Weakly Semiprime Segments in Ordered Semigroups

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Abstract: Let $P_2 \subset P_1$ be a pair of weakly semiprime ideals of an ordered semigroup $(S, \cdot, \leq)$. Then, the pair $P_2 \subset P_1$ is called a weakly semiprime segment of S if $\bigcap_{n \in \mathbb{N}} I^n \subseteq P_2$ for all ideals I of S such that $P_2 \subset I \subset P_1$. In this paper, we classify weakly semiprime segments of an ordered semigroup into four types; those that are simple, exceptional, Archimedean, and decomposable.

Keywords: weakly semiprime; prime; segment; ordered semigroup

1. Introduction

The theory of ideals in rings has been widely studied. Although the ideal theories of semigroups and of rings are similar, the axioms for semigroups are simpler than rings, whence there it is possible to obtain generally only weaker structural results.

The above paragraph clarifies the relationship between the ideal theory of semigroups and that of rings. In fact, a semigroup is a specific case of ordered semigroups, then the ideal theory of rings is always investigated on semigroups, and this is investigated in ordered semigroups later. For instance, Brungs and Törner [1] studied prime segments on rings. Then, this concept was moved to semigroups by Ferrero, Mazurek, and Sant'Ana [2]. Finally, it was extended to ordered semigroups by Luangchaisri, Changphas, and Mazurek in [3].

On a ring $(R, +, \cdot)$, a semiprime segment is a pair $P_2 \subset P_1$ of semiprime ideals of R such that $\bigcap_{n \in \mathbb{N}} I^n \subseteq P_2$ for all ideals I of R with $P_2 \subset I \subset P_1$. This concept was introduced by R. Mazurek and G. Törner in [4]. They showed that semiprime segments on a ring can be classified into four types consisting of simple, exceptional, Archimedean, or decomposable. Moreover, they classified a semiprime segment $P_2 \subset P_1$ when P_1 is a comparizer right ideal of R , and they obtained that this semiprime segment occurred only in three types consisting of simple, exceptional, or Archimedean. Hence, the purpose of this paper is to describe and classify weakly semiprime segments on an ordered semigroup as a result in [4].

An ordered semigroup $(S, \cdot, \leq)$ consists of a semigroup $(S, \cdot)$ together with a partial order $\leq$ that is compatible with the semigroup operation, i.e., for any $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy, xz \leq yz.$$

For $\emptyset \neq A, B \subseteq S$, the set product AB is defined as follows:

$$AB = \{ab \mid a \in A, b \in B\}$$

and we define (A) by:

$$(A) = \{s \in S \mid s \leq a \text{ for some } a \in A\}.$$

For an ordered semigroup $(S, \cdot, \leq)$, it is observed (see [5]) that the following assertions hold:

1. $A \subseteq (A)$;

2. $A \subseteq B \Rightarrow (A] \subseteq (B]$;
3. $(A](B] \subseteq (AB]$;
4. $((A]) = (A]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup, and let A be a nonempty subset of S . Then, A is called a *right ideal* of S if $AS \subseteq A$; and $(A] = A$. We say that A is a *left ideal* of S if $SA \subseteq A$; and $(A] = A$. Finally, A is an *ideal* of S if it is both a left and a right ideal of S . It is easy to observe that:

1. If A and B are ideals of S , so are $(AB]$ and $A \cap B$;
2. $(SaS]$ is an ideal of S for all $a \in S$.

Next, we recall the concepts of prime ideals, weakly prime ideals, semiprime ideals, and weakly semiprime ideals of an ordered semigroup $(S, \cdot, \leq)$.

Definition 1. [5] Let P be an ideal of an ordered semigroup $(S, \cdot, \leq)$. Then:

1. P is prime if for all $A, B \subseteq S$, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$;
2. P is weakly prime if for all ideals A, B of S , $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$;
3. P is semiprime if for each $A \subseteq S$, $A^2 \subseteq P$ implies $A \subseteq P$;
4. P is weakly semiprime if for each ideal A of S , $A^2 \subseteq P$ implies $A \subseteq P$.

Remark 1. Let P be an ideal of an ordered semigroup $(S, \cdot, \leq)$. Then:

1. P is prime if for all $a, b \in S$, $ab \in P$ implies $a \in P$ or $b \in P$;
2. P is weakly prime if for all $a, b \in S$, $aSb \subseteq P$ implies $a \in P$ or $b \in P$;
3. P is semiprime if for each $a \in S$, $a^2 \in P$ implies $a \in P$;
4. P is weakly semiprime if for each $a \in S$, $aSa \subseteq P$ implies $a \in P$.

Lemma 1. [3] An ideal P of an ordered semigroup $(S, \cdot, \leq)$ is prime if and only if P is both semiprime and weakly prime.

Lemma 2. An ideal I of an ordered semigroup $(S, \cdot, \leq)$ is weakly semiprime if and only if it is the intersection of all weakly prime ideals of S containing I .

Proof. Assume that I is weakly semiprime. Let F be the intersection of all weakly prime ideals of S containing I . Suppose that there exists $x_0 \in F \setminus I$. Then:

$$x_0 s_0 x_0 \notin I$$

for some $s_0 \in S$. Let:

$$x_1 = x_0 s_0 x_0.$$

Then, there exists $s_1 \in S$ such that:

$$x_1 s_1 x_1 \notin I.$$

Continuing in a similar manner for $n \geq 2$, then we obtain the set:

$$A = \{x_0, x_1, x_2, \dots\}.$$

Let Q be the union of all ideals of S not containing any element of A . Then, $Q \neq \emptyset$ since $I \subseteq Q$. Let J, K be ideals of S such that $JK \subseteq Q$. If neither $J \subseteq Q$ nor $K \subseteq Q$, then:

$$x_n \in J \text{ and } x_m \in K$$

for some $n, m \in \mathbb{N}$. In the case $n \geq m$, we obtain that:

$$x_{n+1} \in x_n S x_n \subseteq x_n S x_m \subseteq JK \subseteq Q.$$

This is a contradiction. The case $m \geq n$ can be proven similarly. Thus, Q is a weakly prime ideal of S containing I and $x_0 \notin Q$. This contradicts $x_0 \in F$. Hence, $F = I$. The opposite direction is clear. $\square$

2. Results

Let $(S, \cdot, \leq)$ be an ordered semigroup. An element e of S is called an *identity* if $ea = a = ae$ for every $a \in S$. This section introduces a weakly semiprime segment on S , which is defined as a semiprime segment on rings. Then, we classify weakly semiprime segments on an ordered semigroup S according to a classification of semiprime segments on rings. Moreover, we give an example of each type of weakly semiprime segments on S to show that every type of this can occur. From now on, let $(S, \cdot, \leq)$ be an ordered semigroup with identity.

Definition 2. A pair $P_2 \subset P_1$ of weakly semiprime ideals is called a weakly semiprime segment if for every ideal I of S such that $P_2 \subset I \subset P_1$,

$$\bigcap_{n \in \mathbb{N}} (I^n) \subseteq P_2.$$

Example 1. Consider a semigroup $\mathbb{Z}_{30}$ under the usual multiplication. It is clear that $\bar{0}$ and $\bar{6}\mathbb{Z}_{30}$ are semiprime ideals of $\mathbb{Z}_{30}$. Since there are no ideals properly contained in the interval $\bar{0} \subset \bar{6}\mathbb{Z}_{30}$, we have that $\bar{0} \subset \bar{6}\mathbb{Z}_{30}$ is a semiprime segment of a semigroup $\mathbb{Z}_{30}$.

Definition 3. Let $P_2 \subset P_1$ be a weakly semiprime segment of an ordered semigroup $(S, \cdot, \leq)$. Then, $P_2 \subset P_1$ is simple if there is no further ideals of S between P_2 and P_1 , and P_2 is comparable with each ideal of S contained in P_1 .

Definition 4. Let $P_2 \subset P_1$ be a weakly semiprime segment of an ordered semigroup $(S, \cdot, \leq)$. Then, $P_2 \subset P_1$ is exceptional if there exists a weakly semiprime ideal Q of S such that $P_2 \subset Q \subset P_1$ and Q is comparable with each ideal of S contained in P_1 .

Definition 5. Let $P_2 \subset P_1$ be a weakly semiprime segment of an ordered semigroup $(S, \cdot, \leq)$. Then, $P_2 \subset P_1$ is Archimedean if for each $a \in P_1 \setminus P_2$, there exists an ideal I of S with $a \in I \subseteq P_1$ and $\bigcap_{n \in \mathbb{N}} (I^n) \subseteq P_2$.

Definition 6. Let $P_2 \subset P_1$ be a weakly semiprime segment of an ordered semigroup $(S, \cdot, \leq)$. Then, $P_2 \subset P_1$ is decomposable if $P_2 \subset P_1$ is not Archimedean and $P_1 = A \cup B$ where A, B are ideals of S properly contained in P_1 .

The following examples show various types of weakly semiprime segments on ordered semigroups.

Example 2. Let $S = \{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 2\} \cup \{0\}$ be a semigroup under the usual multiplication. Let $P_1 = S$ and $P_2 = \{0\}$. We can observe that $P_2 \subset P_1$ is a semiprime segment of S . For each $a \in P_1 \setminus P_2$, we have that P_1 is an ideal of S containing a , which satisfies the condition:

$$\bigcap_{n=1}^{\infty} P_1^n \subseteq P_2.$$

Thus, a semiprime segment $P_2 \subset P_1$ is Archimedean.

Example 3. Let us consider the semigroup with identity $S = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ under the usual multiplication. If we define $P_1 = S$ and $P_2 = \{0\}$, then a pair $P_2 \subset P_1$ is a semiprime segment of S . Given $I = \{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 2\} \cup \{0\}$, then I is a semiprime ideal of S and $P_2 \subset I \subset P_1$. Thus, a semiprime segment $P_2 \subset P_1$ is exceptional.

Example 4. Let $S = \{0, x, a, b, b^2, b^3, \dots\}$, and define a binary operation on S as the following table:

$\cdot$	0	x	a	b	b^2	b^3	$\dots$
0	0	0	0	0	0	0	$\dots$
x	0	x	0	0	0	0	$\dots$
a	0	0	a	b	b^2	b^3	$\dots$
b	0	0	b	b^2	b^3	b^4	$\dots$
b^2	0	0	b^2	b^3	b^4	b^5	$\dots$
b^3	0	0	b^3	b^4	b^5	b^6	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then, S is a semigroup. Let $P_1 = S$ and $P_2 = \{0, x\}$. It is clear that a pair $P_2 \subset P_1$ is a semiprime segment of S . We can observe that $I = \{0, a, b, b^2, \dots\}$ and S are only ideals of S containing b . Since:

$$\bigcap_{n=1}^{\infty} I^n \not\subseteq P_2 \text{ and } \bigcap_{n=1}^{\infty} S^n \not\subseteq P_2,$$

it follows that $P_2 \subset P_1$ is not Archimedean. Moreover,

$$P_1 = \{0, x\} \cup \{0, a, b, b^2, b^3, \dots\}$$

implies that $P_2 \subseteq P_1$ is decomposable.

Theorem 1. Let $P_2 \subset P_1$ be a weakly semiprime segment of an ordered semigroup $(S, \cdot, \leq)$. Then, exactly one of the following possibilities occurs:

1. There are no further ideals of S between P_2 and P_1 , and P_2 is comparable with each ideal of S contained in P_1 ;
2. There exists a weakly semiprime ideal Q of S such that $P_2 \subset Q \subset P_1$, and Q is comparable with each ideal of S contained in P_1 ;
3. $(P_1 a S \cup S a P_1] \subset (S a S]$ for all $a \in P_1 \setminus P_2$;
4. $P_1 = A \cup B$ for some ideals A, B properly contained in P_1 .

Proof. Let $a \in P_1 \setminus P_2$. First of all, assume that $(P_1^2 \cup P_2] \subset P_1$. If $a \in (P_1 a S \cup S a P_1]$, then:

$$a \in (P_1^n a S^n \cup S^n a P_1^n] \subseteq (P_1^n]$$

for all positive integers n . This implies that:

$$a \in \bigcap_{n=1}^{\infty} (P_1^n] = \bigcap_{n=1}^{\infty} (P_1^{2n}] \subseteq \bigcap_{n=1}^{\infty} ((P_1^2 \cup P_2)^n] \subseteq P_2.$$

This contradicts $a \notin P_2$. Thus, $a \notin (P_1 a S \cup S a P_1]$, and so, $(P_1 a S \cup S a P_1] \subset (S a S]$. Next, assume that $(P_1^2 \cup P_2] = P_1$. Let M be the union of all ideals of S properly contained in P_1 . Then:

$$P_2 \subseteq M \subseteq P_1.$$

If $P_2 = M$, then (1) occurs. Otherwise, we have only two cases to consider:

Case 1. $P_2 \subset M \subset P_1$. Then, M is a weakly semiprime ideal of S , and (2) occurs. Indeed: Let A be an ideal of S such that $A^2 \subseteq M$. Then, $A^2 \subseteq P_1$, which implies $A \subseteq P_1$ since P_1 is weakly semiprime. If $A \subset P_1$, we have that $A \subseteq M$. Otherwise,

$$P_1 = (P_1^2 \cup P_2] = (A^2 \cup P_2] \subseteq M.$$

This implies that $A \subseteq M$.

Case 2. $P_2 \subset M = P_1$. For each $b \in P_1$, we have that $(SbS] \subset P_1$ since $M = P_1$. Suppose that (3) does not occur. Then, there exists $c \in P_1 \setminus P_2$ such that $(P_1cS \cup ScP_1] = (ScS]$. That is:

$$c \in (pcS \cup Scq]$$

for some $p, q \in P_1$. Set $I = (SpS \cup SqS]$. Then, I is an ideal of S and:

$$c \in (pcS \cup Scq] \subseteq (SpScS \cup ScSqS] \subseteq (IcS \cup ScI].$$

If $(I \cup P_2] \subset P_1$, then:

$$c \in \bigcap_{n=1}^{\infty} (I^n] \subseteq \bigcap_{n=1}^{\infty} ((I \cup P_2)^n] \subseteq P_2,$$

a contradiction. Thus, $P_1 = I \cup P_2$. $\square$

By using Theorem 1, a weakly semiprime segment $P_2 \subset P_1$ of an ordered semigroup $(S, \cdot, \leq)$ can be classified into four types.

Theorem 2. If $P_2 \subset P_1$ is a weakly semiprime segment of an ordered semigroup $(S, \cdot, \leq)$, then exactly one of the following possibilities occurs:

1. $P_2 \subset P_1$ is simple;
2. $P_2 \subset P_1$ is exceptional;
3. $P_2 \subset P_1$ is Archimedean;
4. $P_2 \subset P_1$ is decomposable.

Proof. Assume that a pair $P_2 \subset P_1$ of semiprime ideals of S is a semiprime segment of S . By Theorem 1, there are four cases to consider. It is clear that if (1), (2), or (4) occurs, then a semiprime segment $P_2 \subset P_1$ is simple, exceptional, or decomposable, respectively. Assume that:

$$(P_1aS \cup SaP_1] \subset (SaS]$$

for all $a \in P_1 \setminus P_2$. If $P_2 \subset P_1$ is not Archimedean, then there is $b \in P_1 \setminus P_2$ such that for all ideals I of S with $b \in I \subseteq P_1$,

$$\bigcap_{n=1}^{\infty} (I^n] \not\subseteq P_2.$$

Suppose that $((SbS)^2] \cup P_2 \subset P_1$. Then, we have that:

$$\bigcap_{n=1}^{\infty} ((SbS)^n] = \bigcap_{n=1}^{\infty} ((SbS)^{2n}] \subseteq \bigcap_{n=1}^{\infty} (((SbS)^2 \cup P_2)^n] \subseteq P_2.$$

This contradicts $P_2 \subset P_1$ being not Archimedean. Thus, $((SbS)^2] \cup P_2 = P_1$. By assumption, we obtain that:

$$((SbS)^2] \subseteq (P_1bS] \subset (SbS] \subseteq P_1.$$

Hence, a semiprime segment $P_2 \subset P_1$ is decomposable. It is obvious that every type of a semiprime segment is mutually exclusive. $\square$

Definition 7. Let R be a right ideal of an ordered semigroup $(S, \cdot, \leq)$. Then, R is said to be a waist of S if for all right ideal A of S , either $R \subseteq A$ or $A \subset R$.

Definition 8. Let R be a proper right ideal of an ordered semigroup $(S, \cdot, \leq)$. Then, R is said to be a comparizer right ideal if for any $a, b \in S$, either $(aS] \subseteq (bS]$ or $(bR] \subseteq (aS]$.

Lemma 3. Let R be a right ideal of an ordered semigroup $(S, \cdot, \leq)$. Then, R is a comparizer of S if and only if for each right ideal A, B of S ,

$$A \not\subseteq B \text{ implies } (BR] \subset A.$$

Proof. Assume that R is a comparizer right ideal of S . Let A, B be right ideals of S such that $A \not\subseteq B$. Then, there exists $a \in A$ such that:

$$(aS] \not\subseteq (bS]$$

for all $b \in B$. By assumption, we obtain that:

$$(bR] \subseteq (aS] \subseteq A.$$

Thus,

$$(BR] \subseteq A.$$

Conversely, let $a, b \in S$ be such that $(aS] \not\subseteq (bS]$. Then:

$$(bR] \subseteq (bSR] = ((bS]R] \subseteq (aS].$$

Thus, R is a comparizer right ideal of S . $\square$

Theorem 3. For any ordered semigroup $(S, \cdot, \leq)$, the following right ideals are comparizer right ideals.

1. The union of a family of comparizer right ideals of S ;
2. The ideal $I(R)$ of S generated by a comparizer right ideal R of S ;
3. A right ideal of S contained in a comparizer right ideal of S .

Proof. (1) Let $C = \cup\{A_\alpha \mid \alpha \in \Lambda\}$, where A_α is a comparizer right ideal of S . Let $a, b \in S$ such that $(aS] \not\subseteq (bS]$. Then:

$$(bA_\alpha] \subseteq (aS]$$

for all $\alpha \in \Lambda$. That is:

$$(bC] = (b(\bigcup_{\alpha \in \Lambda} A_\alpha)] = (\bigcup_{\alpha \in \Lambda} bA_\alpha] \subseteq (aS].$$

Thus, C is a comparizer right ideal of S .

(2) Let R be a comparizer right ideal of S , and let A, B be right ideals of S such that $A \not\subseteq B$. Then:

$$\begin{aligned} (BI(R)] &= (B(SRS)] \\ &= (BSRS] \\ &= (BR] \\ &\subseteq (A] \\ &= A. \end{aligned}$$

Therefore, $I(R)$ is a comparizer right ideal of S .

(3) Let R' be a right ideal of S contained in a comparizer right ideal R of S . Let $a, b \in S$ be such that:

$$(aS] \not\subseteq (bS].$$

Then:

$$(bR') \subseteq (bR) \subseteq (aS).$$

Thus, R' is a comparizer right ideal of S . $\square$

Theorem 4. Let R be a comparizer right ideal of an ordered semigroup $(S, \cdot, \leq)$. Then, the following statements hold:

1. If $R = (R^2]$, then R is a waist of S ;
2. If P is a weakly prime ideal of S such that $R \not\subseteq P$, then P is a waist of S and $P \subset R$.

Proof. (1) Assume that $R = (R^2]$. Let A be a right ideal of S such that $A \not\subseteq R$. Then:

$$R = (R^2] \subseteq A$$

implies that R is a waist of S .

(2) Assume that P is a weakly prime ideal of S such that $R \not\subseteq P$. Let A be an ideal of S such that $A \not\subseteq P$. Then, $(AR] \not\subseteq P$ since P is weakly prime. It follows that $P \subset A$ since R is a comparizer right ideal of S . Thus, P is a waist of S . $\square$

Note that the set of weakly prime ideals of S contained in a comparizer ideal is linearly ordered by inclusion.

Theorem 5. Let R be a comparizer right ideal of an ordered semigroup $(S, \cdot, \leq)$. If R is an ideal of S , then for each ideal Q of S such that $Q \subset R$, Q is weakly semiprime if and only if Q is weakly prime.

Proof. Assume that Q is weakly semiprime. Then, $Q = \bigcap_{\alpha \in \Lambda} P_\alpha$, where P_α is a weakly prime ideal of S containing Q . By Theorem 4, there exists $\alpha \in \Lambda$ such that $P_\alpha \subset R$. Let A, B be ideals of S such that $AB \subseteq Q$. If neither $A \subseteq Q$ nor $B \subseteq Q$, then $A \not\subseteq P_\beta$ and $B \not\subseteq P_\gamma$ for some $\beta, \gamma \in \Lambda$. Now, we have four cases to consider:

Case 1: If $R \subseteq P_\beta$ and $R \subseteq P_\gamma$, then:

$$A \not\subseteq P_\beta \supseteq R \supset P_\alpha$$

and:

$$B \not\subseteq P_\gamma \supseteq R \supset P_\alpha.$$

These contradict P_α being weakly prime.

Case 2: If $P_\beta \subset R \subseteq P_\gamma$, then:

$$A \not\subseteq P_\beta$$

and:

$$B \not\subseteq P_\gamma \supset P_\beta.$$

These contradict P_β being weakly prime.

Case 3: If $P_\gamma \subset R \subseteq P_\beta$, we can prove this similarly to Case 2.

Case 4: If $P_\gamma \subset R$ and $P_\beta \subset R$, then we have:

$$P_\gamma \subseteq P_\beta \text{ or } P_\beta \subseteq P_\gamma.$$

This case does not occur as Cases 2 and 3.

Thus, $A \subseteq Q$ or $B \subseteq Q$. Therefore, Q is weakly prime. The opposite direction is clear. $\square$

Theorem 6. Let $P_2 \subset P_1$ be a weakly semiprime segment of an ordered semigroup $(S, \cdot, \leq)$. If P_1 is a comparizer right ideal of S , then one of the following possibilities occurs:

1. The weakly semiprime segment $P_2 \subset P_1$ is simple. In this case, P_2 is a weakly prime waist ideal of S ;

2. The weakly semiprime segment $P_2 \subset P_1$ is exceptional. In this case, there exists a weakly prime waist ideal Q of S with $P_2 \subset Q \subset P_1$ such that there are no further ideals between Q and P_1 . Moreover, P_2 is a weakly prime waist ideal of S ;
3. The weakly semiprime segment $P_2 \subset P_1$ is Archimedean. In this case, P_2 is a weakly prime waist ideal of S .

Proof. Assume that a weakly semiprime segment P_1 is a comparizer right ideal of S . By Theorem 5, we have that P_2 is a weakly prime ideal of S . Moreover, P_2 is also a waist of S by Theorem 4. By using Theorem 1, there are four cases that have to be proven. It is clear that Theorem 1(1) implies (1). Assume that Theorem 1(2) holds. Then there is a weakly semiprime ideal Q with $P_2 \subset Q \subset P_1$. According to the proof of Theorem 2, we obtain immediately that there are no further ideals between Q and P_1 . Thus, this case implies that (2) holds.

Next, assume that Theorem 1(3) or Theorem 1(4) occurs. If a weakly semiprime segment $P_2 \subset P_1$ is not Archimedean, by the proof of Theorem 2, then $P_2 \subset P_1$ is decomposable. That is $P_1 = A \cup B$ for some ideals A, B of S properly contained in P_1 . Since P_2 is a waist of S , then there are four cases to consider:

- (α) $P_2 \subset A$ and $P_2 \subset B$. Then, we obtain immediately that $P_2 \subset P_1$ is Archimedean. This contradicts the hypothesis. Therefore, this case does not occur.
- (β) $P_2 \subset A$ and $B \subseteq P_2$. For each $a \in P_1 \setminus P_2$, we have that $a \in A \subseteq P_1$ and $\bigcap_{n=1}^{\infty} A^n \subseteq P_2$. This implies that a weakly semiprime segment $P_2 \subset P_1$ is Archimedean. This is a contradiction.
- (γ) $A \subset P_2$ and $P_2 \subseteq B$. It can be proven similarly to the case (β) that this case does not occur.
- (δ) $A \subseteq P_2$ and $B \subseteq P_2$. Then, $P_1 = A \cup B = P_2$. This case does not occur.

Hence, if Theorem 1(3) or Theorem 1(4) occurs, we can infer that a weakly semiprime segment $P_2 \subseteq P_1$ is Archimedean. $\square$

3. Conclusions

Weakly semiprime segments defined on an ordered semigroup $(S, \cdot, \leq)$ can be classified into four types according to a classification of semiprime segments on rings. In particular, if a weakly semiprime segment $P_2 \subset P_1$ has the property that P_1 is a comparizer right ideal of S , then a weakly semiprime segment $P_2 \subset P_1$ is not decomposable. This implies that a weakly prime segment $P_2 \subset P_1$ can be only simple, exceptional, or Archimedean.

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