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Hopf Bifurcation of Heated Panels Flutter in Supersonic Flow

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Abstract: A differential equation of panel vibration in supersonic flow is established on the basis of the thin-plate large deflection theory under the assumption of a quasi-steady temperature field. The equation is dimensionless, and the derivation of its second-order Galerkin discretization yields a four-dimensional system. The algebraic criterion of the Hopf bifurcation is applied to study the motion stability of heated panels in supersonic flow. We provide a supplementary explanation for the proof process of a theorem, and analytical expressions of flutter dynamic pressure and panel vibration frequencies are derived. The conclusion is that the algebraic criterion of Hopf bifurcation can be applied in high-dimensional problems with many parameters. Moreover, the computational intensity of the method established in this work is less than that of conventional eigenvalue computation methods using parameter variation.

Keywords: panels; hopf-bifurcation; thermal flutter; Hurwitz determinant

1. Introduction

Aircraft siding and skin, which are surrounded and fixed on a skeletal structure by an adhesive or rivet, form the dimensional member of an aerodynamic shape. The panel and skin of the airfoil structure have good strength and stiffness. Thus, they can withstand and transmit aerodynamic loads while maintaining flight stability. The flight stability of high-speed aircraft is related to structural fatigue life and flight safety, and this relationship has attracted increasing attention from aircraft designers [1,2]. Therefore, research on the stability and instability mechanism of the panels is important in overall aircraft design. Panel flutter is a self-excited vibration phenomenon, which is caused by the coupling effect of the elastic force of aircraft surface skin, aerodynamic load and inertia force in supersonic air flow. Xue and Mei incorporated thermal stress into the dynamic equation of the flutter structure, and applied the finite element frequency domain method to study the flutter problem and the fatigue life of two-dimensional panels at any temperature in supersonic airflow [3,4]. Cheng and Mei et al. studied all of the possible types of panel behavior by means of finite time-domain model formulation. In addition, they investigated the effects of airflow declination, temperature change and aerodynamic damping on the stable boundary of panel flutter [5,6]. Azzouz and Mei provided new insights of curved panels flutter, and Newton-Raphson iteration and eigenvalue solutions were used to determine the panel deflection and flutter critical dynamic pressure, respectively[7]. Li and Song investigated the aerothermoelastic characteristics of laminated panels, and a method for the thermal flutter control of composite laminated panels in supersonic air flow [8,9].

Panel flutter is usually interpreted as a limit cycle oscillation. Once Hopf bifurcation occurs, the panel will oscillate for ever, which is known as panel flutter. Zhang et al. investigated both the local and global bifurcations of a rectangular, thin plate [10]. Ye studied the nonlinear aeroelastic flutter and stability of the panel [11].Yang et al. also provided a study on the nonlinear thermal flutter of heated curved panels in supersonic air flow using the Newton iterative approach and Runge-Kutta method [12]. In Hopf bifurcation, the equilibrium point changes from stable to unstable, and grows out of the limit cycle when the nonlinear system parameter changes by a critical value. Li et al. studied the dynamic behaviors of panel structures on supersonic aircrafts, by seeking the eigenvalus of the Jacobi matrix of the dynamic system at bifurcation points [13]. Monfared Z. devoted to study the partial differential equation governing panel motion from the Hopf bifurcation point of view by the fourth and fifth-order Runge-Kutta method [14]. Zhang investigated the stability and bifurcation behaviors of a two-dimensional, nonlinear, viscoelastic panel in supersonic flow, using analytical and numerical methods [15]. Avramov et al. analyzed the dynamic instability of the parabolic shells in a supersonic gas flow numerically [16]. Chen et al. studied the coefficients of the characteristic equation of the first approximation system and its corresponding Hurwitz determinant, which is used to derive the algebraic criterion of Hopf bifurcation. This method transforms the problem of searching for the system bifurcation point into the problem of solving the root of the nonlinear equation [17,18]. Zhang et al. developed the analytic expression of critical speed for the serpentine movement of a nonlinear vehicle wheelset system on the basis of this method [19].

In this study, a differential equation of panel vibration in supersonic flow is established on the basis of the thin-plate large deflection theory. The equation is dimensionless, and is transformed into a four-dimensional ordinary differential equation system by second-order Galerkin discretization derivation. The flutter dynamic pressure happens to correspond to the Hopf bifurcation point. The traditional method of finding this Hopf bifurcation point is by means of solving the characteristic equation, and judging when a pair of complex conjugate eigenvalues of the Jacobi matrix pass through the imaginary axis, while all the others have negative real parts. Even though a numerical computation of eigenvalues is feasible, the method mentioned above is difficult and tedious. Moreover, computation is expensive. Therefore, it is more ideal to have a method stated in terms of the coefficients of the characteristic equations, which is called as the algebraic criterion of Hopf bifurcation. The main objective of this paper is to study the motion stability of heated panels in supersonic flow by the algebraic criterion. The Hopf bifurcation point is found by solving an algebraic equation of the bifurcation parameter, and thus analytical expressions of flutter dynamic pressure and panel vibration frequencies are derived. Furthermore, we provide a supplementary explanation for the proof process of a theorem.

2. Differential Equation of Heated Panel Flutter in Supersonic Flow

Figure 1 shows a two-dimensional panel with infinitely extended sides (see [11,20]), where L is length, h is thickness, and ρ is density. The upper surface of the panel is subject to supersonic airflow along the x-direction with density ρ_∞, speed U_∞, and match number M_∞.

By using the thin-plate large deflection theory, assuming that the temperature field is quasi-steady, and considering only lateral panel vibration, then the differential equations for the oscillation of two-dimensional heated panels with infinitely elongated supersonic flow can be obtained as (see [20]):

$$Dw^{(4)} - \left[\frac{Eh}{2(1-\mu^2)L}\int_0^L (w')^2 dx - N_{\mathrm{T}}h\right]w'' + \rho h\ddot{w} - q_a = 0 \tag{1}$$

where $D = \frac{Eh^3}{12(1-\mu^2)}$ is the bending stiffness of the panel, and E and μ stand for the modulus of elasticity and Poisson's ratio of the panel material, respectively.

In addition, w is the displacement along the z-direction. $w^{(4)}$, w'' and $\ddot{w}$ correspond to the fourth-order partial derivative of w to x, second-order partial derivative of w to x, and second-order partial derivative of w to t, respectively. $N_{\mathrm{T}} = \frac{E}{1-\mu}\alpha\Delta T$ represents the temperature stress caused by the temperature variation ΔT, and α is the coefficient of linear thermal expansion. q_a is aerodynamic force. The expansion of aerodynamic force q_a based on the first-order piston theory can be given as follows:

$$q_a = -\frac{2q_\infty}{\beta}\left(\frac{\partial w}{\partial x} + \frac{M_\infty^2 - 2}{M_\infty^2 - 1}\frac{1}{U_\infty}\frac{\partial w}{\partial t}\right) \tag{2}$$

where $q_\infty = \frac{\rho_\infty U_\infty^2}{2}$ is flow pressure, $\beta = \sqrt{M_\infty^2 - 1}$ is the Prandtl–Glauert factor.

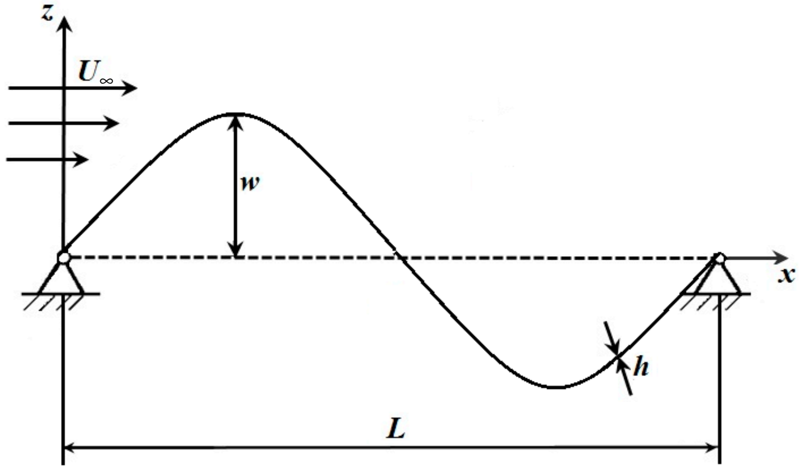

Figure 1. Mechanical model of heated panels in supersonic flow.

For convenient analysis, the dimensionless parameter is incorporated and given by

$$\overline{w} = \frac{w}{h}, \overline{x} = \frac{x}{L}, \tau = \frac{t}{T} \tag{3}$$

where $w = h\overline{w}$, $x = L\overline{x}$, $t = T\tau$. Then the dimensionless differential equations for the oscillation of two-dimensional heated panels in supersonic flow can be derived as follows:

$$\overline{w}^{(4)} - \left[6\int_0^1 \left(\overline{w}'\right)^2 d\overline{x} - R_T\right]\overline{w}'' + \ddot{\overline{w}} + p_1\overline{w}' + p_2\dot{\overline{w}} = 0 \tag{4}$$

>where p_1 and p_2 stand for the aerodynamic stiffness coefficient and aerodynamic damping coefficient, respectively. $R_T = \frac{N_T h L^2}{D}$ is a dimensionless parameter. The corresponding expressions are given by

$$\begin{aligned} p_1 &= \frac{L^3\rho_\infty U_\infty^2}{D\sqrt{M_\infty^2 - 1}} = \frac{L^3 a}{D}, \quad a = \frac{\rho_\infty U_\infty^2}{\sqrt{M_\infty^2 - 1}} \\ p_2 &= \frac{L^2\rho_\infty U_\infty\left(M_\infty^2 - 2\right)}{\sqrt{\rho h D}\left(M_\infty^2 - 1\right)^{\frac{3}{2}}} = \frac{L^2 b}{\sqrt{\rho h D}}, \quad b = \frac{\rho_\infty U_\infty\left(M_\infty^2 - 2\right)}{\left(M_\infty^2 - 1\right)^{\frac{3}{2}}} \end{aligned} \tag{5}$$

where the two-dimensional panel is simply supported on the opposite side, and the boundary condition is given by

$$\overline{w}(0,\tau) = \overline{w}(1,\tau) = 0, \quad \frac{\partial^2\overline{w}(0,\tau)}{\partial\overline{x}^2} = \frac{\partial^2\overline{w}(1,\tau)}{\partial\overline{x}^2} = 0 \tag{6}$$

$\varphi_i(\overline{x}) = \sin i\pi\overline{x}$ is set as the trial function that satisfies boundary condition (6), and $q_i(\tau)$ is the generalized coordinates. Then the displacement variable of the two dimensional panel is expressed as follows:

$$\overline{w}(\overline{x},\tau) = \sum_{i=1}^{N} q_i(\tau)\sin(i\pi\overline{x}) = \sum_{i=1}^{N} q_i(\tau)\varphi_i(\overline{x}) \tag{7}$$

where N is the Galerkin truncated order. $\boldsymbol{q} = \begin{bmatrix} q_1(\tau) \\ q_2(\tau) \end{bmatrix}$ and $\boldsymbol{\varphi} = \begin{bmatrix} \varphi_1(\bar{x}) \\ \varphi_2(\bar{x}) \end{bmatrix}$ are set. Then, Equation (4) is given by:

$$\left[\boldsymbol{\varphi}^{(4)}\right]^T \boldsymbol{q} - \left[6\int_0^1 \left(\boldsymbol{\varphi}'^T \boldsymbol{q}\right)^2 d\bar{x} - R_T\right]\boldsymbol{\varphi}''^T \boldsymbol{q} + \boldsymbol{\varphi}^T \ddot{\boldsymbol{q}} + p_1 \boldsymbol{\varphi}'^T \boldsymbol{q} + p_2 \boldsymbol{\varphi}^T \dot{\boldsymbol{q}} = 0 \tag{8}$$

Equation (8) is multiplied at both ends and integrated upon the interval [0,1] on the basis of the Galerkin method and the orthogonality of main mode. Then, it can be expressed as follows:

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + p_2 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} \pi^4 - R_T\pi^2 & -\frac{8}{3}p_1 \\ \frac{8}{3}p_1 & 16\pi^4 - 4R_T\pi^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = 0 \tag{9}$$

where $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 3\pi^4 q_1^3 + 12\pi^4 q_1 q_2^2 \\ 12\pi^4 q_1^2 q_2 + 48\pi^4 q_2^3 \end{bmatrix}$.

If we set $\dot{q}_1 = q_3$, $\dot{q}_2 = q_4$, then $\ddot{q}_1 = \dot{q}_3$, $\ddot{q}_2 = \dot{q}_4$. Thus, the controlling equation of the two-dimensional panel system is transformed into a four-dimensional first-order nonlinear ordinary differential equation

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\pi^4 - R_T\pi^2\right) & \frac{8}{3}p_1 & -p_2 & 0 \\ -\frac{8}{3}p_1 & -\left(16\pi^4 - 4R_T\pi^2\right) & 0 & -p_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ F_1 \\ F_2 \end{bmatrix} \tag{10}$$

3. Algebraic Criterion of the Hopf Bifurcation

For a system with parameters

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu) \tag{11}$$

where $\boldsymbol{x} \in R^n$ is the state variable, and $\mu \in R$ is bifurcation parameter. The isolated balance point of system (11) is $\boldsymbol{x} = \boldsymbol{x}_0(\mu)$ $(f(\boldsymbol{x}_0(\mu), \mu) = 0)$, which can consistently be transformed to the coordinate origin. $\boldsymbol{f}(\boldsymbol{x}, \mu)$ is assumed to be analyzed to $\boldsymbol{x}$ and μ in the neighborhood of $(0, 0) \in R^{n\times 1}$, and $\boldsymbol{f}(\boldsymbol{x}, \mu) \equiv 0$. The Jacobi matrix of system (11) at balance point $\boldsymbol{x} = 0$ is given by

$$A(\mu) = D_x(0, \mu) \tag{12}$$

The classical Hopf theorem [19] is expressed as follows:

Theorem 1. *(1) $A(\mu) = D_x(0, \mu)$ has a pair of complex roots λ and $\overline{\lambda}$, $\lambda(\mu) = \alpha(\mu)+i\omega(\mu)$, where $\omega(\mu_0) = \omega_0 > 0, \alpha(\mu_0) = 0, \alpha'(\mu_0) \neq 0$; (2) The remaining n-2 roots of $A(\mu_0)$ have negative real parts. Then, the system (11) will obtain a Hopf branch at parameter $\mu = \mu_0$. Thus, periodic motion solution exists at $\mu = \mu_0$.*

The characteristic equation $\det[A(\mu) - \lambda E] = 0$ of the Jacobi matrix $A(\mu)$ can be expressed as follows:

$$\lambda^n + a_1(\mu)\lambda^{n-1} + a_2(\mu)\lambda^{n-2} + \cdots + a_{n-1}(\mu)\lambda + a_n(\mu) = 0 \tag{13}$$

Theorem 2. *The necessary and sufficient condition for the real coefficient algebraic Equation (13) to have a pair of pure imaginary zeros and the remaining n-2 roots have negative real parts is*

$$(1) \quad a_1 > 0,\, a_2 > 0,\, \cdots, a_n > 0;$$

$$(2) \quad \Delta_{n-1} = 0,\, \Delta_i > 0 (i = n-3,\, n-5, \cdots)$$

where $a_i(i = 1, 2, \cdots, n)$ is the coefficient of Equation (13), and $\Delta_i(i = 1, 2, \cdots, n)$ is the Hurwitz determinant derived from $a_i(i = 1, 2, \cdots, n)$, which is given by:

$$\Delta_m = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{2m-1} & a_{2m-2} & a_{2m-3} & a_{2m-4} & \cdots & a_{2m} \end{vmatrix} \quad (m = 1, 2, \cdots, n) \tag{14}$$

If $i > n$, then $a_i = 0$.

Theorem 3. *If the real coefficient algebraic Equation (13) has a pair of pure virtual roots and the remaining n-2 roots have negative real parts, then*

$$\omega^2 = \frac{\Delta_{n-3}}{\Delta_{n-2}} a_n \tag{15}$$

Proof. The real coefficient algebraic Equation (13) has a pair of pure *imaginary zeros*. Then Equation (13) can be written as

$$\left(\lambda^2 + \omega^2\right)\left(\lambda^{n-2} + b_1\lambda^{n-3} + b_2\lambda^{n-4} + \cdots + b_{n-3}\lambda + b_{n-2}\right) = 0 \tag{16}$$

Following the title, all the roots of the equation given by

$$\lambda^{n-2} + b_1\lambda^{n-3} + b_2\lambda^{n-4} + \cdots + b_{n-3}\lambda + b_{n-2} = 0 \tag{17}$$

have negative real parts. The Equation (16) is expanded and compared with the coefficient of Equation (13), which can be derived as:

$$\begin{array}{l} a_1 = b_1, \quad a_2 = b_2 + \omega^2 \\ a_i = b_i + \omega^2 b_{i-2}, \quad i = 3, 4, \cdots, n-2 \\ a_{n-1} = \omega^2 b_{n-3}, \quad a_n = \omega^2 b_{n-2} \end{array} \tag{18}$$

according to the definition of Equation (14), the Hurwitz determinant derived from Equation (17) is $\overline{\Delta}_i(i = 1, 2, \cdots, n-2)$. The highest-order Hurwitz determinant of Equation (17) with n−2 order is the main determinant$\overline{\Delta}_{n-2}$, which has the following relationship with (n−2)-1 order diagonal determinant $\overline{\Delta}_{n-3}$, and is given by

$$\overline{\Delta}_{n-2} = b_{n-2}\overline{\Delta}_{n-3} \tag{19}$$

No relationship similar to that of Equation (19) exists between the other n-4 adjacent two Hurwitz determinants of Equation (17).

By contrast, Equation (18) is incorporated into Equation (14) to derive the following Equation (20), which is based on the determinant characteristic.

$$\overline{\Delta}_i = \Delta_i(i = 1, 2, \cdots, n-2) \tag{20}$$

Thus

$$b_{n-2} = \frac{\overline{\Delta}_{n-2}}{\overline{\Delta}_{n-3}} = \frac{\Delta_{n-2}}{\Delta_{n-3}} \tag{21}$$

Substitute Equation (21) into $a_n = \omega^2 b_{n-2}$ in Equation (18), noting that $\overline{\Delta}_0 = \Delta_{-1} = 1$, then Equation (15) is derived. □

The above proof process, with respect to the literature [18], provides the appropriate explanations and supplements.

Theorem 4. *The feature Equation (13) of the Jacobi matrix $A(\mu)$ of system (11) is assumed to have feature roots with negative real parts at $\mu = \mu_0$, and the Hurwitz determinant satisfies the following equation:*

$$\Delta_{n-3}(\mu_c) > 0$$

where the definition of μ_c is the same that in Equation (18). Then Equation (13) has a pair of pure virtual roots $\pm i\omega_c$ at $\mu = \mu_c$, and the remaining $n-2$ roots have negative real parts. Assumptions W and V are the left and right eigenvectors of the Jacobi matrix $A(\mu_c)$ that belong to the eigenvalue $i\omega_c$, respectively, and WV = 1. If

$$\mathrm{Re}(WBV) \neq 0$$

where $B = \left.\frac{dA(\mu)}{d\mu}\right|_{\mu=\mu_c}$, $\mu_c = min\left\{\left|\mu - \mu_0\right| : \Delta_{n-1}(\mu) = 0\right\}$, then system (11) will present Hopf bifurcation at $\mu = \mu_c$.

4. Hopf Bifurcation and Flutter of Heated Panels in Supersonic Flow

4.1. Panel Flutter in Supersonic Flowobtained by Hopf Bifurcation Criterion

First, the Hopf bifurcation point of heated panels in supersonic flow will be identified.

The balance point of system (10) is $X_0(0,0,0,0)$, and the corresponding Jacobi matrix of the balance point is

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\left(\pi^4 - R_T\pi^2\right) & \frac{8}{3}p_1 & -p_2 & 0 \\ -\frac{8}{3}p_1 & -\left(16\pi^4 - 4R_T\pi^2\right) & 0 & -p_2 \end{bmatrix} \tag{22}$$

The corresponding characteristic equation is

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{23}$$

where the coefficient of each term is

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 2p_2 \\ a_2 &= 17\pi^4 - 5\pi^2R_T + p_2^2 \\ a_3 &= \left(17\pi^4 - 5\pi^2R_T\right)p_2 \\ a_4 &= 16\pi^8 - 20\pi^6R_T + 4\pi^4R_T^2 + \frac{64}{9}p_1^2 \end{aligned} \tag{24}$$

Owing to the aerodynamic damping coefficient $p_2 = \frac{L^2b}{\sqrt{\rho hD}} > 0$, we only need to set $R_T < \frac{17}{5}\pi^2$.

Then the first condition (14*a*) of theorem 2 will be satisfied, that is, the coefficients $a_i > 0 (i = 0, 1, 2, 3, 4)$ of all terms of Equation (24) established. Second, the corresponding Hurwitz determinants of each order are computed as follows:

$$\begin{aligned} \Delta_1 &= a_1 = 2p_2 \\ \Delta_2 &= p_2\left(17\pi^4 - 5\pi^2R_T\right) + 2p_2^3 \\ \Delta_3 &= p_2^2\left(225\pi^8 - 90\pi^6R_T + 9\pi^4R_T^2 + 34\pi^4p_2^2 - 10\pi^2R_Tp_2^2 - \frac{256}{9}p_1^2\right) \end{aligned} \tag{25}$$

$\Delta_1 > 0$ and $\Delta_2 > 0$, then the critical flow speed could be computed as follows when $\Delta_3 = 0$:

$$U_{cr}^2 = \frac{9}{256}\frac{D}{\rho h L^2}\left(17\pi^4 - 5\pi^2 R_T\right) + \frac{9}{256}\frac{\pi^2 D M_\infty}{\rho_\infty L^3}\sqrt{N} \tag{26}$$

where $N = \frac{\rho_\infty^2 L^2}{(\rho h)^2 M_\infty^2}\left(17\pi^2 - 5R_T\right)^2 + \left(80\pi^2 - 16R_T\right)^2$.

When $p_1 = p_{1cr} = \frac{L^3\rho_\infty}{M_\infty}U_{cr}^2$, Equation (23) will have a pair of pure imaginary zeros $\pm i\omega$. On the basis of Theorem 3, we can derive the following result

$$\omega^2 = \frac{\Delta_1}{\Delta_2}a_4 = \frac{2\left(16\pi^8 - 20\pi^6 R_T + 4\pi^4 R_T^2 + \frac{64}{9}p_{1\mathrm{cr}}^2\right)}{17\pi^4 - 5\pi^2 R_T + 2p_2^2} \tag{27}$$

where $Q = 17\pi^4 - 5\pi^2 R_T$. Vector $W = (x_1, x_2, x_3, x_4)$ and $V = (y_1, y_2, y_3, y_4)^T$ are assumed to be the normalized left and right eigenvectors of matrix A that belong to eigenvalues $\pm i\omega$. On the basis of $WA = i\omega W$, $AV = i\omega V$, $WV = 1$, W and V can be given by

$$W = u\begin{pmatrix} \frac{3}{8p_1}\left[\left(16\pi^4 - 4R_T\pi^2\right)(p_2 + i\omega) + i\omega(p_2 + i\omega)^2\right], p_2 + i\omega, \\ \frac{3}{8p_1}\left[\left(16\pi^4 - 4R_T\pi^2\right) + i\omega(p_2 + i\omega)\right], 1 \end{pmatrix} \tag{28}$$

$$V = \left(1, \frac{3}{8p_1}\left[\left(\pi^4 - R_T\pi^2\right) - \left(\omega^2 - i\omega p_2\right)\right], i\omega, \frac{3}{8p_1}\left[i\omega\left(\pi^4 - R_T\pi^2\right) - \left(i\omega^3 + \omega^2 p_2\right)\right]\right)^T \tag{29}$$

where $u = \frac{8p_1}{3p_2(Q-6\omega^2) + 6i\omega\left(Q - 2\omega^2 + p_2^2\right)}$ On the basis of theorem 4, $B = \left.\frac{dA(p_1)}{dp_1}\right|_{\mu=p_{1cr}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & 0 & 0 \\ -\frac{8}{3} & 0 & 0 & 0 \end{bmatrix}$, the pure virtual roots $\pm i\omega$ at the neighborhood of parameter $p_1 = p_{1cr} = \frac{L^3\rho_\infty}{M_\infty}U_{cr}^2$ can be expressed as

$$\zeta_{1,2}(p_1) = \alpha(p_1) \pm i\omega(p_1) \tag{30}$$

where $\alpha(p_1) = 0$, $\omega(p_1) > 0$. Then

$$W(p_1)A(p_1)V(p_1) = \alpha(p_1) + i\omega(p_1) \tag{31}$$

when $p_1 = p_{1cr} = \frac{L^3\rho_\infty}{M_\infty}U_{cr}^2$

$$\begin{aligned} \alpha'(p_{1cr}) &= \mathrm{Re}(W(p_{1cr})B(p_{1cr})V(p_{1cr})) \\ &= \frac{\widetilde{a} + \widetilde{b}}{9p_1\left[p_2^2 S^2 + 4\omega^2\left(Q - 2\omega^2 + p_2^2\right)^2\right]} > 0 \end{aligned} \tag{32}$$

where $S = Q - 6\omega^2$, $\widetilde{a} = 18\left(Q - 2\omega^2 + p_2^2\right)\omega^2\left(Qp_2 - 2\omega^2 p_2\right)$, $\widetilde{b} = p_2 S\left[9\left(S_1 - Q\omega^2 + \omega^4 - p_2^2\omega^2\right) - 64p_1^2\right]$, $S_1 = \left(16\pi^4 - 4R_T\pi^2\right)\left(\pi^4 - R_T\pi^2\right)$. Therefore, the system (10) will present a Hopf bifurcation at $p_1 = p_{1cr} = \frac{L^3\rho_\infty}{M_\infty}U_{cr}^2$. Thus, the panel system will undergo flutter in supersonic flow at $U = U_{\mathrm{cr}}$.

4.2. Numerical Example

Take the mechanical properties of aluminum alloy, for example [12]. The mechanical properties are shown in Table 1.

Table 1. The mechanical properties of aluminum alloy.

Young's Modulus E/GPa	Poisson's Ratio μ	Mass Density ρ/(kg·m^{-3})	Thermal Expression Coefficient α/(10^{-6}K^{-1})
78.55	0.3	2710	22.68

The geometry of the panel is defined by length $L = 0.5$m, and thickness $h = 0.002$m. Suppose the flight height of a certain aircraft is 11 km.Therefore, the parameters of air flow are defined by mass density, sound velocity $a_\infty = 295.065\text{m/s}$, and Mach number $M_\infty = 5$. According to the analysis mentioned above, the bending stiffness $D = 57.546\text{N} \cdot \text{m}$ can be obtained. Further, the values of the parameters of the characteristic polynomical of the Jacobi matrix, which is showed in Equation (24), and that of Hurwitz determinant in Equation (25) are computed as follows:

$$a_0 = 1,\ a_1 = 2.51248,\ a_2 = 1308.34,\ a_3 = 1641.61,\ a_4 = 427937.36\Delta_1 = 2.51248,\ \Delta_2 = 1645.57$$

As is seen, the coefficients of Equation (23), Δ_1 and Δ_2 are all positive numbers.

The critical speed value $U_{\text{cr}} = 1219.0157\text{m/s}$ is done by means of making $\Delta_3 = 0$. The flutter frequency$\omega = 25.5613$Hz and $\alpha'(p_{1\text{cr}}) = 1.01735 > 0$ are obtained. In this case, the panel undergoes its Hopf bifurcation at the equilibrium $X_0(0,0,0,0)$. In other words, the heated panel flutter in supersonic flow occurred at dynamic pressure $p_{1\text{cr}} = 234.99$. Its Hopf bifurcation point is found by Equation (26), which is the analytical expression of the flutter dynamic pressure. The analytical expressions of panel vibration frequencies are derived by Equation (27) at the same time. When the dimensionless vibration dynamic pressure $p_1 < p_{1\text{cr}}$ is satisfied, the heated panel in supersonic flow is stable at the equilibrium point. While the dimensionless vibration dynamic pressure $p_1 > p_{1\text{cr}}$ is satisfied, the heated panel in supersonic flow is unstable.

On the other hand, eigenvalues of the Jacobi matrix for different parameters are calculated by the aid of software. Suppose that the dimensionless vibration dynamic pressures are 200, 234.99 and 260. The eigenvalues of the Jacobi matrix for different pressures are shown in Table 2.

Table 2. Eigenvalues of the Jacobi matrix for different pressures.

Dimensionless Vibration Dynamic Pressure p_1	Eigenvalues of Jacobi Matrix
200	$-0.579477 \pm 31.3121i$ $-0.579477 \pm 18.0455i$
234.99	$-1.25624 \pm 25.5613i$ $\pm 25.5613i$
260	$-6.36051 \pm 26.1808i$ $5.0391 \pm 26.1808i$

The calculation results show that there exists a pair of pure imaginary zeros, while the other eigenvalues have negative parts for the matrix, where the dimensionless vibration dynamic pressure is 234.99. This has verified the results obtained from the the algebraic criterion of the Hopf bifurcation. Moreover, the function of eigenvalues for the Jacobi matrix, with the flutter dynamic pressure as an independent variable, is hard to get. Difficult and tedious numerical simulation must be carried out, and then a numerical solution is obtained. Just as in Table 2, when the dimensionless vibration dynamic pressure is $200 < p_{1\text{cr}}$, all the eigenvalues have negative parts. That is to say, any disturbance at the equilibrium point will converge the equilibrium position by passing the time. When the dimensionless vibration dynamic pressure is $260 > p_{1\text{cr}}$, the eigenvalues have positive parts. The equilibrium point is unstable, and is surrounded by a stable limit cycle. That means the amplitude of panel vibration will increase and finally vibrate forever.

5. Conclusions

In this study, we established a differential equation for the vibration of heated panels in supersonic flow on the basis of the thin-plate large deflection theory under the assumption of a quasi-steady temperature field. According to the Galerkin method, the equation was applied to a second-order Galerkin discretization, that yielded a four-dimensional ordinary differential system. We provided a supplementary explanation for the proof process of a theorem. Different from the traditional method of finding the Hopf bifurcation point, the analytic expressions of critical speed (or flutter critical dynamic pressure) and flutter frequency under panel flutter, are obtained by means of the algebraic criterion of the Hopf bifurcation, which need not calculate all eigenvalues of our Jacobi matrix for any parameter, thus saving large amounts of computer time. A numerical example is taken to verify the results. The eigenvalues of the Jacobi matrix for different pressures are calculated, which are in accordance with the above results. It is hard to get the analytic relationship between eigenvalues of this Jacobi matrix and the flutter dynamic pressure. Thus, difficult and tedious numerical simulation must be carried out. Our method shows that the algebraic criterion of Hopf bifurcation can be effectively applied in high-dimensional problems with many parameters. The computational intensity of the method present in this paper will be greatly reduced relative to that of traditional methods, in which we should calculate the eigenvalues of different parameters and judge their real parts as negative.

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