x \in W} g(x)$ with $u \neq v$. So $g(u) = g(v) = \inf_{z \in W} g(z)$. Since W is convex, we have $\frac{1}{2}u + \frac{1}{2}v \in W$. By the strict convexity of g, we get

$$g\left(\frac{1}{2}u + \frac{1}{2}v\right) < \frac{1}{2}g(u) + \frac{1}{2}g(v) = \inf_{z \in W} g(z)$$

which leads a contradiction. Therefore $\arg\min_{x \in W} g(x)$ is a singleton set. The proof is completed. □

Finally, by applying Theorem 5, we establish a new fixed point theorem which is original and quite different from the well-known generalizations in the literature.

Theorem 6. *Let W be a nonempty weakly compact and convex subset of a normed linear space $(E, \|\cdot\|)$ with origin θ and $T : W \to W$ be a affine and continuous mapping. If $\inf_{x \in W} \|Tx - x\| = 0$, then T admits a fixed point in X.*

Proof. Define $g : W \to [0, +\infty)$ by

$$g(x) = \|Tx - x\| \quad \text{for } x \in W.$$

Since T is affine and continuous, g is convex, continuous and bounded below function. By Theorem 5, $\arg\min_{x \in W} g(x) \neq \emptyset$. Therefore, there exists $v \in W$ such that

$$\|Tv - v\| = g(v) = \inf_{z \in W} g(z) = \inf_{z \in W} \|Tz - z\| = 0.$$

Hence, we get $Tv = v$. The proof is completed. □

Remark 1. *Theorems 1–6 and Corollaries 1–6 are completely original and quite different from the known results in the relevant literature.*

4. Conclusions

The convexity of functions or sets plays a significant role in almost all branches of mathematics, physics, economics and engineering. In this paper, we introduce the concepts of K-adjustability convexity and strictly K-adjustability convexity which respectively generalize and extend the concepts of K-convexity and strictly K-convexity. Some new existence and uniqueness theorems of zeros for vector-valued functions with K-adjustability convexity are established. As their applications, we obtain existence theorems for minimization problem and fixed point problem which are original and quite different from the known results in the relevant literature.

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