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Ground State Solution of Pohožaev Type for Quasilinear Schrödinger Equation Involving Critical Exponent in Orlicz Space

Jianqing Chen † and Qian Zhang *

College of Mathematics and Informatics & FJKLMAA, Fujian Normal University, Fuzhou 350117, China
* Correspondence: fxsx@fjnu.edu.cn
† These authors contributed equally to this work.

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Abstract: We study the following quasilinear Schrödinger equation involving critical exponent $-\Delta u + V(x)u - \Delta(u^2)u = A(x)|u|^{p-1}u + \lambda B(x)u^{\frac{3N+2}{N-2}}$, $u(x) > 0$ for $x \in \mathbb{R}^N$ and $u(x) \to 0$ as $|x| \to \infty$. By using a monotonicity trick and global compactness lemma, we prove the existence of positive ground state solutions of Pohožaev type. The nonlinear term $|u|^{p-1}u$ for the well-studied case $p \in [3, \frac{3N+2}{N-2})$, and the less-studied case $p \in [2, 3)$, and for the latter case few existence results are available in the literature. Our results generalize partial previous works.

Keywords: quasilinear Schrödinger equation; ground state solution; pohožaev identity

1. Introduction and Main Results

In this paper, we consider the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u - \Delta(u^2)u = A(x)|u|^{p-1}u + \lambda B(x)u^{22^*-1}, & x \in \mathbb{R}^N, \\ u(x) \to 0 \text{ as } |x| \to \infty, \quad u(x) > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geq 3$, $22^* := 2 \times 2^* = \frac{4N}{N-2}$, $1 < p < 22^* - 1$, $\lambda > 0$. The solutions of Equation (1) are related to the existence of standing waves of the following quasilinear elliptic equations

$$i\partial_t z = -\Delta z + V(x)z - l(|z|^2)z - k\Delta g(|z|^2)g'(|z|^2)z, \ x \in \mathbb{R}^N, \tag{2}$$

where V is a given potential, $k \in \mathbb{R}$, l and g are real functions. Quasilinear Equation (2) has been derived as models of several physical phenomena (see e.g., [1–3] and the references therein). In recent years, extensive studies have been focused on the existence of solutions for quasilinear Schrödinger equations of the form

$$-\Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) = g(x, u), \ x \in \mathbb{R}^N. \tag{3}$$

One of the main difficulties of Equation (3) is that there is no suitable space on which the energy functional is well defined and belongs to C^1-class except for $N = 1$ (see [4]). In [5], for pure power nonlinearities, Liu and Wang proved that Equation (3) has a ground state solution by using a change of variables and treating the new problem in an Orlicz space when $3 \leq p < 22^* - 1$ and the potential $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies

(v_1) $\inf_{x \in \mathbb{R}^N} V(x) \geq a > 0$ and for each $M > 0$, $\text{meas}\{x \in \mathbb{R}^N \mid V(x) \leq M\} < +\infty$.

Such kind of hypotheses was firstly introduced by Bartsch and Wang [6] to ensure the compactness of embeddings of $E_0 := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)u^2 < \infty\} \hookrightarrow L^s(\mathbb{R}^N)$, where $2 < s < 2^*$. In [7], for $g(x,u) = |u|^{p-1}u$, $3 \le p < 22^* - 1$, Liu and Wang established the existence of both one-sign and nodal ground states of soliton type solutions for Equation (3) by the Nehari method under the assumptions on $V(x)$,

$$(v_2)\ V(x) \in C(\mathbb{R}^N, \mathbb{R}),\ \text{ and }\ 0 < \inf_{\mathbb{R}^N} V(x) \le V_\infty := \lim_{|x|\to\infty} V(x) < +\infty,$$

$$(v_3)\ \text{ there are positive constants } M,\ K \text{ and } m \text{ such that for } |x| \ge M,\ V(x) \le V_\infty - \frac{K}{1+|x|^m}.$$

Very recently, when $A(x) \equiv 1, p \in [3, 22^* - 1)$, Equation (1) without $\lambda B(x)|u|^{22^*-1}$, Xu and Chen [8] studied the existence of positive ground state solution with the help of global compactness Lemma. See also related results obtained in [9–11]. All the ground state solutions obtained in [5,7,8] are only valid for $|u|^{p-1}u, p \in [3, 22^* - 1)$. In [12], under the assumption that

$$(v_4)\ 0 < V_0 \le V(x) \le V_\infty = \lim_{|x|\to\infty} V(x) < +\infty,\ (\nabla V(x), x) \in L^\infty(\mathbb{R}^N),$$

$$(v_5)\ s \mapsto s^{\frac{N+2}{N+p+1}} V(s^{\frac{1}{N+p+1}} x) \quad \text{is concave for any } x \in \mathbb{R}^N.$$

Ruiz and Siciliano showed Equation (3) with the subcritical growth has ground state solutions for $N \ge 3, g(x,u) = u^{p-1}u, 1 < p < 22^* - 1$ via Nehari-Pohožaev manifold.

To the best of our knowledge, there is no result in the literature on the existence of positive ground state solutions of Pohožaev type to the problem in Equation (1) with critical term. The first purpose of the present paper is to prove the existence of positive ground state solutions of Pohožaev type to the problem in Equation (1) with critical term. Since the approaches in [5,7,8,13], when applied to the monomial nonlinearity $f(u) = |u|^{p-1}u$, are only valid for $p \in [3, 22^* - 1)$, we want to provide an argument which covers the case $p \in [2,3)$ and this is the second purpose of the present paper. Moreover, our argument does not depend on existence of the Nehari manifold.

Before state our main results, we make the following assumptions.

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R}^+), 0 < \inf_{x\in\mathbb{R}^N} V(x) =: V_0 \le V(x) \le V_\infty = \lim_{|x|\to\infty} V(x) < +\infty$ and $V(x) \not\equiv V_\infty$;
(V_2) $\langle \nabla V(x), x\rangle \in L^\infty(\mathbb{R}^N)$, $\langle \nabla V(x), x\rangle \le 0$, $x \in \mathbb{R}^N$;
(A) $A \in C(\mathbb{R}^N, \mathbb{R})$, $\lim_{|x|\to\infty} A(x) = A_\infty \in (0,\infty), A(x) \ge A_\infty, 0 \le \langle \nabla A(x), x\rangle \in L^\infty(\mathbb{R}^N), x \in \mathbb{R}^N$;
(B) $B \in C(\mathbb{R}^N, \mathbb{R})$, $\lim_{|x|\to\infty} B(x) = B_\infty \in (0,\infty), B(x) \ge B_\infty, 0 \le \langle \nabla B(x), x\rangle \in L^\infty(\mathbb{R}^N), x \in \mathbb{R}^N$.

It is worth noting that the similar hypotheses on $V(x)$ as above (V_1) and (V_2) are introduced in [14–16] and have physical meaning. Moreover, there are indeed many functions satisfying (V_1) and (V_2). For instance, $V(x) = V_0 + \frac{1}{1+|x|}$. Under conditions analogous to (A), (B), Zhao and Zhao [17] obtained the positive solutions of Schrödinger-Maxwell equations with the case $p \in (2, 2^*)$.

Our main result reads as follows.

Theorem 1. *Let $V(x)$, $A(x)$ abd $B(x)$ be positive constants. If $\lambda > 0$ is sufficiently large, then the problem in Equation (1) has a positive ground state solution for $N \ge 3, 1 < p < 22^* - 1$.*

Theorem 2. *Under the assumptions (V_1), (V_2), (A) and (B), the problem in Equation (1) has a positive ground state solution for $N \ge 3, 2 \le p < 22^* - 1$ and sufficiently large $\lambda > 0$.*

Remark 1. *As mentioned above, the results and methods in [5,7,8,18], when applied to the subcritical nonlinearity $f(u) \sim |u|^{p-1}u$, are only valid for $p \in [3, 22^* - 1)$; however, our result covers the case $p \in [2, 22^* - 1)$. Hence, our results extend those established in the literature.*

Remark 2. *The novelty of this works with respect to some recent results is that we treat the existence by using Pohožaev manifold method in an Orlicz space. The idea of Pohožaev manifold has been used in [8,12], where the authors studied problems with subcritical nonlinearity. It is worthy noting that their argument cannot be applied to our problem due to the presence of the critical term.*

The rest of the paper is organized as follows. In Section 2, we state the variational framework of our problem and some preliminary results. The proof of Theorem 1 is contained in Section 3. Section 4 is devoted to establishing a global compactness lemma and proving Theorem 2.

2. Preliminaries and Functional Setting

Let $L^s(\mathbb{R}^N)(1 \le s < +\infty)$ be the usual Lebesgue space with norm $\|\cdot\|_{L^s} := \int_{\mathbb{R}^N} |\cdot|^s$. $H^1(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) \mid \nabla u \in L^2(\mathbb{R}^N)\}$ is the standard Sobolev space with norm $\|u\|_H^2 := \int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)$. We formally formulate the problem in Equation (1) in a variational structure as follows

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1+2u^2)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} A(x)|u|^{p+1} - \frac{\lambda}{22^*}\int_{\mathbb{R}^N} B(x)|u|^{22^*} \tag{4}$$

for $u \in H^1(\mathbb{R}^N)$. From a variational point of view, J is not well defined in $H^1(\mathbb{R}^N)$, which prevents us from applying variational methods directly. To overcome this difficulty, we employ an idea from Colin and Jeanjean [19]. First, we make a change of variables $v = f^{-1}(u)$, where $f(t)$ is defined by $f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}$ on $[0, +\infty)$ and $f(-t) = -f(t)$ on $(-\infty, 0]$. By the following lemma, we collect some properties of f.

Lemma 1. *([5]) The function f satisfies the following properties:*
(f_1) *f is uniquely defined C^∞ and invertible;*
(f_2) $0 < f'(t) \le 1, \ t \in \mathbb{R}$;
(f_3) $0 < |f(t)| \le |t|, \ t \in \mathbb{R}$;
(f_4) $\lim\limits_{t\to 0} \frac{f(t)}{t} = 1$;
(f_5) $\lim\limits_{t\to\infty} \frac{f(t)}{\sqrt{t}} = 2^{\frac{1}{4}}$;
(f_6) $\frac{f(t)}{2} \le tf'(t) \le f(t), \ t \ge 0$;
(f_7) $|f(t)| \le 2^{\frac{1}{4}}\sqrt{|t|}, \ t \in \mathbb{R}$;
(f_8) *the function $f^2(t)$ is strictly convex;*
(f_9) *there exists a positive constant θ such that*

$$|f(t)| \ge \begin{cases} \theta|t|, & |t| \le 1, \\ \theta\sqrt{|t|}, & |t| \ge 1; \end{cases}$$

(f_{10}) *there exist positive constant C_1 and C_2 such that*

$$|t| \le C_1|f(t)| + C_2|f(t)|^2, \ t \in \mathbb{R};$$

(f_{11}) $|f(t)f'(t)| \le \frac{1}{\sqrt{2}}, \ t \in \mathbb{R}$.

Thus, after the above change of variables, we can write the functional $J(u)$ as

$$I_V(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v) - \frac{1}{p+1}\int_{\mathbb{R}^N} A(x)|f(v)|^{p+1} - \frac{\lambda}{22^*}\int_{\mathbb{R}^N} B(x)|f(v)|^{22^*}. \tag{5}$$

Under the assumptions (V_1), (V_2), (A) and (B), I_V is well defined and $I_V \in C^1(E,\mathbb{R})$ on the Orlicz space ([20])

$$E := \left\{ v \in \mathbb{R}^N \;\middle|\; \int_{\mathbb{R}^N} V(x)f^2(v) < +\infty \right\}$$

endowed with the norm

$$\|v\| = |\nabla v|_{L^2} + \inf_{\xi>0}\left[1 + \int_{\mathbb{R}^N} V(x)f^2(\xi v)\right]$$

and

$$\begin{aligned}\langle I_V'(v), w\rangle = &\int_{\mathbb{R}^N} (\nabla v \nabla w + V(x)f(v)f'(v)w) - \int_{\mathbb{R}^N} A(x)|f(v)|^{p-1}f(v)f'(v)w \\ &- \lambda\int_{\mathbb{R}^N} B(x)|f(v)|^{22^*-2}f(v)f'(v)w\end{aligned} \tag{6}$$

for any $w \in E$. Moreover, if v is a critical point for the functional I_V, then v is a solution for the equation

$$-\Delta v = f'(v)(-V(x)f(v) + A(x)|f(v)|^{p-1}f(v) + \lambda B(x)|f(v)|^{22^*-2}f(v)) \text{ in } \mathbb{R}^N. \tag{7}$$

Therefore, $u = f(v)$ is a solution of the problem in Equation (1) ([19]).

Lemma 2. *([7,21]) Under (V_1), the map: $v \to f(v)$ from E into $L^s(\mathbb{R}^N)$ is continuous for $2 \le s \le 22^*$, and E is continuously embedded into $L^s(\mathbb{R}^N)$ for $2 \le s < 22^*$; If $N \ge 2$, $V(x)$ is radially symmetric, i.e., $V(x) = V(|x|)$, the above map is compact for $2 < s < 22^*$.*

Next, we prove a Pohožaev identity with respect to the problem in Equation (7), which plays a significant role in constructing a new manifold.

Lemma 3. *Under the assumptions (V_1), (V_2), (A) and (B), if $v \in E$ is a weak solution of Equation (7), then $f(v)$ satisfies the following Pohožaev identity:*

$$\begin{aligned}0 = &\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V(x)|f(v)|^2 + \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |f(v)|^2 \\ &- \frac{N}{p+1}\int_{\mathbb{R}^N} A(x)|f(v)|^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^N} \langle \nabla A(x), x\rangle |f(v)|^{p+1} \\ &- \frac{\lambda N}{22^*}\int_{\mathbb{R}^N} B(x)|f(v)|^{22^*} - \frac{\lambda}{22^*}\int_{\mathbb{R}^N} \langle \nabla B(x), x\rangle |f(v)|^{22^*}.\end{aligned} \tag{8}$$

Proof. We only prove it formally. For any given positive constant R, $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$. Let $u_i := \frac{\partial u}{\partial x_i}$ and $\mathbf{n}$ be the unit outer normal at ∂B_R. By the divergence theorem, we have

$$\begin{aligned}\int_{B_R} \operatorname{div}((x\cdot\nabla u)\nabla u) &= \int_{B_R} \Delta v(x\cdot\nabla u) + \int_{B_R} |\nabla u|^2 + \frac{1}{2}\int_{B_R} \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}\left(|\nabla u|^2\right) \\ &= \int_{\partial B_R} \left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 R dS.\end{aligned} \tag{9}$$

Next, by using

$$\operatorname{div}\left(\frac{1}{2}|\nabla u|^2 x\right) = \frac{N}{2}|\nabla u|^2 + \frac{1}{2}\sum_{k=1}^N x_i \frac{\partial}{\partial x_i}\left(|\nabla u|^2\right),$$

and the divergence theorem

$$\begin{aligned}\int_{B_R} \operatorname{div}\left(\frac{1}{2}|\nabla u|^2 x\right) &= \frac{N}{2}\int_{B_R}|\nabla u|^2 + \frac{1}{2}\int_{B_R}\sum_{k=1}^{N} x_i \frac{\partial}{\partial x_i}\left(|\nabla u|^2\right) \\ &= \frac{1}{2}\int_{\partial B_R}|\nabla u|^2 R dS.\end{aligned} \tag{10}$$

By Equations (9) and (10), one has

$$\int_{B_R} \Delta u(x\cdot\nabla u) = \frac{N-2}{2}\int_{B_R}|\nabla u|^2 + \int_{\partial B_R}\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 R dS - \frac{1}{2}\int_{\partial B_R}|\nabla u|^2 R dS. \tag{11}$$

Note that u is a solution of Equation (1); it follows from integration by parts that

$$-\int_{B_R} \Delta u(x\cdot\nabla u) = -\frac{N-2}{2}\int_{B_R}|\nabla u|^2 - \int_{\partial B_R}\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 R dS + \frac{1}{2}\int_{\partial B_R}|\nabla u|^2 R dS. \tag{12}$$

$$\begin{aligned}&\int_{B_R}\left[-V(x)u + \Delta(u^2)u + A(x)|u|^{p-1}u + \lambda B(x)u^{2(2^*)-1}\right](x\cdot\nabla u) \\ =& -\int_{B_R} V(x)u(x\cdot\nabla u) + \int_{B_R}(2u^2\Delta u + 2u|\nabla u|^2)u(x\cdot\nabla u) + \int_{B_R} A(x)|u|^{p-1}u(x\cdot\nabla u) \\ &+ \int_{B_R}\lambda B(x)u^{2(2^*)-1}(x\cdot\nabla u) \\ =& -\frac{1}{2}\int_{\partial B_R} V(x)|u|^2 R dS + \frac{N}{2}\int_{B_R} V(x)|u|^2 dx + \frac{1}{2}\int_{B_R}\langle\nabla V(x), x\rangle|u|^2 \\ &+ (N-2)\int_{B_R}|u|^2|\nabla u|^2 + 2\int_{\partial B_R} u^2\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 R dS - \int_{\partial B_R} u^2|\nabla u|^2 R dS \\ &+ \frac{1}{p+1}\int_{\partial B_R} A(x)|u|^{p+1} R dS - \frac{N}{p+1}\int_{B_R} A(x)|u|^{p+1} \\ &- \frac{1}{p+1}\int_{B_R}\langle\nabla A(x), x\rangle|u|^{p+1} + \frac{\lambda}{22^*}\int_{\partial B_R} B(x)|u|^{22^*} R dS - \frac{\lambda N}{22^*}\int_{B_R} B(x)|u|^{22^*} \\ &- \frac{\lambda}{22^*}\int_{B_R}\langle\nabla B(x), x\rangle|u|^{22^*}.\end{aligned} \tag{13}$$

We get by Equations (7) and (12) that

$$\begin{aligned}&\frac{N-2}{2}\int_{B_R}|\nabla u|^2 + (N-2)\int_{B_R}|u|^2|\nabla u|^2 + \frac{N}{2}\int_{B_R} V(x)|u|^2 + \frac{1}{2}\int_{B_R}\langle\nabla V(x), x\rangle|u|^2 \\ &- \frac{N}{p+1}\int_{B_R} A(x)|u|^{p+1} - \frac{\lambda}{p+1}\int_{B_R}\langle\nabla A(x), x\rangle|u|^{p+1} - \frac{\lambda N}{22^*}\int_{B_R} B(x)|u|^{22^*} \\ &- \frac{\lambda}{22^*}\int_{B_R}\langle\nabla B(x), x\rangle|u|^{22^*} \\ =& R\int_{\partial B_R}\left[\frac{|\nabla u|^2 - V(x)|u|^2}{2} - u^2|\nabla u|^2 + (2u^2-1)\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 + \frac{A(x)|u|^{p+1}}{p+1} + \frac{\lambda B(x)|u|^{22^*}}{22^*}\right] dS.\end{aligned} \tag{14}$$

Next, we show that the right hand side of Equation (14) converges to 0 for at least one suitably chosen sequence $R_n \to +\infty$. Since

$$
\begin{aligned}
+\infty &> \int_{\mathbb{R}^N} \left| \frac{|\nabla u|^2}{2} - \frac{V(x)|u|^2}{2} - u^2|\nabla u|^2 + (2u^2-1)\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 + \frac{A(x)|u|^{p+1}}{p+1} + \frac{\lambda B(x)|u|^{22^*}}{22^*} \right| \\
&= \int_0^{+\infty} \left(\int_{\partial B_R} \left| \frac{|\nabla u|^2}{2} - \frac{V(x)|u|^2}{2} - u^2|\nabla u|^2 + (2u^2-1)\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 + \frac{A(x)|u|^{p+1}}{p+1} \right.\right. \\
&\quad \left.\left. + \frac{\lambda B(x)|u|^{22^*}}{22^*} \right| dS \right) dR,
\end{aligned}
\tag{15}
$$

there exists a sequence $R_n \to +\infty$ such that

$$
\begin{aligned}
&R_n \int_{\partial B_{R_n}} \left| \frac{|\nabla u|^2}{2} - \frac{V(x)|u|^2}{2} - u^2|\nabla u|^2 + (2u^2-1)\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 + \frac{A(x)|u|^{p+1}}{p+1} + \frac{\lambda B(x)|u|^{22^*}}{22^*} \right| dS \\
&\to 0 \ \text{ as } n \to +\infty.
\end{aligned}
$$

Indeed, if

$$
\begin{aligned}
&\liminf_{R\to+\infty} R \int_{\partial B_R} \left| \frac{|\nabla u|^2}{2} - \frac{V(x)|u|^2}{2} - u^2|\nabla u|^2 + (2u^2-1)\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 + \frac{A(x)|u|^{p+1}}{p+1} \right. \\
&\quad \left. + \frac{\lambda B(x)|u|^{22^*}}{22^*} \right| dS = \alpha > 0,
\end{aligned}
$$

then there exists $0 < \alpha' < \alpha$ such that if $R >> 1$,

$$
\begin{aligned}
\Phi(R) := &\int_{\partial B_R} \left| \frac{|\nabla u|^2}{2} - \frac{V(x)|u|^2}{2} - u^2|\nabla u|^2 + (2u^2-1)\left(\frac{\partial u}{\partial \mathbf{n}}\right)^2 + \frac{A(x)|u|^{p+1}}{p+1} \right. \\
&\quad \left. + \frac{\lambda B(x)|u|^{22^*}}{22^*} \right| dS > \frac{\alpha'}{R},
\end{aligned}
$$

therefore, $\Phi(R)$ would not be in $L^1(0,+\infty)$, which contradicts Equation (15), implying that

$$
\begin{aligned}
&\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (N-2)\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V(x)|u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u|^2 \\
&- \frac{N}{p+1}\int_{\mathbb{R}^N} A(x)|u|^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^N} \langle \nabla A(x), x\rangle |u|^{p+1} - \frac{N}{22^*}\int_{\mathbb{R}^N} B(x)|u|^{22^*} \\
&- \frac{1}{22^*}\int_{\mathbb{R}^N} \langle \nabla B(x), x\rangle |u|^{22^*} = 0,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V(x)|f(v)|^2 + \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |f(v)|^2 \\
&- \frac{N}{p+1}\int_{\mathbb{R}^N} A(x)|f(v)|^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^N} \langle \nabla A(x), x\rangle |f(v)|^{p+1} - \frac{N}{22^*}\int_{\mathbb{R}^N} B(x)|f(v)|^{22^*} \\
&- \frac{1}{22^*}\int_{\mathbb{R}^N} \langle \nabla B(x), x\rangle |f(v)|^{22^*} = 0.
\end{aligned}
$$

The proof is finished. □

In particular, if $V(x), A(x), B(x)$ are positive constant V, A, B, the above-mentioned Pohožaev identity can be rewritten as follows

$$P(v) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V|f(v)|^2 - \frac{N}{p+1}\int_{\mathbb{R}^N} A|f(v)|^{p+1} - \frac{\lambda N}{22^*}\int_{\mathbb{R}^N} B|f(v)|^{22^*} = 0. \tag{16}$$

Lemma 4. *The functional I is not bounded from below on E.*

Proof. Let $v_t(x) := v(t^{-1}x), t > 0$. Since $N \geq 3$, we have

$$I(v_t) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{Vt^N}{2}\int_{\mathbb{R}^N} f^2(v) - \frac{At^N}{p+1}\int_{\mathbb{R}^N} |f(v)|^{p+1} - \frac{\lambda Bt^N}{22^*}\int_{\mathbb{R}^N} |f(v)|^{22^*} \to -\infty$$

as $t \to +\infty$ for all $v \in E\backslash\{0\}$ and large enough λ. □

Lemma 4 means that we can not obtain the boundedness of the (PS) sequence by usual method. We need to consider a constrained minimization on a suitable manifold.

To give the definition of such a manifold, we need the following lemma.

Lemma 5. *Let a_i $(i = 1, 2, 3, 4)$ be positive constants. Define $h(t) := a_1 t^{N-2} + a_2 t^N - a_3 t^N - a_4 \lambda t^N$ for $t \geq 0$. Then, h has a unique critical point which corresponds to its maximum.*

Proof. For large enough $\lambda > 0$ such that $a_4\lambda - a_2 + a_3 > 0$, consider derivatives of h :

$$h'(t) = a_1(N-2)t^{N-3} + a_2 N t^{N-1} - a_3 N t^{N-1} - a_4 N \lambda t^{N-1}.$$

Note that $h'(t) \to -\infty$ as $t \to +\infty$ and is positive for $t > 0$ small since $N \geq 3$. Then, there exists $t > 0$ such that $h'(t) = 0$. The uniqueness of the critical point of h follows from the fact that the equation

$$h'(t) = a_1(N-2)t^{N-3} + a_2 N t^{N-1} - a_3 N t^{N-1} - a_4 N \lambda t^{N-1}, \quad t > 0$$

has a unique positive solution $\sqrt{\frac{a_1(N-2)}{N(a_4\lambda - a_2 + a_3)}}$ since $a_4\lambda - a_2 + a_3 > 0$. The proof is complete. □

Motivated by [8], we introduce the following Pohožave manifold

$$M = \{v \in E\backslash\{0\} \mid P(v) = 0\},$$

where $P(v)$ is defined by Equation (16).

Lemma 6. *For any $v \in E\backslash\{0\}$, there exists a unique $\hat{t} > 0$, such that $v_{\hat{t}} \in M$, where $v_{\hat{t}}(x) = v(\hat{t}^{-1}x)$. Moreover, $I(v_{\hat{t}}) = \max_{t>0} I(v_t)$.*

Proof. For every $v \in E\backslash\{0\}$ and $t > 0$, keeping the definition of v_t in mind. Denote

$$\chi(t) := I(v_t) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{t^N}{2}\int_{\mathbb{R}^N} V f^2(v) - \frac{t^N}{p+1}\int_{\mathbb{R}^N} A|f(v)|^{p+1} - \frac{\lambda t^N}{22^*}\int_{\mathbb{R}^N} B|f(v)|^{22^*}.$$

By Lemma 5, we have that χ has a unique critical point $\hat{t} > 0$ corresponding to its maximum, i.e., $\chi(\hat{t}) = \max_{t>0} \chi(t), \chi'(\hat{t}) = 0$. Thus,

$$\frac{N-2}{2}\hat{t}^{N-2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N\hat{t}^N}{2}\int_{\mathbb{R}^N} V f^2(v) - \frac{N\hat{t}^N}{p+1}\int_{\mathbb{R}^N} A|f(v)|^{p+1} - \frac{\lambda N\hat{t}^N}{22^*}\int_{\mathbb{R}^N} B|f(v)|^{22^*} = 0,$$

which implies that $P(v_{\hat{t}}) = 0$ and $v_{\hat{t}} \in M$. $\square$

Lemma 7. *The M is a natural C^1 manifold and every critical point of $I|_M$ is a critical point of I in $H^1(\mathbb{R}^N)$.*

Proof. By Lemma 6, it is easy to check that $M \neq \emptyset$. The proof consists of four steps.

Step 1. $0 \notin \partial M$.

Set $S(\rho) = \{v \in E \mid \int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V f^2(v) = \rho^2\}$. Note that, for any $v \in M$, using Lemma 1, Sobolev embedding inequality and choosing a number $\rho > 0$, then there exist $r > 0$, C_1 and $C_2 > 0$ such that

$$\begin{aligned} P(v) &= \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V f^2(v) - \frac{N}{p+1}\int_{\mathbb{R}^N} A|f(v)|^{p+1} - \frac{\lambda N}{22^*}\int_{\mathbb{R}^N} B|f(v)|^{22^*} \\ &\geq \frac{N-2}{2}\rho^2 - C_1\rho^{p+1} - C_2\rho^{22^*} > r > 0, \end{aligned}$$

for ρ small enough and $\lambda > 0$, so that $M, \partial M \subset E \backslash B_\rho(0)$.

Step 2. The M is a C^1 manifold.

Since $P(v)$ is a C^1 functional, to prove M is a C^1 manifold, it suffices to prove that $P'(v) \neq 0$ for all $v \in M$. Indeed, suppose on the contrary that $P'(v) = 0$ for some $v \in M$. Let

$$\alpha := \int_{\mathbb{R}^N} |\nabla v|^2, \quad \beta := \int_{\mathbb{R}^N} V f^2(v), \quad \gamma := \int_{\mathbb{R}^3} A|f(v)|^{p+1}, \quad \theta := \lambda \int_{\mathbb{R}^3} B|f(v)|^{22^*}.$$

The equation $P'(v) = 0$ can be written as

$$-(N-2)\Delta v + N f'(v)(V f(v) - A|f(v)|^{p-1} f(v) f'(v) - \lambda B|f(v)|^{22^*-2} f(v)) = 0, \tag{17}$$

and v satisfies the following Pohožaev identity

$$\frac{(N-2)^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N^2}{2}\int_{\mathbb{R}^3} V f^2(v) - \frac{N^2}{p+1}\int_{\mathbb{R}^3} A|f(v)|^{p+1} - \frac{\lambda N^2}{22^*}\int_{\mathbb{R}^N} B|f(v)|^{22^*} = 0.$$

We then obtain

$$\begin{cases} \dfrac{N-2}{2}\alpha + \dfrac{N}{2}\beta - \dfrac{N}{p+1}\gamma - \dfrac{N}{22^*}\theta = 0, \\ \dfrac{(N-2)^2}{2}\alpha + \dfrac{N^2}{2}\beta - \dfrac{N^2}{p+1}\gamma - \dfrac{N^2}{22^*}\theta = 0. \end{cases}$$

From above system, we have

$$2(N-2)\alpha = 0,$$

then $\alpha = 0$ since $N \geq 3$, which is a contradiction. Thus, $P'(v) \neq 0$ for any $v \in M$. This completes the proof of **Step 2**.

Step 3. Every critical point of $I|_M$ is a critical point of I in E.

If v is a critical point of $I|_M$, i.e., $v \in M$ and $(I|_M)'(v) = 0$. Thanks to the Lagrange multiplier rule, there exists $\rho \in \mathbb{R}$ such that $I'(v) = \rho P'(v)$. We prove that $\rho = 0$. Firstly, in a weak sense, the equation $I'(v) = \rho P'(v)$ can be written as

$$-(1-\rho(N-2))\Delta v + (1-\rho N)(V f(v) - A|f(v)|^{p-1} f(v) - \lambda B|f(v)|^{22^*-2} f(v)) f'(v) = 0,$$

and v satisfies the following Pohožaev identity

$$\begin{aligned} &\frac{(N-2)(1-\rho(N-2))}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N(1-\rho N)}{2}\int_{\mathbb{R}^N} V f^2(v) - \frac{N(1-\rho N)}{p+1}\int_{\mathbb{R}^3} A|f(v)|^{p+1} \\ &- \frac{N(1-\rho N)}{22^*}\lambda \int_{\mathbb{R}^3} B|f(v)|^{22^*} = 0. \end{aligned}$$

Using notations α, β, γ and θ as in **Step 3**, we obtain that

$$\begin{cases} \dfrac{N-2}{2}\alpha + \dfrac{N}{2}\beta - \dfrac{N\gamma}{p+1} - \dfrac{N}{22^*}\theta = 0, \\ \dfrac{(N-2)(1-\rho(N-2))}{2}\alpha + \dfrac{N(1-\rho N)}{2}\beta - \dfrac{N(1-\rho N)}{p+1}\gamma - \dfrac{N(1-\rho N)}{22^*}\theta = 0. \end{cases}$$

It is deduced from the above equations that

$$\rho(N-2)\alpha = 0.$$

If $\rho \neq 0$, then $\alpha = 0$ since $N \geq 3$, which is impossible. Therefore, $\rho = 0$ and $I'(u) = 0$. □

Lemma 8. *Let $r > 0$, $q \in [2, 22^*)$. If $\{v_n\}$ is bounded in E and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |f(v_n)|^q = 0,$$

then we have $v_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 22^*)$.

Proof. We use an idea from [22]. Let $q < s < 22^*$. Since $\{v_n\}$ is bounded in E and $E \hookrightarrow H^1(\mathbb{R}^N)$ is continuous, $\{v_n\}$ is also bounded in $H^1(\mathbb{R}^N)$. It follows from the Hölder and Sobolev inequalities that

$$\begin{aligned} |f(v_n)|_{L^s(B_R(y))} &\leq |f(v_n)|^{1-\mu}_{L^q(B_R(y))} |f(v_n)|^{\mu}_{L^{22^*}(B_R(y))} \\ &\leq C|f(v_n)|^{1-\mu}_{L^q(B_R(y))} \left(\int_{B_R(y)} (|\nabla v_n|^2 + v_n^2) \right)^{\frac{\mu}{4}}, \end{aligned}$$

where $\frac{1}{s} = \frac{1-\mu}{q} + \frac{\mu}{22^*}$, then $\mu = \frac{s-q}{22^*-q}\frac{22^*}{s}$. Choosing $\mu = \frac{4}{s}$, we obtain

$$\int_{B_R(y)} |f(v_n)|^s \leq C^s |f(v_n)|^{(1-\mu)s}_{L^q(B_R(y))} \left(\int_{B_R(y)} (|\nabla v_n|^2 + v_n^2) \right).$$

Covering $\mathbb{R}^N$ by a family of balls $\{B_R(y_i)\}$ such that each point is contained in at most k such balls and summing up these inequalities over this family of balls we obtain

$$\int_{\mathbb{R}^N} |f(v_n)|^s \leq kC^s \sup_{y\in\mathbb{R}^N} \left(\int_{B_R(y)} |f(v_n)|^q \right)^{(1-\mu)\frac{s}{q}} \left(\int_{\mathbb{R}^N} (|\nabla v_n|^2 + v_n^2) \right).$$

Under the assumption of the lemma, $f(v_n) \to 0$ in $L^s(\mathbb{R}^N)$. Since $2 < s < 22^*$, $f(v_n) \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 22^*$, by Sobolev and Hölder inequalities. □

Lemma 9. *([22], Lemma 1.32) Let Ω be an open subset of $\mathbb{R}^N$ and let $\{u_n\} \subset L^p(\Omega), 1 \leq p < \infty$. If $\{u_n\}$ is bounded in $L^p(\Omega)$ and $u_n \to u$ a.e. on Ω, then $\lim\limits_{n\to\infty}(|u_n|^p_{L^p} - |u_n - u|^p_{L^p}) = |u|^p_{L^p}$.*

3. Ground State of Equation (1) with Constant Coefficient

In this section, we study the existence of positive ground state solutions of Pohožaev type to Equation (1) with constant coefficient.

Lemma 10. *For $N \geq 3$, then there exists a minimizer v of $\inf\limits_{M} I$. Moreover, $I'(v) = 0$ in E.*

Proof. Inspired by [8], we divide the proof into three steps.

Step 1. Let $\{v_n\} \subset M$ be a sequence such that $I(v_n) \to \inf_M I$. We claim that $\{v_n\}$ is bounded. Indeed, by using $P(v_n) = 0$, one has that

$$1 + \inf_M I > I(v_n) = I(v_n) - \frac{1}{N} P(v_n) = \frac{N+2}{2N} \int_{\mathbb{R}^N} |\nabla v_n|^2,$$

for large enough n. Therefore, we conclude the boundedness of $\{|\nabla v_n|_{L^2}\}$. In the following, we prove $\{\int_{\mathbb{R}^N} V f^2(v_n)\}$ is also bounded. Using the boundedness of $\{|\nabla v_n|_{L^2}\}$, Hölder inequality, Sobolev inequality, and (f_3) and (f_7) of Lemma 1, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(v_n)|^{p+1} &\leq \left(\int_{\mathbb{R}^N} |f(v_n)|^2 \right)^{\frac{\xi(p+1)}{2}} \left(\int_{\mathbb{R}^N} |f^2(v_n)|^{2^*} \right)^{1-\frac{\xi(p+1)}{2}} \\
&\leq C_1 \left(\int_{\mathbb{R}^N} |f(v_n)|^2 \right)^{\frac{\xi(p+1)}{2}} \left(\int_{\mathbb{R}^N} |\nabla f^2(v_n)|^2 \right)^{\frac{2^*(1-\frac{\xi(p+1)}{2})}{2}} \\
&\leq C_2 |f(v_n)|_{L^2}^{\frac{\xi(p+1)}{2}} |\nabla v_n|_{L^2}^{\frac{2^*(2-\xi(p+1))}{2}},
\end{aligned} \tag{18}$$

$$|f(v_n)|_{L^{22^*}}^{22^*} = |f^2(v_n)|_{L^{2^*}}^{22^*} \leq C_3 |\nabla f^2(v_n)|_{L^2}^{2^*} \leq C_4 |\nabla v_n|_{L^2}^{2^*} \leq C_5, \tag{19}$$

where $1 = \xi + \frac{2^*(2-\xi(p+1))}{p+1}$ and $\xi = \frac{22^* - (p+1)}{(p+1)(2^*-1)}$. By $v_n \in M$, the boundedness of $\{|\nabla v_n|_{L^2}\}$ and (18) we obtain that

$$\begin{aligned}
\frac{N}{2} \int_{\mathbb{R}^N} V f^2(v_n) &= \frac{N}{p+1} \int_{\mathbb{R}^N} A |f(v_n)|^{p+1} + \frac{\lambda N}{22^*} \int_{\mathbb{R}^N} B |f(v_n)|^{22^*} - \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \\
&\leq \frac{AN}{p+1} \left(\varepsilon \int_{\mathbb{R}^N} |f(v_n)|^2 + C_\varepsilon \left(\int_{\mathbb{R}^N} |\nabla v_n|^2 \right)^{2^*} \right) + C_6.
\end{aligned}$$

Choosing small enough ε, we obtain $\{\int_{\mathbb{R}^N} V f^2(v_n)\}$ is bounded too. Therefore, $\{\int_{\mathbb{R}^N} |\nabla v_n|^2 + V f^2(v_n)\}$ is bounded. From $0 \leq |f(t)| \leq |t|, \ t \in \mathbb{R}^N$, there holds

$$\int_{\mathbb{R}^N} V |f(\xi v_n)|^2 \leq \xi^2 \int_{\mathbb{R}^N} V |v_n|^2, \ \ \xi \geq 0,$$

from which we obtain that

$$\inf_{\xi > 0} \frac{1}{\xi} \left\{ 1 + \int_{\mathbb{R}^N} V |f(\xi v_n)|^2 \right\} \leq \inf_{\xi > 0} \left\{ \frac{1}{\xi} + LV\xi \right\},$$

where $L = \int_{\mathbb{R}^N} |v_n|^2$. Now, let us consider the function

$$g(\xi) = \frac{1}{\xi} + LV\xi, \ \xi > 0.$$

A direct computation implies that g has a global minimum at $\xi_0 = \frac{1}{\sqrt{LV}} > 0$, and

$$g(\xi_0) = \sqrt{LV} + LV \frac{1}{\sqrt{LV}} = 2\sqrt{LV}.$$

It is now deduced that

$$\begin{aligned}
\|v_n\| &= |\nabla v_n|_2 + \inf_{\xi > 0} \frac{1}{\xi} \left[1 + \int_{\mathbb{R}^N} V f^2(\xi v_n) \right] \\
&\leq C \left(\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V f^2(v_n)) \right)^{\frac{1}{2}},
\end{aligned}$$

which implies that $\{v_n\}$ is bounded in E.

Step 2. Since $\{v_n\}$ is bounded in E, passing to a subsequence, we may assume $v_n \rightharpoonup v$ in E, $v_n \rightharpoonup v$ in $L^s(\mathbb{R}^N)$ for $2 \leq s \leq 22^*$. We prove that $v \in M$ and $v_n \to v$ in E. Thus, $I|_M$ attains its minimum at v. By Lemma 2, we get that

$$\int_{\mathbb{R}^N} |f(v_n)|^{p+1} \to \int_{\mathbb{R}^N} |f(v)|^{p+1},\ 1 < p < 22^* - 1.$$

Using the Ekeland's Variational Principle in Ekeland [23], we can assume that $I(v_n) \to \inf\limits_M I$ and $I'(v_n) \to 0$. Thus, by Fatou's Lemma, we obtain

$$\int_{\mathbb{R}^N} (|\nabla v_n|^2 + Vf^2(v_n)) \leq \liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} (|\nabla v_n|^2 + Vf^2(v_n)) \right).$$

Arguing by a contradiction, supposing that

$$\int_{\mathbb{R}^N} (|\nabla v_n|^2 + Vf^2(v_n)) < \liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} (|\nabla v|^2 + Vf^2(v)) \right),$$

$$\begin{aligned}
\inf_M I &\leq I(v) - \frac{1}{2^*}\langle I'(v), v\rangle \\
&= \frac{2^*-2}{22^*}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{2^*-2}{22^*}\int_{\mathbb{R}^N} Vf^2(v) - \frac{p+1-2^*}{2^*(p+1)}\int_{\mathbb{R}^N} A|f(v)|^{p+1} \\
&\quad + \frac{1}{2^*}\int_{\mathbb{R}^N} \lambda B|f(v)|^{22^*} \\
&< \liminf_{n\to\infty} \left(\frac{2^*-2}{22^*}\int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{2^*-2}{22^*}\int_{\mathbb{R}^N} Vf^2(v_n) - \frac{p+1-2^*}{2^*(p+1)}\int_{\mathbb{R}^N} A|f(v_n)|^{p+1} \right. \\
&\quad \left. + \frac{1}{2^*}\int_{\mathbb{R}^N} \lambda B|f(v_n)|^{22^*} \right) \\
&= \liminf_{n\to\infty} \left(I(v_n) - \frac{1}{2^*}\langle I'(v_n), v_n\rangle \right) = \inf_M I,
\end{aligned}$$

which is a contradiction. Then, $\int_{\mathbb{R}^N} (|\nabla v_n|^2 + Vf^2(v_n)) = \liminf\limits_{n\to\infty} \left(\int_{\mathbb{R}^N} (|\nabla v_n|^2 + Vf^2(v_n)) \right)$ and $P(v) = \liminf\limits_{n\to\infty} P(v_n) = 0$. Therefore, $v \in M$ and $v_n \to v$ in E.

Step 3. We now show that $I'(v) = 0$. Thanks to the Lagrange multiplier rule, there exists $\tau \in \mathbb{R}$ so that $I'(v) = \tau P'(v) = 0$. As in the proof of **Step 4** in Lemma 7, we can prove that $\tau = 0$. Thus, $I'(v) = 0$. □

Proof of Theorem 1. For $N \geq 3$ and large enough $\lambda > 0$, it is deduced from Lemma 10 that there exists $v \in M$ such that $I(v) = \inf I|_M$ and $I'(v) = 0$. Then, v is a nontrivial critical point of $I|_M$. Hence, by Lemma 7, the v is a nontrivial ground state solution of (7) with $V(x) = V$, $A(x) = A$ and $B(x) = B$. Thus, $u = f(v)$ is nontrivial ground state solution of Equation (1) in the case of $V(x) = V$, $A(x) = A$ and $B(x) = B$. Furthermore, it is easy to see that $|u|$ is also a ground state solution of Equation (1) since the functional $I(v)$ and $P(v)$ are even. Therefore, we may assume that such a ground state solution does not change sign, i.e. $u \geq 0$. The strong maximum principle and standard arguments [24] imply that $u(x) > 0$ for all $x \in \mathbb{R}^N$ and the proof is completed. □

4. Ground State of Equation (1) with Nonconstant Coefficient

In this section, we investigate Equation (1) in the case that $V(x)$, $A(x)$ and $B(x)$ are nonconstant. A starting point is the following lemma.

Lemma 11. *([25]) Let $(X, \|\cdot\|)$ be a Banach space and $T \in \mathbb{R}^+$ be an interval. Consider a family of C^1 functionals on X of the form*

$$\Phi_\delta(u) = C(u) - \delta D(u), \ \textit{for all} \ \delta \in T,$$

with $D(u) \geq 0$ and either $C(u) \to +\infty$ or $D(u) \to +\infty$, as $\|u\| \to \infty$. Assume that there are two points $v_1, v_2 \in X$ such that

$$c_\delta = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \Phi_\delta(\gamma(s)) > \max\{\Phi_\delta(v_1), \Phi_\delta(v_2)\}, \quad \textit{for any} \ \delta \in T,$$

where

$$\Gamma = \{\gamma \in C([0,1], X) | \ \gamma(0) = v_1, \ \gamma(1) = v_2\}.$$

Then, for almost every $\delta \in T$, there is a bounded $(PS)_{c_\delta}$ sequences in X.

For $\delta \in [\frac{1}{2}, 1]$, we consider the functional $I_{V,\delta} : E \to \mathbb{R}$ defined by

$$I_{V,\delta}(v) = C(v) - \delta D(v) \frac{\delta}{p+1} \int_{\mathbb{R}^N} A(x)|f(v)|^{p+1} - \frac{\lambda\delta}{22^*} \int_{\mathbb{R}^N} B(x)|f(v)|^{22^*}, v \in E, \tag{20}$$

where $C(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v)$, $D(v) = \frac{1}{p+1}\int_{\mathbb{R}^N} A(x)|f(v)|^{p+1} + \frac{\lambda}{22^*}\int_{\mathbb{R}^N} B(x)|f(v)|^{22^*}$. It is clear that this functional is of C^1. Moreover, for every $v, w \in E$,

$$\begin{aligned} \langle I'_{V,\delta}(v), w\rangle = & \int_{\mathbb{R}^N} (\nabla v \nabla w + V(x) f(v) f'(v) w) - \delta \int_{\mathbb{R}^N} A(x)|f(v)|^{p-1} f'(v) w \\ & - \lambda\delta \int_{\mathbb{R}^N} B(x)|f(v)|^{22^*-2} f'(v) w. \end{aligned} \tag{21}$$

We also need to consider the associated limit problem

$$-\Delta v + V_\infty f(v) f'(v) = \delta A_\infty |f(v)|^{p-1} f(v) f'(v) + \delta\lambda B_\infty |f(v)|^{22^*-1} f(v) f'(v), \ v \in E. \tag{$(QS)_\infty$}$$

It is clear that $(QS)_\infty$ is the Euler–Lagrange equations of the functional

$$I_{\infty,\delta}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty f^2(v) - \frac{\delta}{p+1}\int_{\mathbb{R}^N} A_\infty |f(v)|^{p+1} - \frac{\delta\lambda}{22^*}\int_{\mathbb{R}^N} B_\infty |f(v)|^{22^*}. \tag{22}$$

The following lemma ensures that $I_{V,\delta}$ has the mountain pass geometry with the corresponding mountain pass level denoted by $c_{V,\delta}$.

Lemma 12. *If (V_1), (V_2), (A) and (B) hold. Then,*
(1) there exists $v_0 \in E\backslash\{0\}$ such that $I_{V,\delta}(v_0) < 0$, for $\delta \in [\frac{1}{2}, 1]$;
(2) $c_{V,\delta} := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I_{V,\delta}(\gamma(s)) > \max\{I_{V,\delta}(0), \ I_{V,\delta}(v)\}$ for $\delta \in [\frac{1}{2}, 1]$, where

$$\Gamma = \{\gamma \in C([0,1], E) | \ \gamma(0) = 0, \ \gamma(1) = v\}.$$

Proof. (1) For any $v \in E\backslash\{0\}, \delta \in [\delta, 1]$.

$$\begin{aligned} I_{V,\delta}(v_t) & \leq I_{\infty,\delta}(v_t) \\ & = \int_{\mathbb{R}^N} \left(\frac{t^{N-2}}{2} |\nabla v|^2 + \frac{t^N}{2} V_\infty f^2(v) - \frac{\delta t^N}{p+1} A_\infty |f(v)|^{p+1} - \frac{\delta\lambda t^N}{22^*} B_\infty |f(v)|^{22^*} \right) \to -\infty \end{aligned}$$

as $t \to +\infty$. Taking $v = v_t$ for t large, this shows at once that $I_{V,\delta}(v) \leq I_{\infty,\delta}(v) < 0$.

(2) Recalling Lemma 1 and **Step 1** of Lemma 7, we get

$$\begin{aligned} I_{V,\delta}(v) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla v|^2 + V(x)|f(v)|^2) - \frac{\delta}{p+1}\int_{\mathbb{R}^3} A(x)|f(v)|^{p+1} - \frac{\delta\lambda}{22^*}\int_{\mathbb{R}^3} B(x)|f(v)|^{22^*} \\ &\geq \frac{1}{2}C_1\rho^2 - C_2\rho^{p+1} - C_3\rho^{22^*}, \end{aligned}$$

for sufficiently small $\rho > 0$, there exists $\tau > 0$ such that $I_{V,\delta}(v) \geq \tau > 0$, then $c_{V,\delta} > 0$. □

Lemma 12 means that, if $I_{V,\delta}(v)$ satisfies the assumptions of Lemma 11 with $X = E$ and $\Phi_\delta = I_{V,\delta}$, we then obtain immediately, for a.e. $\delta \in [\frac{1}{2}, 1]$, there exists a bounded sequence $\{u_n\} \subset E$ such that $I_{V,\delta}(u_n) \to c_{V,\delta}$, $I'_{V,\delta}(v_n) \to 0$ in E.

Lemma 13. *([25], Lemma 2.3) Under the assumptions of Lemma 11, the map $\delta \to c_\delta$ is non-increasing and left continuous.*

Introduce the following manifold

$$M_{\infty,\delta} = \{v \in E\backslash\{0\} \mid P_{\infty,\delta}(v) = 0\},$$

where

$$P_{\infty,\delta}(v) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V_\infty f^2(v) - \delta N \int_{\mathbb{R}^N} A_\infty |f(v)|^{p+1} - \delta\lambda N \int_{\mathbb{R}^N} B_\infty |f(v)|^{22^*}.$$

Set

$$m_{\infty,\delta} := \inf_{v \in M_{\infty,\delta}} I_{\infty,\delta}(v).$$

According to Section 3, $M_{\infty,\delta}(v)$ has some similar properties to those of the manifold M, such as containing all the nontrivial critical points of $I_{\infty,\delta}(v)$.

Lemma 14. *If $N \geq 3$ and $\delta \in [\frac{1}{2}, 1]$, $m_{\infty,\delta}$ is obtained at some $v_{\infty,\delta} \in M_{\infty,\delta}$. Moreover,*

$$I_{\infty,\delta}(v_{\infty,\delta}) = m_{\infty,\delta} = \inf\{I_{\infty,\delta}(v) \mid v \neq 0, I'_{\infty,\delta}(v) = 0\}.$$

Proof. The proof is similar to that of Theorem 1, and is omitted here. □

Lemma 15. *Suppose that (V_1), (V_2), (A) and (B) hold. Then, $c_{V,\delta} < m_{\infty,\delta}$ for $\delta \in [\frac{1}{2}, 1]$.*

Proof. Let $v_{\infty,\delta}$ be a minimizer of $m_{\infty,\delta}$. By Lemma 5, $I_{\infty,\delta}(v_{\infty,\delta}) = \max_{t>0} I_{\infty,\delta}(v(t^{-1}x))$. Then, we see that, for $\delta \in [\frac{1}{2}, 1]$,

$$c_{\infty,\delta} \leq \max_{t>0} I_{V,\delta}(v_{\infty,\delta}(t^{-1}x)) < \max_{t>0} I_{\infty,\delta}(v_{\infty,\delta}(t^{-1}x)) = I_{\infty,\delta}(v_{\infty,\delta}) = m_{\infty,\delta}.$$

□

Next, we need the following global compactness lemma, which is adopted to prove that the functional $I_{\infty,\delta}$ satisfies $(PS)_{c_{V,\delta}}$ condition for a.e. $\delta \in [\frac{1}{2}, 1]$.

Lemma 16. *Suppose that (V_1), (V_2), (A) and (B) hold. For every $\delta \in [\frac{1}{2}, 1]$, let $\{v_n\}$ be a bounded $(PS)_{c_{V,\delta}}$ sequence for $I_{V,\delta}$ Then, there exist a subsequence of $\{v_n\}$, still denote $\{v_n\}$, v_0 and integer $\eta \in \mathbb{N} \cup \{0\}$, sequence $\{y_n^j\}$, $w_j \subset H^1(\mathbb{R}^N)$ for $1 \leq j \leq \eta$ such that* *(i)* $v_n \rightharpoonup v_0$ *with* $I'_{V,\delta}(v_0) = 0$;

(ii) $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ *if* $i \neq j, n \to +\infty$;

(iii) $w^j \neq 0$ *and* $I'_{\infty,\delta}(w^j) = 0$ *for* $1 \leq j \leq \eta$;

(iv) $\left\| v_n - v_0 - \sum_{j=1}^{\eta} w^j(\cdot - y_n^j) \right\| \to 0$; *and*

(v) $I_{V,\delta}(v_n) \to I_{V,\delta}(v_0) + \sum_{j=1}^{\eta} I_{\infty,\delta}(w^j)$.

Here, we agree that in the case $\eta = 0$ *the above holds without* w^j *and* $\{y_n^j\}$.

Proof. We complete the proof in two steps.

Step 1. Since $\{v_n\}$ is bounded in E, up to subsequence, there exists v_0 such that $v_n \rightharpoonup v_0$ in E,

$$v_n \to v_0 \quad \text{in } L^r_{loc}(\mathbb{R}^N), \;\; f(v_n) \to f(v_0) \quad \text{in } L^r_{loc}(\mathbb{R}^N) \; (2 \leq r < 22^*). \tag{23}$$

Arguing as in [26], let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\Upsilon := supp(\varphi)$. Then, $v_n \to v_0$ a.e. on Υ and $|v_n(x)| \leq w_r(x)$ for every $n \in \mathbb{N}$ and a.e. on Υ with $w_r(x) \in L^r(\Upsilon)$ (see Lemma A.1, [22]). Consequently,

$$V(x)f(v_n)f'(v_n) \to V(x)f(v_0)f'(v_0) \;\; a.e. \; on \; \Upsilon$$

$$A(x)|f(v_n)|^{p-1}f(v_n)f'(v_n) \to A(x)|f(v_0)|^{p-1}f(v_0)f'(v_0) \;\; a.e. \; on \; \Upsilon,$$
$$B(x)|f(v_n)|^{22^*-2}f(v_n)f'(v_n) \to B(x)|f(v_0)|^{22^*-2}f(v_0)f'(v_0) \;\; a.e. \; on \; \Upsilon.$$

Now, we show that $I'_{V,\delta}(v_0) = 0$. In fact, it suffices to prove that $\langle I'_{V,\delta}(v_0), \varphi \rangle = 0$. It follows from Equation (23) that for any fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla v_n \nabla \varphi = \int_{\mathbb{R}^N} \nabla v_0 \nabla \varphi. \tag{24}$$

Using (f_3) of Lemma 1 and (V_1), we have that

$$|V(x)f(v_n)f'(v_n)\varphi| \leq \sup_{\Upsilon} V(x)|w_2||\varphi|.$$

The Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)f(v_n)f'(v_n)\varphi = \int_{\mathbb{R}^N} V(x)f(v_0)f'(v_0)\varphi. \tag{25}$$

Similarly, since $B(x)|f(v_n)|^{22^*-2}f(v_n)f'(v_n)\varphi| \leq \sup_{\Upsilon} B(x)|w_{22^*-1}|^{22^*-1}|\varphi|$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} B(x)|f(v_n)|^{22^*-2}f(v_n)f'(v_n)\varphi = \int_{\mathbb{R}^N} B(x)|f(v_0)|^{22^*-2}f(v_0)f'(v_0)\varphi.$$

If $|v_n(x)| \leq 1$, using (f_2) and (f_3) of Lemma 1, we have

$$A(x)|f(v_n)|^{p-1}f(v_n)f'(v_n)\varphi| \leq |f(v_n)|^p|\varphi| \leq \sup_{\Upsilon} A(x)|\varphi|. \tag{26}$$

If $|v_n(x)| > 1$, using (f_2), (f_3) and (f_7) of Lemma 1, we have

$$\left| A(x)|f(v_n)|^{p-1}f(v_n)f'(v_n)\varphi \right| \leq \sup_{\Upsilon} A(x)|f(v_n)|^p|\varphi| < 2^{\frac{p}{4}}|w_{\frac{22^*-1}{2}}|^{\frac{22^*-1}{2}}|\varphi|. \tag{27}$$

Thus, combining Equation (26) with Equation (27), one deduces that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} A(x)|f(v_n)|^{p-1}f(v_n)f'(v_n)\varphi = \int_{\mathbb{R}^N} A(x)|f(v_0)|^{p-1}f(v_0)f'(v_0)\varphi. \tag{28}$$

It follows from Equations (24), (25) and (28) that

$$\begin{aligned}
&\langle I'_{V,\delta}(v_n),\varphi\rangle - \langle I'_{V,\delta}(v_0),\varphi\rangle \\
=& \int_{\mathbb{R}^N}\nabla(v_n-v_0)\nabla\varphi + \int_{\mathbb{R}^N} V(x)(f(v_n)f'(v_n)-f(v_0)f'(v_0))\varphi \\
&-\delta\int_{\mathbb{R}^N} A(x)(|f(v_n)|^{p-1}f(v_n)f'(v_n)-|f(v_0)|^{p-1}f(v_0)f'(v_0)) \\
&-\lambda\delta B(x)(|f(v_n)|^{22^*-2}f(v_n)f'(v_n)-|f(v_0)|^{22^*-2}f(v_0)f'(v_0))\varphi \to 0.
\end{aligned}\tag{29}$$

Thus, $I'_{V,\delta}(v_0)=0$.
Step 2. We prove that $I_{V,\delta}(v_0)\geq 0$.
From (V_2) and $N\geq 3$, we deduce that

$$I(v_0) = I(v_0) - \frac{1}{N}P(v_0) = \frac{N+2}{2N}\int_{\mathbb{R}^N}|\nabla v_0|^2 - \frac{1}{2N}\int_{\mathbb{R}^N}\langle\nabla V(x),x\rangle f^2(v_0)\geq 0. \tag{30}$$

Step 3. Set $w_n^1 = v_n - v_0$, then we get $w_n^1 \rightharpoonup 0$ in E.
Let us define

$$\mu = \lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{\mathbb{R}^N}|f(w_n^1)|^2.$$

Vanishing: If $\mu=0$, then it follows from Lemma 8 that

$$f(w_n^1)\to 0 \tag{31}$$

in $L^s(\mathbb{R}^N)$ for $s\in(2,22^*)$. By $I'_{V,\delta}(v_0)=0$ and Fatou's Lemma, we have

$$\begin{aligned}
c_{V,\delta} &\leq I_{V,\delta}(v_0) - \frac{1}{2^*}\langle I'_{V,\delta}(v_0),v_0\rangle \\
&= \frac{2^*-2}{22^*}\int_{\mathbb{R}^N}|\nabla v_0|^2 + \frac{2^*-2}{22^*}\int_{\mathbb{R}^N}V(x)f^2(v_0) \\
&\quad -\frac{p+1-2^*}{2^*(p+1)}\int_{\mathbb{R}^N}A(x)|f(v_0)|^{p+1} + \frac{1}{2^*}\int_{\mathbb{R}^N}\lambda B(x)|f(v_0)|^{22^*} \\
&\leq \liminf_{n\to\infty}\Big(\frac{2^*-2}{22^*}\int_{\mathbb{R}^N}|\nabla v_n|^2 + \frac{2^*-2}{22^*}\int_{\mathbb{R}^N}V(x)f^2(v_n) \\
&\quad -\frac{p+1-2^*}{2^*(p+1)}\int_{\mathbb{R}^N}A(x)|f(v_n)|^{p+1} + \frac{1}{2^*}\int_{\mathbb{R}^N}\lambda B(x)|f(v_n)|^{22^*}\Big) \\
&= \liminf_{n\to\infty}\Big(I_{V,\delta}(v_n) - \frac{1}{2^*}\langle I'_{V,\delta}(v_n),v_n\rangle\Big) = c_{V,\delta},
\end{aligned}\tag{32}$$

which means that $\|w_n^1\|\to 0$.

Non-vanishing: If $\mu>0$, we can find a sequence $\{y_n^1\}\subset\mathbb{R}^N$ such that

$$\int_{B_1(0)} f^2(\tilde{w}_n^1) = \int_{B_1(y_n)} f^2(w_n^1) > \frac{\mu}{2} > 0, \tag{33}$$

where $\tilde{w}_n^1 = w_n^1(\cdot + y_n^1)$. Note that $\|\tilde{w}_n^1\| = \|w_n^1(\cdot+y_n^1)\|$, we see that $\{\tilde{w}_n^1\}$ is bounded. Going if necessary to a subsequence, we have a $v^1\in E$ such that $\tilde{w}_n^1\rightharpoonup v^1$ in E. Since $\int_{B_1(0)}|\tilde{w}_n^1|^2 \geq \int_{B_1(0)}|f(\tilde{w}_n^1)|^2 > \frac{\mu}{2}$, we see that $v^1\neq 0$. Moreover, $w_n^1\rightharpoonup 0$ in E implies that $|y_n^1|\to+\infty$. Next, we prove that $I'_{\infty,\delta}(v^1) = 0$. Similar to the proof of **Step 1**, for any fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$, it suffices to show that $\langle I'_{\infty,\delta}(\tilde{w}_n^1), \varphi\rangle \to 0$. By (V_1), (A), (B) and $|y_n^1| \to +\infty$, as $n \to \infty$, we have that

$$\int_{\mathbb{R}^N} (V(x+y_n^1) - V_\infty) f(\tilde{w}_n^1) f'(\tilde{w}_n^1)\varphi \to 0, \tag{34}$$

$$\int_{\mathbb{R}^N} (A(x+y_n^1) - A_\infty)|f(\tilde{w}_n^1)|^{p-1} f(\tilde{w}_n^1) f'(\tilde{w}_n^1)\varphi \to 0, \tag{35}$$

$$\int_{\mathbb{R}^N} (B(x+y_n^1) - B_\infty)|f(\tilde{w}_n^1)|^{22^*-2} f(\tilde{w}_n^1) f'(\tilde{w}_n^1)\varphi \to 0. \tag{36}$$

Since $w_n^1 \rightharpoonup 0$ in E, one has that $\langle I'_{V,\delta}(w_n^1), \varphi(\cdot - y_n^1)\rangle \to 0$, i.e.

$$\begin{aligned}&\int_{\mathbb{R}^N} \nabla \tilde{w}_n^1 \nabla \varphi + \int_{\mathbb{R}^N} V(x+y_n^1) f(\tilde{w}_n^1) f'(\tilde{w}_n^1)\varphi - \delta \int_{\mathbb{R}^N} A(x+y_n^1)|f(\tilde{w}_n^1)|^{p-1} f(\tilde{w}_n^1) f'(\tilde{w}_n^1)\varphi \\ &- \lambda\delta \int_{\mathbb{R}^N} B(x+y_n^1)|f(\tilde{w}_n^1)|^{22^*-2} f(\tilde{w}_n^1) f'(\tilde{w}_n^1)\varphi \to 0\end{aligned} \tag{37}$$

as $n \to \infty$. Thus, using Equations (34)–(37), one has $\langle I'_{\infty,\delta}(\tilde{w}_n^1), \varphi\rangle \to 0$. Therefore, $I'_{\infty,\delta}(v^1) = 0$. In the following, we prove that

$$I_{V,\delta}(w_n^1) = c_{V,\delta} - I_{V,\delta}(v_0) + o(1) \tag{38}$$

and

$$I_{V,\delta}(v_n) - I_{V,\delta}(v_0) - I_{\infty,\delta}(w_n^1) \to 0. \tag{39}$$

Firstly, we claim that the relation below holds:

$$\int_{\mathbb{R}^N} |f(w_n^1)|^l = \int_{\mathbb{R}^N} |f(v_n)|^l - \int_{\mathbb{R}^N} |f(v_0)|^l + o(1), \ \ 2 \le l \le 22^*. \tag{40}$$

We have by (f_2) and (f_3) of Lemma 1 that

$$\int_{\mathbb{R}^N} |\nabla f(w_n^1)|^2 \le \int_{\mathbb{R}^N} |f'(w_n^1)|^2 |\nabla w_n^1|^2 \le \int_{\mathbb{R}^N} |\nabla w_n^1|^2, \ \int_{\mathbb{R}^N} |f(w_n^1)|^2 \le \int_{\mathbb{R}^N} |w_n^1|^2. \tag{41}$$

Thus, $\{f(w_n^1)\}$ is bounded in E and $f(w_n^1) \in L^l(\mathbb{R}^N)$. Because of the local compactness of the Sobolev embedding theorem, we have, up to a subsequence, $f(w_n^1) \to f(v_0)$ almost everywhere on $\mathbb{R}^N$. Then, the conclusion follows from the Brrézis-Lieb Lemma. This implies that Equation (40) holds. Using similar arguments above, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we also obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} |f(w_n^1)|^{p-1} f(w_n^1) f'(w_n^1)\varphi \\ =& \int_{\mathbb{R}^N} |f(v_n)|^{p-1} f(v_n) f'(v_n)\varphi - \int_{\mathbb{R}^N} |f(v_0)|^{p-1} f(v_0) f'(v_0)\varphi + o(1).\end{aligned} \tag{42}$$

In addition, by Lemma 9, we have

$$\int_{\mathbb{R}^N} |\nabla w_n^1|^2 = \int_{\mathbb{R}^N} |\nabla v_n|^2 - \int_{\mathbb{R}^N} |\nabla v_0|^2 + o(1). \tag{43}$$

Now, from Equations (40) and (43), we know that Equation (38) holds. We deduce from Equations (20) and (22) that

$$\begin{aligned}
&I_{V,\delta}(v_n) - I_{V,\delta}(v_0) - I_{\infty,\delta}(v_n - v_0) \\
&= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v_n|^2 - |\nabla v_0|^2 - |\nabla(v_n - v_0)|^2) \\
&\quad + \frac{1}{2}\left(\int_{\mathbb{R}^N} V(x)(f^2(v_n) - f^2(v_0)) - \int_{\mathbb{R}^N} V_\infty f^2(v_n - v_0)\right) \\
&\quad - \frac{\delta}{p+1}\left(\int_{\mathbb{R}^N} A(x)(|f(v_n)|^{p+1} - |f(v_0|^{p+1}) - \int_{\mathbb{R}^N} A_\infty |f(v_n - v_0)|^{p+1}\right) \\
&\quad - \frac{\lambda\delta}{22^*}\left(\int_{\mathbb{R}^N} B(x)(|f(v_n)|^{22^*} - |f(v_0)|^{22^*}) - \int_{\mathbb{R}^N} B_\infty |f(v_n - v_0)|^{22^*}\right).
\end{aligned} \tag{44}$$

It is deduced from Equations (40)–(44) that Equation (39) holds.

Step 4. Set $w_n^2 = w_n^1 - v^1(\cdot - y_n)$, then $w_n^2 \rightharpoonup 0$ in E. It follows from Equations (40)–(42) that

$$\begin{aligned}
&|\nabla w_n^2|_{L^2}^2 = |\nabla v_n|_{L^2}^2 - |\nabla v_0|_{L^2}^2 - |\nabla v^1(\cdot - y_n)|_{L^2}^2 + o(1), \\
&|f(w_n^2)|_{L^{p+1}}^{p+1} = |f(v_n)|_{L^{p+1}}^{p+1} - |f(v_0)|_{L^{p+1}}^{p+1} - |f(v^1(\cdot - y_n))|_{L^{p+1}}^{p+1} + o(1), \\
&\int_{\mathbb{R}^N} V(x)|f(w_n^2)|^2 \\
&= \int_{\mathbb{R}^N} V(x)|f(v_n)|^2 - \int_{\mathbb{R}^N} V(x)|f(v_0)|^2 - \int_{\mathbb{R}^N} V(x)|f(v^1(\cdot - y_n))|^2 + o(1), \\
&\int_{\mathbb{R}^N} A(x)|f(w_n^2)|^{p-1} f(w_n^2) f'(w_n^2)\varphi \\
&= \int_{\mathbb{R}^N} A(x)|f(v_n)^{p-1} f(v_n) f'(v_n)\varphi - \int_{\mathbb{R}^N} A(x)|f(v_0)|^{p-1} f(v_0) f'(v_0)\varphi \\
&\quad - \int_{\mathbb{R}^N} A(x)|f(v^1(\cdot - y_n))^{p-1} f(v^1(\cdot - y_n)) f'(v^1(\cdot - y_n))\varphi + o(1), \\
&\int_{\mathbb{R}^N} \lambda B(x)|f(w_n^2)|^{22^*-2} f(w_n^2) f'(w_n^2)\varphi \\
&= \int_{\mathbb{R}^N} \lambda B(x)|f(v_n)|^{22^*-2} f(v_n) f'(v_n)\varphi - \int_{\mathbb{R}^N} \lambda B(x)|f(v_0)|^{22^*-2} f(v_0) f'(v_0)\varphi \\
&\quad - \int_{\mathbb{R}^N} \lambda B(x)|f(v^1(\cdot - y_n))|^{22^*-2} f(v^1(\cdot - y_n)) f'(v^1(\cdot - y_n))\varphi + o(1).
\end{aligned}$$

By similar argument, we can deduce that

$$\begin{aligned}
I_{V,\delta}(w_n^2) &= I_{V,\delta}(v_n) - I_{V,\delta}(v_0) - I_{\infty,\delta}(v^1) + o(1), \\
I_{V,\delta}(w_n^2) &= I_{V,\delta}(w_n^1) - I_{\infty,\delta}(v^1) + o(1), \\
\langle I'_{V,\delta}(w_n^2), \varphi\rangle &= \langle I'_{V,\delta}(v_n), \varphi\rangle - \langle I'_{V,\delta}(v_0), \varphi\rangle - \langle I'_{\infty,\delta}(v^1), \varphi\rangle + o(1) = o(1)
\end{aligned}$$

and then

$$I_{V,\delta}(v_n) = I_{V,\delta}(v_0) + I_{\infty,\delta}(w_n^1) + o(1) = I_{V,\delta}(v_0) + I_{\infty,\delta}(w_n^2) + I_{\infty,\delta}(v^1) + o(1).$$

Similar to the proof in **Step 2** of Lemma 16, we obtain that $I_{\infty,\delta}(v^1) \geq 0$. Then, we get from Equation (30) that

$$I_{V,\delta}(w_n^2) = c_{V,\delta} - I_{V,\delta}(v_0) - I_{\infty,\delta}(v^1) + o(1) \leq c_{V,\delta}.$$

Repeating the same type of arguments explored in **Step 3**, set

$$\mu_1 = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} |f(w_n^2)|^2.$$

If vanishing occurs, then $\|w_n^2\| \to 0$ in E. Thus, Lemma 16 holds with $j = 1$. If w_n^2 is non vanishing, then there exists a sequence $\{y_n^2\}$ and $v^2 \in E$ such that $\tilde{w}_n^2 = w_n^2(\cdot + y_n^2) \rightharpoonup v^2$ in E and $I'_{\infty,\delta}(v^2) = 0$. Furthermore, $v_n^2 \rightharpoonup 0$ in E means that $|y_n^2| \to +\infty$ and $|y_n^1 - y_n^2| \to +\infty$. By iterating this technique, we obtain $w_n^j = w_n^{j-1} - v^{j-1}$ with $j \geq 1$ such that $w_n^j \to v^j$, $I'_{\infty,\delta}(v^j) = 0$ and sequences $y_n^j \subset \mathbb{R}^N$ such that $|y_n^j| \to +\infty$ and $|y_n^i - y_n^j| \to +\infty$ if $i \neq j$ as $n \to +\infty$, and using the properties of the weak convergence, we have

$$\|v_n\|^2 - \|v_0\|^2 - \sum_{k=1}^{j-1} \|v^k(\cdot - y_n^k)\|^2 = \left\| v_n - v_0 - \sum_{k=1}^{j-1} v^k(\cdot - y_n^k) \right\|^2 + o(1), \tag{45}$$

$$I_{V,\delta}(v_n) \to I_{V,\delta}(v_0) + \sum_{k=1}^{j-1} I_{\infty,\delta}(v^{k-1}) + I_{\infty,\delta}(w_n^j). \tag{46}$$

Equation (46) implies that the iteration stops at some finite index $\eta + 1$. Therefore, $w_n^{\eta+1} \to 0$ in E. We can verify that (iv) and (v) hold by Equations (45) and (46). This proves the lemma. □

Lemma 17. *Assume that* (V_1), (V_2), (A) *and* (B) *hold;* $2 \leq p < 22^* - 1$. *Let* $\{v_n\}$ *be a bounded* $(PS)_{c_{V,\delta}}$ *sequence of* $I_{V,\delta}$. *Then, there exists a nontrivial* $v_{V,\delta} \in E$ *such that* $I'_{V,\delta}(v_{V,\delta}) = 0$ *and* $I_{V,\delta}(v_{V,\delta}) = c_{V,\delta}$ *for almost all* $\delta \in [\frac{1}{2}, 1]$.

Proof. For $\delta \in [\frac{1}{2}, 1]$, let $v_{\infty,\delta}$ be the minimizer of $m_{\infty,\delta}$. By Lemma 13, we have that

$$c_{\infty,\delta} < m_{\infty,\delta}. \tag{47}$$

It follows from Lemma 16 that there exists $v_{V,\delta} \in E$, $\eta \in \mathbb{N} \cup \{0\}$ and sequences $\{y_n^j\} \subset \mathbb{R}^N$, $v^j \subset E$ for $j \in \{1, 2, \cdots, \eta\}$ such that

$$I'_{V,\delta}(v_{V,\delta}) = 0, \ \ v_n \rightharpoonup v_{V,\delta}, \text{ and } I_{V,\delta}(v_n) \to I_{V,\delta}(v_{V,\delta}) + \sum_{j=1}^{\eta} I_{\infty,\delta}(v^j), \tag{48}$$

where v^j is a critical point of $I_{\infty,\delta}(v_{V,\delta})$. Similar to the argument of Equation (30), by (V_2) and $2 \leq p < 22^* - 1$, we also have $I_{\infty,\delta}(v_{V,\delta}) \geq 0$. If $\eta \neq 0$, and then, by Equation (48), one obtains that

$$c_{V,\delta} = I_{V,\delta}(u_{V,\delta}) + \sum_{j=1}^{\eta} I_{\infty,\delta}(w^j) \geq m_{\infty,\delta},$$

which contradicts Equation (47). Thus, $\eta = 0$, which implies $v_n \to v_{V,\delta}$ in E and $I_{V,\delta}(v_{V,\delta}) = c_{V,\delta}$. □

Proof of Theorem 2. The proof contains two steps.

Step 1. From Lemmas 11 and 12, for almost every $\delta \in [\frac{1}{2}, 1]$, there exists a bounded $(PS)_{c_{V,\delta}}$ sequence for $I_{V,\delta}$. Then, Lemma 7 implies that there exists $v_{V,\delta} \in E \backslash \{0\}$ such that $I'_{V,\delta}(v_{V,\delta}) = 0$ and $I_{V,\delta}(v_{V,\delta}) = c_{V,\delta}$. Choose $\delta_n \to 1$ such that I_{V,δ_n} has a critical point v_{V,δ_n} still denoted by $\{v_n\}$. Now, we show that $\{v_n\}$ is bounded in E. Denote

$$\begin{cases} a_n := \int_{\mathbb{R}^3} |\nabla v_n|^2, & b_n := \int_{\mathbb{R}^3} V(x) f^2(v_n), \qquad \bar{b}_n := \int_{\mathbb{R}^3} (\nabla V(x), x) f^2(v_n), \\ c_n := \int_{\mathbb{R}^3} A(x)|f(v_n)|^{p+1}, & \bar{c}_n := \int_{\mathbb{R}^3} (\nabla A(x), x)|f(v_n)|^{p+1},\ d_n := \int_{\mathbb{R}^3} \lambda B(x)|f(v_n)|^{22^*}, \\ \bar{d}_n := \int_{\mathbb{R}^3} \lambda(\nabla B(x), x)|f(v_n)|^{22^*},\ A_n := \dfrac{1}{1+2f^2(v_n)}. \end{cases}$$

Then,

$$\begin{cases} \frac{1}{2}a_n + \frac{1}{2}b_n - \frac{\delta_n}{p+1}c_n - \frac{\delta_n}{22^*}d_n = c_{V,\delta_n}, \\ \frac{N-2}{2}a_n + \frac{N}{2}b_n + \frac{1}{2}\bar{b}_n - \frac{N\delta_n}{p+1}c_n - \frac{\delta_n}{p+1}\bar{c}_n - \frac{N\delta_n}{22^*}d_n - \frac{\delta_n}{22^*}\bar{d}_n = 0, \\ A_n a_n + b_n - \delta_n c_n - \delta_n d_n = 0. \end{cases} \tag{49}$$

From these relations, (V_2), (A) and (B), one has that

$$\left(\frac{5}{2} - A_n\right) a_n + \frac{1}{2}b_n - \frac{1}{2}\bar{b}_n + \frac{p-2}{p+1}\delta_n c_n + \frac{22^*-3}{22^*}\delta_n d_n + \frac{1}{p+1}\delta_n \bar{c}_n + \frac{1}{22^*}\delta_n \bar{d}_n = (N+3)c_{V,\delta},$$

which implies that $\{a_n + b_n\}$ is bounded since $2 \le p < 22^* - 1$ and $0 < A_n \le 1$. Therefore, $\{\int_{\mathbb{R}^N}(|\nabla v_n|^2 + V(x)f^2(v_n))\}$ is bounded. Using **Step 1** of Lemma 10, we deduce that $\{v_n\}$ is bounded in E. Moreover, using Lemma 13, we deduce that

$$\lim_{n\to\infty} I_V(v_n) = \lim_{n\to\infty} \left\{ I_{V,\delta_n}(v_n) + (\delta_n - 1)\left[\int_{\mathbb{R}^N} \frac{1}{p+1}|f(v_n)^{p+1} + \frac{\lambda}{22^*}\int_{\mathbb{R}^N} |f(v_n)|^{22^*} \right] \right\}. \tag{50}$$

Since the sequence $\{v_n\}$ is bounded in E, we have that $\{f(v_n)\}$ is bounded in $L^s(\mathbb{R}^N)$ for $2 \le s \le 22^*$. Then,

$$\begin{aligned} &\lim_{n\to\infty} (\delta_n - 1)\left[\int_{\mathbb{R}^N} \frac{1}{p+1}|f(v_n)^{p+1} + \frac{\lambda}{22^*}\int_{\mathbb{R}^N} |f(v_n)|^{22^*} \right] \\ &\le \lim_{n\to\infty} C(\delta_n - 1)(\|v_n\|^{p+1} + \|v_n\|^{22^*}) = 0. \end{aligned} \tag{51}$$

It is deduced from Equations (50) and (51) that

$$\lim_{n\to\infty} I_V(v_n) = \lim_{n\to\infty} c_{V,\delta_n} = c_{V,1}. \tag{52}$$

Similar to the argument for Equation (52), we get that

$$\begin{aligned} &\lim_{n\to\infty} \left\langle I_V'(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle \\ &= \lim_{n\to\infty} \left\{ \left\langle I'_{V,\delta_n}(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle + (\delta_n - 1)\left[\int_{\mathbb{R}^N} |f(v_n)^{p+1} + \lambda \int_{\mathbb{R}^N} |f(v_n)|^{22^*} \right] \right\} \\ &= 0. \end{aligned} \tag{53}$$

Equations (52) and (53) show that $\{v_n\}$ is a bounded $(PS)_{c_{V,1}}$ sequence for $I_V := I_{V,1}$. Then, by Lemma 17, there exists a nontrivial critical point $v_0 \in E$ for I_V and $I_V(v_0) = c_{V,1}$.

Step 2. Now, we prove the existence of a ground state solution for Equation (1). Set

$$m_V := \inf\{I_V(v) \mid v \neq 0,\ I_V'(v) = 0\}.$$

As in the proof of **Step 2** of Lemma 16, we can see that every critical point of I_V has nonnegative energy. Thus, $0 \leq m_V \leq I_V(v_0) < c_{V,1} < +\infty$. Let $\{v_n\}$ be a sequence of nontrivial critical points of I_V satisfying $I_V(v_n) \to m_V$. Since $I_V(v_n)$ is bounded, using the similar arguments as Equation (49), we can conclude that $\{v_n\}$ is bounded $(PS)_{m_V}$ sequence of I_V. Similar arguments in Lemma 17, there exists a positive and nontrivial $v^* \in E$ such that $I_V(v^*) = m_V$, which implies that $u^* = f(v^*)$ is a ground state solution for Equation (1). By strong maximum principle, $u^* = f(v^*)$ is a positive ground state solution for Equation (1). The proof is complete. □

5. Discussion

Our results generalize partial results in Xu and Chen [8] and Zhao and Zhao [16]. The case of $p \in [1, 2)$ is still unknown, which can be a problem for further study.

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