mathematics

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Split Variational Inclusion Problem and Fixed Point Problem for a Class of Multivalued Mappings in $CAT(0)$ Spaces

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Received: 27 June 2019; Accepted: 12 August 2019; Published: 16 August 2019

Abstract: The aim of this paper is to introduce a modified viscosity iterative method to approximate a solution of the split variational inclusion problem and fixed point problem for a uniformly continuous multivalued total asymptotically strictly pseudocontractive mapping in $CAT(0)$ spaces. A strong convergence theorem for the above problem is established and several important known results are deduced as corollaries to it. Furthermore, we solve a split Hammerstein integral inclusion problem and fixed point problem as an application to validate our result. It seems that our main result in the split variational inclusion problem is new in the setting of $CAT(0)$ spaces.

Keywords: split variational inclusion problem; fixed point problem; $CAT(0)$ space; total asymptotically strictly pseudocontractive mapping

1. Introduction

1.1. Cat(0) Space

Let (X,d) be a metric space. A geodesic path joining x and y is a map $c : [0,l] \subset \mathbb{R} \to X$ such that

i. $c(0) = x, c(l) = y$ and $d(x,y) = l$.
ii. c is an isometry: $d(c(t), c(s)) = |t-s|$ for all $t,s \in [0,l]$.

In this case, $c([0,l])$ is called a geodesic segment joining x and y which when unique is denoted by $[x,y]$.

The space (X,d) is said to be a geodesic space if any two points of X are joined by a geodesic segment.

A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ).

A comparison triangle for geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\bar{\Delta}(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in $(\mathbb{R}^2,d)$ such that $d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. Such a triangle always exists (Bridson and Haefliger [2]).

A metric space X is said to be a $CAT(0)$ space if it is geodesically connected and every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane.

Let Δ be a geodesic triangle in X, and let $\bar{\Delta}$ be its comparison triangle in $\mathbb{R}^2$. Then, X is said to satisfy $CAT(0)$ inequality, if, for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

If $x, y_1, y_2 \in X$, and y_0 is the midpoint of the segment $[y_1, y_2]$, then, the $CAT(0)$ inequality implies

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{1}$$

It is well known that the following spaces are $CAT(0)$ spaces: a complete, simply connected Riemannian manifold with non-positive sectional curvature, Pre-Hilbert spaces [2], Euclidean buildings [3], R-trees [18], and Hilbert ball with a hyperbolic metric [10,16].

1.2. Some Basic Concepts in Hilbert Space

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed and convex subset of H.

The inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ generates norm via

$$\langle x, x \rangle = \|x\|^2$$

for all $x \in H$.

A mapping $T : C \to C$ is said to be total asymptotically strictly pseudocontractive (see [4]), if there exists a constant $\gamma \in [0,1]$ such that

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + \gamma\|x - T^n x\|^2 + \kappa_n \varphi(\|x - y\|) + \mu_n$$

holds for all $x, y \in C$, the sequences $\kappa_n, \mu_n \in [0,\infty)$ satisfy $\lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} \mu_n = 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is strictly increasing and continuous mapping with $\varphi(0) = 0$.

For concepts such as bounded linear operator and its adjoint operator, maximal monotone operator and metric projection, we refer to Chidume [5].

The metric projection is parity and scale invariant (cf. Proposition 1.26(e) in [30]) in the sense that

$$\lambda P_C x = P_{\lambda C} \lambda x, \ for\ every\ \lambda \geq 0,\ x \in H,$$

consequently,

$$\lambda(P_{C-x}(x) - x) = P_{\lambda C - \lambda x}(\lambda x) - \lambda x, \ for\ every\ \lambda \geq 0,\ x \in H.$$

1.3. Counterpart of the above Concepts in the Setting of a Cat(0) Space

A mapping $T : C \to C$ is said to be total asymptotically strictly pseudocontractive if there exists $\gamma \in [0,1]$ such that

$$d(T^n x, T^n y)^2 \leq d(x,y)^2 + \gamma d(x, T^n x)^2 + \kappa_n \varphi(d(x,y)) + \mu_n \tag{2}$$

holds for all $x, y \in C$, the sequences $\kappa_n, \mu_n \in [0,\infty)$ satisfy $\lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} \mu_n = 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is strictly increasing and continuous mapping with $\varphi(0) = 0$.

Define an addition $(x,y) \mapsto x \oplus y$ and a scalar multiplication $(\alpha, x) \mapsto \alpha \cdot x$ in the space X as follows: for any $z \in X$ and $\alpha, \beta \in \mathbb{R}$, we denote the point $z = \alpha x \oplus \beta y$ such that $d(x,z) = d((1-\alpha)x, \beta y)$.

A mapping $A : X \to X$ is said to be linear if for $x, y \in X$, we have

$$A(\alpha x \oplus \beta y) = \alpha A(x) \oplus \beta A(y).$$

A mapping $A : X \to X$ is said to be bounded if for all $x, y \in X$, there exists $M \geq 0$ such that

$$d(Ax, Ay)^2 \leq Md(x, y)^2,$$

Let C be a nonempty subset of a $CAT(0)$ space X.

In [1], a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to R$ is said to be quasi-linearization in X if

$$\langle \overrightarrow{pq}, \overrightarrow{rs} \rangle \;=\; \frac{1}{2}(d(p,s)^2 + d(q,r)^2 - d(p,r)^2 - d(q,s)^2), \tag{3}$$

holds for all $p, q, r, s \in X$; here a pair $(p, q) \in X \times X$ is denoted by a vector $\overrightarrow{pq}$. Consequently, we have

1. $\langle \overrightarrow{pq}, \overrightarrow{pq} \rangle \;=\; d(p, q)^2$,
2. for all $p, q, r, s, t, u, v, w \in X$,

$$\left\langle \overrightarrow{\overrightarrow{pq}\,\overrightarrow{rs}}, \overrightarrow{\overrightarrow{tu}\,\overrightarrow{vw}} \right\rangle \;=\; \left\langle \overrightarrow{pq}, \overrightarrow{tu} \right\rangle - \langle \overrightarrow{pq}, \overrightarrow{vw} \rangle - \left\langle \overrightarrow{rs}, \overrightarrow{tu} \right\rangle + \langle \overrightarrow{rs}, \overrightarrow{vw} \rangle. \tag{4}$$

A mapping $A^* : X \to X$ is said to be adjoint operator of A if for all $x, y, w, z \in X$, we have

$$\langle \overrightarrow{AxAw}, \overrightarrow{yz} \rangle \;=\; \langle A\overrightarrow{xw}, \overrightarrow{yz} \rangle \;=\; \langle \overrightarrow{xw}, A^* \overrightarrow{yz} \rangle \;=\; \langle \overrightarrow{xw}, \overrightarrow{A^*yA^*z} \rangle. \tag{5}$$

Clearly, A^* is a linear operator when so is A. As in a Hilbert space, we have

$$d(A^*y, A^*z)^2 \;=\; d(Ax, Aw)^2 \leq Md(x, w)^2,$$

and hence, A^* is bounded in X.

For any $x \in X$, there exists a unique point $x_0 \in C$ such that

$$d(x, x_0) \leq d(x, y) \;\forall\, y \in C,$$

and the mapping $P_C : X \to C$ defined by $P_C x \;=\; x_0$ is called the metric projection of X onto C (cf. Proposition 2.4 in [2]). Equivalently, in view of the characterization of Hossein and Jamal [12], we have

$$\langle \overrightarrow{x_0 x}, \overrightarrow{y x_0} \rangle \geq 0,$$

consequently,

$$P_{\overrightarrow{Cx}} : X \to \overrightarrow{Cx} \text{ is defined by } \overrightarrow{P_{\overrightarrow{Cx}}(x)x} \;=\; \overrightarrow{x_0 x},$$

equivalently,

$$\begin{aligned}
&\left\langle \overrightarrow{\overrightarrow{x_0 x}\,\overrightarrow{xx}}, \overrightarrow{\overrightarrow{yx}\,\overrightarrow{x_0 x}} \right\rangle \geq 0, \\
\Leftrightarrow &\left\langle \overrightarrow{x_0 x}, \overrightarrow{\overrightarrow{yx}\,\overrightarrow{x_0 x}} \right\rangle \geq 0 \text{ (because } \overrightarrow{xx} \text{ is an additive identity element)}, \\
\Leftrightarrow &\left(\langle \overrightarrow{x_0 x}, \overrightarrow{yx} \rangle - \langle \overrightarrow{x_0 x}, \overrightarrow{x_0 x} \rangle\right) \geq 0 \text{ (by (4))},
\end{aligned} \tag{6}$$

where $\overrightarrow{x_0 x}, \overrightarrow{yx} \in \overrightarrow{Cx}$.

The metric projection is parity and scale invariant in the sense that

$$\lambda P_C x \;=\; P_{\lambda C} \lambda x, \;\; for\ every\ \lambda \geq 0,\; x \in X,$$

consequently,

$$\lambda \overrightarrow{P_{\overrightarrow{Cx}}(x)x} = \overrightarrow{P_{\overrightarrow{\lambda C\lambda x}}(\lambda x)\lambda x}, \; for\ every\ \lambda \geq 0,\ x \in X. \tag{7}$$

1.4. Fixed Point Theory in a $Cat(0)$ Space

Fixed point theory in a $CAT(0)$ space has been introduced by Kirk (see for example [19]). He established that a nonexpansive mapping defined on a bounded, closed and convex subset of a complete $CAT(0)$ space has a fixed point. Consequently, fixed point theorems in $CAT(0)$ spaces have been developed by many mathematicians; see for example [8,29]. More so, some of these theorems in $CAT(0)$ spaces are applicable in many fields of studies such as, graph theory, biology and computer science (see for example [9,18,20,31]).

Let $T : X \to 2^X$ be a multivalued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$ and $F(T) = \{x \in X : x \in Tx\}$ is called the fixed point set of T.

1.5. Our Motivation

As a generalized version of the well known split common fixed point problem, Moudafi [25] introduced the following split monotone variational inclusion (SMVI) by using maximal monotone mappings;

$$\begin{aligned} &find\ x^* \in H_1\ such\ that\ 0 \in f(x^*) + B_1(x^*),\\ &y^* = Ax^* \in H_2\ solves\ 0 \in g(y^*) + B_2(y^*), \end{aligned}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are given single-valued operators.

In 2000, Moudafi [26] proposed the viscosity approximation method by considering the approximate well-posed problem of a nonexpansive mapping S with a contraction mapping f over a nonempty closed and convex subset; in particular he showed that given an arbitrary x_1 in a nonempty closed and convex subset, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n,$$

where $\{\alpha_n\} \subset (0, 1)$ with $\alpha_n \to 0$ as $n \to \infty$, converges strongly to the fixed point set of S, $F(S)$.

In [28], viscosity approximation method for split variational inclusion and fixed point problems in Hilbert Spaces was presented as follows.

$$\begin{cases} u_n &= J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n);\\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)T^n(u_n), \forall n \geq 1, \end{cases} \tag{8}$$

where B_1 and B_2 are maximal monotone operators, $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are resolvent mappings of B_1 and B_2 respectively, f is a Meir-Keeler mapping, T a nonexpansive mapping, A^* is an adjoint of A, $\gamma_n, \alpha_n \in (0, 1)$ and $\lambda > 0$.

In this paper, motivated by (8), we present a modified viscosity algorithm sequence and prove strong convergence theorem for split variational inclusion problem and fixed point problem of a total asymptotically strictly pseudocontractive mapping in the setting of two different $CAT(0)$ spaces. It seems that our main result is new in the setting of $CAT(0)$ spaces.

2. Preliminaries

Denote by $CB(X)$, the collection of all nonempty closed and bounded subsets of X and let H be the Hausdorff metric with respect to the metric d; that is,

$$H(A,B) \;=\; \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\} \tag{9}$$

for all $A, B \in CB(X)$, where $d(a,B) \;=\; \inf_{b\in B} d(a,b)$ is the distance from the point a to the subset B.

Let X be a complete $CAT(0)$ space with its dual X^* (for details, see [17]). A mapping $G : D(G) \subset X \to 2^X$ is said to be monotone if

$$\langle \overrightarrow{xy}, \overrightarrow{x^*y^*} \rangle \geq 0 \; \forall x, y \in D(G),\; x^* \in Gx,\; y^* \in Gy.$$

A mapping $G : D(G) \subset X \to 2^X$ is said to be maximal monotone if it is monotone and also has no monotone extension, that is, its graph $gr(G) : \;=\; \{(x, x^*) \in X \times X^* : x^* \in G(x)\}$ is not properly contained in the graph of any other monotone operator on X.

For $\gamma > 0$, a mapping $B_\gamma^G \;=\; (I + \gamma G)^{-1} : X \to 2^X$ defined by $B_\gamma^G(x) \;=\; \{z \in X : [\frac{1}{\gamma}\overrightarrow{zx}] \in G(z)\}$ is said to be a resolvent of G.

The operator G is said to satisfy the range condition if for every $\gamma > 0$, $D(B_\gamma^G) \;=\; X$.

Let X be a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in X. Then the asymptotic center of $\{x_n\}$ is defined by

$$A(\{x_n\}) \;=\; \{x \in X : \limsup_{n\to\infty} d(x, x_n) \leq \limsup_{n\to\infty} d(z, x_n), \forall z \in X\}.$$

The asymptotic center $A(\{x_n\})$, consists of exactly one point ([6]).

Definition 1. *A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of any subsequence $\{x_{n_k}\} \subset \{x_n\}$. Symbolically, we write it as $\Delta - \lim_{n\to\infty} x_n \;=\; x$ [21,22].*

Lemma 1. *Let $\{x_n\}$ be a bounded sequence in a complete $CAT(0)$ space X [21]. Then*

i. *$\{x_n\}$ has a Δ-convergent subsequence.*
ii. *the asymptotic center of $\{x_n\} \subset C \subset X$ is in C, where C is nonempty, closed and convex.*

Lemma 2. *Let $\{x_n\}$ be a bounded sequence in a complete $CAT(0)$ space and $A(\{x_n\}) \;=\; \{x\}$. Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$ and $A(\{x_{n_k}\}) \;=\; \{y\}$. If $\lim_{n\to\infty} d(x_n, y)$ exists, then $x \;=\; y$ [7].*

Let C be closed and convex subset of a $CAT(0)$ space X and $\{x_n\}$ a bounded sequence in C. Then the relation $x_n \rightharpoonup x$ is described by

$$\limsup_{n\to\infty} d(x_n, x) \;=\; \inf_{y\in C} \limsup_{n\to\infty} d(x_n, y).$$

Lemma 3. *[27] Let C be closed and convex subset of a $CAT(0)$ space X and $\{x_n\}$ a bounded sequence in C. Then $\Delta - \lim_{n\to\infty} x_n \;=\; x$ implies that $x_n \rightharpoonup x$.*

Lemma 4. *[7] Let X be a $CAT(0)$ space and $x, y, z \in X$. Then*

i. $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z), t \in [0,1]$,
ii. $d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2, t \in [0,1]$.

Lemma 5. *[13] Let X be a complete $CAT(0)$ space, $\{x_n\}$ a sequence in X and $x \in X$. Then $\{x_n\}$, $\Delta-$converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle \leq 0$ for all $y \in X$.*

Lemma 6. *[34] Let X be a complete $CAT(0)$ space. Then for all $x, y, z \in X$, the following inequality holds*

$$d(x,z)^2 \leq d(y,z)^2 + 2\langle \overrightarrow{xy}, \overrightarrow{xz} \rangle.$$

3. Main Results

Let X_1 and X_2 be two $CAT(0)$ spaces, $C \subset X_1$ be a closed and convex subset, $A : X_1 \to X_2$ bounded linear and unitary operator, $U : X_1 \to 2^{X_1}$ and $S : X_2 \to 2^{X_2}$ be uniformly continuous and maximal monotone operators, $f : X_1 \to X_1$ contraction mapping and $T : C \to CB(C)$ be uniformly continuous multivalued total asymptotically strictly pseudocontractive mapping defined as

$$H(T^n x, T^n y)^2 \leq d(x,y)^2 + \gamma d(x, T^n x)^2 + \kappa_n \varphi(d(x,y)) + \mu_n$$

where $x, y \in C$ and the sequences $\kappa_n, \mu_n \in [0, \infty)$ satisfy $\sum_{n=1}^{\infty} \kappa_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Suppose that $\gamma \in [0,1]$ and $\varphi : [0,\infty) \to [0,\infty)$ is strictly increasing and continuous mapping such that $\varphi(0) = 0$, and $P_{AC} : X_2 \to AC$ and $P_{\overrightarrow{ACAx}} : X_2 \to \overrightarrow{ACAx}$ are the metric projections onto, respectively, nonempty closed and convex subset AC and $\overrightarrow{ACAx}$ of X_2, where $AC = \{Ax, \forall x \in C\}$ and $\overrightarrow{ACAx} = \{\overrightarrow{AyAx}, \forall y \in C$ and Ax fixed$\}$. Let, for $\gamma > 0$, $B_\gamma^U : 2^{X_1} \to X_1$ and $B_\gamma^S : 2^{X_2} \to X_2$ be resolvent operators for U and S, respectively. Denoted by $VIP(U, \gamma)$ and $VIP(S, \gamma)$, and $F(T)$ the solution set of variational inequality problems with respect to U and S and fixed point problem with respect to T.

As in [25], we define the split variational inclusion (SVI) as follows:

$$\text{find } x \in X_1 \textit{ such that } \overrightarrow{xx} \in U(x) \text{ and } Ax \in X_2 \textit{ solves } \overrightarrow{AxAx} \in S(Ax),$$

where $\overrightarrow{xx}$ and $\overrightarrow{AxAx}$ are the additive identity elements in X_1 and X_2, respectively.

Denoted by $F(T)$ is the fixed point set of a map T, let $F(T) \neq \emptyset$ and $p \in F(T)$. Then T is multivalued total quasi-asymptotically strictly pseudocontractive mapping if

$$H(T^n x, T^n p)^2 \leq d(x,p)^2 + \gamma d(x, T^n x)^2 + \kappa_n \varphi(d(x,p)) + \mu_n.$$

Remark 1. *Please note that a multivalued total asymptotically strictly pseudocontractive mapping is multivalued total quasi-asymptotically strictly pseudocontractive provided, its fixed point set is nonempty.*

Throughout this paper we shall strictly employ the above terminology.

For a bounded sequence $\{x_n\}$ in C, we employ the notion:

$$\limsup_{n\to\infty} d(x_n, x) = \inf_{y \in C} \limsup_{n\to\infty} d(x_n, y), \tag{10}$$

equivalently x is the asymptotic center of each subsequence of $\{x_n\}$.

Following Karapinar et al [14], we first establish a demiclosedness principle based on (10).

Lemma 7. *(Demiclosedness of T) Let T be a multivalued total asymptotically strictly pseudocontractive mapping on a closed and convex subset C of a $CAT(0)$ space X. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta - \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $x \in Tx$.*

Proof. By the hypothesis $\Delta - \lim_{n\to\infty} x_n = x$ and so by Lemma 3, we get $\{x_n\} \rightharpoonup x$. Then by Lemma 1 (ii), we arrive at $A(\{x_n\}) = \{x\}$. Let $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. So we obtain

$$\limsup_{n\to\infty} d(x_n, y) = \limsup_{n\to\infty} d(y, Tx_n), \tag{11}$$

for all $y \in C$. From the hypothesis that T is multivalued total asymptotically strictly pseudocontractive mapping and by (11), choosing $y \in Tx$, we have

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, y)^2 &= \limsup_{n\to\infty} d(y, Tx_n)^2 \le \limsup_{n\to\infty} H(Tx_n, Tx)^2 \\ &\le \limsup_{n\to\infty} \{d(x_n, x)^2 + \gamma d(x_n, Tx_n)^2 + \kappa_n \varphi(d(x_n, x)) + \mu_n\} \\ &= \limsup_{n\to\infty} d(x_n, x)^2. \end{aligned} \tag{12}$$

□

By (1), we get

$$d\left(x_n, \frac{x \oplus y}{2}\right)^2 \le \frac{1}{2} d(x_n, x)^2 + \frac{1}{2} d(x_n, y)^2 - \frac{1}{4} d(x, y)^2.$$

Let $n \to \infty$ and take superior limit on the both sides of the above inequality and get

$$\begin{aligned}\limsup_{n\to\infty} d\left(x_n, \frac{x \oplus y}{2}\right)^2 \le& \frac{1}{2} \limsup_{n\to\infty} d(x_n, x)^2 \\ &+ \frac{1}{2} \limsup_{n\to\infty} d(x_n, y)^2 - \frac{1}{4} d(x, y)^2.\end{aligned}$$

Since $A(\{x_n\}) = \{x\}$, therefore we have

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, x)^2 \le \limsup_{n\to\infty} d\left(x_n, \frac{x \oplus y}{2}\right)^2 \le& \frac{1}{2} \limsup_{n\to\infty} d(x_n, x)^2 \\ &+ \frac{1}{2} \limsup_{n\to\infty} d(x_n, y)^2 - \frac{1}{4} d(x, y)^2,\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} d(x_n, x)^2 \le \limsup_{n\to\infty} d(x_n, y)^2. \tag{13}$$

By (12) and (13), we conclude that $x = y$ and therefore $x \in Tx$, as desired.

Next, we prove our main result as follows.

Theorem 1. *Let $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} y_n = B^U_{\gamma_n}\left(\alpha_n x_n \oplus (1 - \alpha_n)\lambda_n A^* \overrightarrow{P_{AC_n Ax_n} B^S_{\gamma_n}(Ax_n) Ax_n}\right) \\ x_{n+1} = \beta_n f(x_n) \oplus (1 - \beta_n) z_n,\ z_n \in \{T_n y_n\}, n \ge 1, \end{cases} \tag{14}$$

where A^* is the adjoint operator of A, and $M, \lambda_n, \alpha_n, \beta_n \in [0, 1]$. Suppose that AC is closed and convex, $P_{AC} B^S_\gamma$ is demiclosed and $\Gamma = \{x \in VIP(U, \gamma) : Ax \in VIP(S, \gamma)\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. *there exists constant $N > 0$ such that $\varphi(r) \le Nr$, $r \ge 0$;*
2. $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \alpha_n = 0$;
3. *T satisfies the asymptotically regular condition $\lim_{n\to\infty} d(y_n, T_n y_n) = 0$.*

Then $\{x_n\}$ converges strongly to a point $x \in \Gamma$, where $P_{AC} B^S_\gamma(Ax) = B^S_\gamma(Ax)$.

Proof. We will divide the proof into three steps.

Step one . We prove that $\{x_n\}$ is bounded.
If $p \in \Gamma$, then by Lemma 4(ii) and (9) we obtain

$$\begin{aligned} d(y_n,p)^2 &= d\left(B^U_{\gamma_n}\left(\alpha_n x_n \oplus (1-\alpha_n)\lambda_n A^* \overrightarrow{P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)Ax_n}\right), p\right)^2 \\ &\leq \alpha_n d\,(x_n,p)^2 + (1-\alpha_n) d\left(\lambda_n A^* \overrightarrow{P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)Ax_n}, p\right)^2 \end{aligned} \tag{15}$$

whereas, by (6), (4), (5), and boundedness, linearity and unitary property of A, we have,

$$\begin{aligned} & d\left(\overrightarrow{\left(\lambda_n A^* P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)\right) Ax_n}, p\right)^2 \\ &= \left\langle \overrightarrow{\overrightarrow{\left(\lambda_n A^* P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)\right) Ax_n} p}, \overrightarrow{\overrightarrow{\left(\lambda_n A^* P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)\right) Ax_n} p} \right\rangle \\ &= \left\langle \overrightarrow{\overrightarrow{\left(\lambda_n P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)\right) Ax_n} Ap}, \overrightarrow{\overrightarrow{\left(\lambda_n P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)\right) Ax_n} Ax_n} \right\rangle \\ &- \left\langle \overrightarrow{ApAx_n}, \overrightarrow{\overrightarrow{\left(\lambda_n P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)\right) Ax_n} Ax_n} \right\rangle \\ &- \left\langle \overrightarrow{\overrightarrow{\left(\lambda_n P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)\right) Ax_n} Ax_n}, \overrightarrow{ApAx_n} \right\rangle + \left\langle \overrightarrow{ApAx_n}, \overrightarrow{ApAx_n} \right\rangle \\ &\leq -\left\langle \overrightarrow{\overrightarrow{\left(\lambda_n P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)\right) Ax_n} Ax_n}, \overrightarrow{ApAx_n} \right\rangle + Md\,(x_n,p)^2. \end{aligned} \tag{16}$$

Substituting (16) into (15), we get

$$\begin{aligned} d(y_n,p)^2 &\leq -(1-\alpha_n)\left\langle \overrightarrow{\overrightarrow{\left(\lambda_n P_{\overrightarrow{AC_nAx_n}} B^{S_n}_{\gamma_n}(Ax_n)\right) Ax_n} Ax_n}, \overrightarrow{ApAx_n} \right\rangle \\ &+ (M(1-\alpha_n)+\alpha_n) d\,(x_n,p)^2 \qquad (17) \\ &\leq d\,(x_n,p)^2. \qquad (18) \end{aligned}$$

By Remark 1, (14), (2), Lemma 4(ii) and (9), we get

$$\begin{aligned} d(x_{n+1},p)^2 &= d\,(\beta_n f(x_n) \oplus (1-\beta_n) z_n, p)^2 \\ &\leq \beta_n d\,(f(x_n),p)^2 + (1-\beta_n) d\,(z_n,p)^2 - \beta_n(1-\beta_n) d\,(f(y_n),z_n)^2 \\ &\leq \beta_n (d\,(f(x_n),f(p))^2 + d\,(f(p),p)^2) + (1-\beta_n) H\,(T_n y_n, T_n p)^2 \\ &\leq \beta_n d\,(f(p),p)^2 + (1-(1-\zeta)\beta_n)(1+\kappa_n N)) d\,(x_n,p)^2 \\ &+ (1-\beta_n)\gamma d\,(y_n, T_n y_n)^2 + (1-\beta_n)\mu_n. \end{aligned} \tag{19}$$

Since $\sum_{n=1}^{\infty} \kappa_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$ and γ is arbitrary in $[0,1]$, therefore by (19), we get

$$\begin{aligned} d(x_{n+1},p)^2 &\leq \beta_n d\,(f(p),p)^2 + (1-(1-\zeta)\beta_n) d\,(x_n,p)^2 \\ &\leq \max\{d\,(x_n,p)^2, \frac{1}{1-\zeta} d\,(f(p),p)^2\} \\ &\;\;\vdots \\ &\leq \max\{d\,(x_1,p)^2, \frac{1}{1-\zeta} d\,(f(p),p)^2\}. \end{aligned} \tag{20}$$

By (18) and (20), we have that $\{x_n\}$ and $\{y_n\}$ are bounded. Hence $\{T_n y_n\}$ and $\{f(x_n)\}$ are also bounded.

Step two. We will show that $\lim_{n\to\infty} d(P_{AC_n} B^S_{\gamma_n}(Ax_n), Ax_n) = 0.$

By Lemmas 1 and 2, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{n\to\infty} x_{n_j} = x \in C$. Thus, $\Delta - \lim_{n\to\infty} x_n = x$. By Lemmas 5 and 6, we get

$$\begin{aligned} d\left(x_{n+1}, x_n\right)^2 &\leq 2d\left(x_{n+1}, x\right)^2 + 2d\left(x, x_n\right)^2 \\ &= 2\left\langle \overrightarrow{x_{n+1}x}, \overrightarrow{x_{n+1}x} \right\rangle + 2\left\langle \overrightarrow{x_n x}, \overrightarrow{x_n x}\right\rangle \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{21}$$

This implies that $x_n \to x$ as $n \to \infty$.

In addition, by Lemma 4(ii) we have

$$\begin{aligned} d\left(y_n, x_{n+1}\right)^2 &= d\left(y_n, \beta_n f(x_n) \oplus (1-\beta_n) z_n\right)^2 \\ &\leq \beta_n d\left(y_n, f(x_n)\right)^2 + (1-\beta_n) d\left(y_n, z_n\right)^2 \\ &\leq \beta_n d\left(y_n, f(x_n)\right)^2 + (1-\beta_n) d\left(y_n, T_n y_n\right)^2 \\ &\longrightarrow 0 \text{ as } n \to \infty, \end{aligned} \tag{22}$$

and therefore by (21), (22) and Lemma 6, we get

$$\begin{aligned} d\left(y_n, x_n\right)^2 &\leq d\left(y_n, x_{n+1}\right)^2 + d\left(x_{n+1}, x_n\right)^2 \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{23}$$

This implies that $y_n \to x$ as $n \to \infty$.

As λ_n is arbitrary in $[0,1]$, so by (17), (3) and (7) we arrive at

$$\begin{aligned} &(1-\alpha_n)\left\langle \overrightarrow{\lambda_n P_{\overrightarrow{AC_n Ax_n}} B^S_{\gamma_n}(Ax_n) Ax_n Ax_n}, \overrightarrow{ApAx_n} \right\rangle \leq d(x_n, p)^2 - d(y_n, p)^2 \\ &\Longrightarrow 2(1-\alpha_n) d\left(\lambda_n \overrightarrow{P_{\overrightarrow{AC_n Ax_n}} B^S_{\gamma_n}(Ax_n) Ax_n}, Ax_n\right)^2 \leq d(x_n, p)^2 - d(y_n, p)^2 \\ &\Longrightarrow 2(1-\alpha_n) d\left(\overrightarrow{P_{\overrightarrow{\lambda_n AC_n \lambda_n Ax_n}} B^S_{\gamma_n}(\lambda_n Ax_n) \lambda_n Ax_n}, Ax_n\right)^2 \leq d(x_n, p)^2 - d(y_n, p)^2 \\ &\Longrightarrow 2(1-\alpha_n) d\left(\overrightarrow{P_{\overrightarrow{AC_n Ax_n}} B^S_{\gamma_n}(Ax_n) Ax_n}, Ax_n\right)^2 \leq d(x_n, p)^2 - d(y_n, p)^2 \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{24}$$

It follows from (24) that

$$d\left(P_{AC_n} B^S_{\gamma_n}(Ax_n), Ax_n\right) \longrightarrow 0 \text{ as } n \to \infty. \tag{25}$$

Step three. We show that $x_n \to x \in \Gamma$.
By (14), we obtain

$$\begin{aligned}
& \alpha_n x_n \oplus (1-\alpha_n)\lambda_n A^* \overrightarrow{P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)Ax_n} \in y_n \oplus \gamma_n U_n(y_n) \\
\Longrightarrow & \\
& x_n \in y_n \oplus \gamma_n U_n(y_n) \text{ (since } \alpha_n \text{ is arbirary in } [0,1]). \\
\Longrightarrow & \\
& \overrightarrow{x_n y_n} \in \gamma_n U_n(y_n) \qquad (26)
\end{aligned}$$

Since U and S are uniformly continuous, therefore it follows by (26), as $n \to \infty$, that $\overrightarrow{xx} \in U(x)$. In addition, it is clear that $\Delta - \lim\limits_{n\to\infty} Ax_n = Ax$. So by using (25) and applying the demiclosedness of $P_{AC}B^S_\gamma$, we have that $\overrightarrow{AxAx} \in SAx$, as $P_{AC}B^S_\gamma Ax = B^S_\gamma Ax$. On the other hand, by Lemma 7 and $\Delta - \lim\limits_{n\to\infty} y_n = x$ (by (23)), we have by the hypothesis $\lim\limits_{n\to\infty} d(Ty_n, y_n) = 0$ that $x \in Tx$, as T is uniformly continuous. Hence, $x \in \Gamma$. $\square$

The proof is completed.

If $U : X_1 \to X_1$ and $S : X_2 \to X_2$ are total asymptotically strictly pseudocontractive in Theorem 1 and their fixed point sets $F(U)$ and $F(S)$ are nonempty, then we get:

Corollary 1. *Let $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} y_n = U_n\left(\alpha_n x_n \oplus (1-\alpha_n)\lambda_n A^* \overrightarrow{P_{\overrightarrow{AC_nAx_n}} S_n(Ax_n)Ax_n}\right) \\ x_{n+1} = \beta_n f(x_n) \oplus (1-\beta_n) z_n,\ z_n \in \{T_n y_n\}, n \geq 1, \end{cases}$$

where A^* is the adjoint operator of A, and $M, \lambda_n, \alpha_n, \beta_n \in [0,1]$. Suppose that AC is closed and convex, $\Gamma = \{x \in F(U) : Ax \in F(S)\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. *there exists constant $N > 0$ such that $\varphi(r) \leq Nr, r \geq 0$;*
2. $\lim\limits_{n\to\infty} \beta_n = \lim\limits_{n\to\infty} \alpha_n = 0$;
3. *T satisfies the asymptotically regular condition $\lim\limits_{n\to\infty} d(y_n, T_n y_n) = 0$.*

Then $\{x_n\}$ converges strongly to a point $x \in \Gamma$, where $P_{AC}S(Ax) = S(Ax)$.

Remark 2. *Corollary 1 is about split common fixed point problem and fixed point problem. Hence, this result is new in the literature; in particular, it generalizes similar results in [24,33] from Banach space setting to $CAT(0)$ spaces.*

In Theorem 1, let $\overrightarrow{P_{\overrightarrow{AC_nAx_n}}(Ax_n)Ax_n} = \overrightarrow{P_{\overrightarrow{AC_nAx_n}} B^S_{\gamma_n}(Ax_n)Ax_n}$ and $P_C = B^U_{\gamma_n}$, where $P_C : X_1 \to C$ is the metric projection of X_1 onto C. Then we get the following result.

Corollary 2. *Let $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} y_n = P_{C_n}\left(\alpha_n x_n \oplus (1-\alpha_n)\lambda_n A^* \overrightarrow{P_{\overrightarrow{AC_nAx_n}}(Ax_n)Ax_n}\right) \\ x_{n+1} = \beta_n f(x_n) \oplus (1-\beta_n) z_n,\ z_n \in \{T_n y_n\}, n \geq 1, \end{cases}$$

where A^* is the adjoint operator of A, and $M, \lambda_n, \alpha_n, \beta_n \in [0,1]$. Suppose that AC is closed and convex, $\Gamma = \{x \in C : Ax \in AC\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. *there exists a constant $N > 0$ such that $\varphi(r) \leq Nr, r \geq 0$;*
2. $\lim\limits_{n\to\infty} \beta_n = \lim\limits_{n\to\infty} \alpha_n = 0$;

3. *T satisfies the asymptotically regular condition* $\lim_{n\to\infty} d(y_n, T_n y_n) = 0$.

   Then $\{x_n\}$ *converges strongly to a point* $x \in \Gamma$.

Remark 3. *As Corollary 2 deals with split feasibility problem and fixed point problem so it is a new result in the literature. It also extends similar results in Banach spaces [15,32] to the case of $CAT(0)$ spaces.*

4. Application to Split Hammerstein Integral Inclusion and Fixed Point Problem

An integral equation of Hammerstien-type is of the form

$$u(x) + \int_C k(x,y) f(y, u(y)) dy = g(x)$$

(see [11]).

By writing the above equation in the following form

$$u + KFu = g,$$

without loss of generality, we have

$$u + KFu = 0. \tag{27}$$

If instead of the singlevalued maps f and k, we have the multivalued functions f and k, then we obtain Hammerstein integral inclusion in the form $0 \in u \oplus KFu$, where $F : X_1 \to CB(X_1)$ defined by $Fu(y) := \{v(y) : v$ is some selection of $f(\cdot, u(\cdot))\}$ and $K : X_1 \to CB(X_1)$ defined by $Kv(x) := \{w(x) : w$ is some selection of $k(\cdot, y)\}$, are bounded and maximal monotone operators (see for example [23]).

So the split Hammerstein integral inclusion problem is formulated as: find $x^*, y^* \in X_1 \times X_1$ such that, for $v(\cdot) \in Fu(\cdot)$ and $w(\cdot) \in Kv(\cdot)$

$$x^* \oplus w(v(x^*)) = 0 \text{ with } v(x^*) = y^* \text{ and } w(y^*) \oplus x^* = 0$$

and $Ax^*, Ay^* \in X_2 \times X_2$ such that, for $v'(\cdot) \in F'u'(\cdot)$ and $w'(\cdot) \in K'v'(\cdot)$,

$$Ax^* \oplus w'(v'(Ax^*)) = 0 \text{ with } v'(Ax^*) = Ay^* \text{ and } w'(Ay^*) \oplus Ax^* = 0$$

where $F' : X_2 \to CB(X_2)$ and $K' : X_2 \to CB(X_2)$, defined as F and K, respectively, are also bounded and maximal monotone.

Lemma 8. *Let X be a $CAT(0)$ space, $E := X \times X$ and let $F : dom(F) \subseteq X \to CB(X)$, $K : dom(K) \subseteq X \to CB(X)$ be two multivalued maps. Define $D : dom(F) \times dom(K) \to CB(E)$ by $D(x,y) := \left(\overrightarrow{Fxy}\right) \times (Ky \oplus x)$ $\forall x, y \in dom(F) \times dom(K) = \left\{(\overrightarrow{v(y)y}, w(x) \oplus x) : v(y) \in Fu(y), w(x) \in Kv(x)\right\}$. Suppose that F and K are monotone. Then D is monotone.*

Proof. Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in E$ and let $\bar{w}_1 \in D(z_1), \bar{w}_2 \in D(z_2)$. Then $\bar{z}_1 = (\overrightarrow{v_1(y_1)y_1}, w_1(x_1) \oplus x_1)$, $\bar{z}_2 = (\overrightarrow{v_2(y_2)y_2}, w_2(x_2) \oplus x_2)$, for some $v_1(y_1) \in Fu_1$, $v_2(y_2) \in Fu_2$, $w_1(x_1) \in Ky_1$ and $w_2(x_2) \in Ky_2$. Therefore, by monotonicity of F and K, we get

$$\begin{aligned}
\langle \overrightarrow{z_1z_2}, \overrightarrow{\bar{z}_1\bar{z}_2} \rangle &= \langle (\overrightarrow{x_1x_2}, \overrightarrow{y_1y_2}), (\overrightarrow{\overrightarrow{v_1(y_1)v_2(y_2)}y_1 \oplus y_2}, \overrightarrow{w_1(x_1)w_2(x_2)} \oplus \overrightarrow{x_1x_2}) \rangle \\
&= \langle \overrightarrow{x_1x_2}, \overrightarrow{\overrightarrow{v_1(y_1)v_2(y_2)}y_1 \oplus y_2} \rangle + \langle \overrightarrow{y_1y_2}, \overrightarrow{w_1(x_1)w_2(x_2)} \oplus \overrightarrow{x_1x_2} \rangle \\
&= \langle \overrightarrow{x_1x_2}, \overrightarrow{\overrightarrow{v_1(y_1)v_2(y_2)}y_1} \rangle - \langle \overrightarrow{x_1x_2}, \overrightarrow{y_1y_2} \rangle + \langle \overrightarrow{y_1y_2}, \overrightarrow{w_1(x_1)w_2(x_2)} \rangle \\
&+ \langle \overrightarrow{y_1y_2}, \overrightarrow{x_1x_2} \rangle = \langle \overrightarrow{x_1x_2}, \overrightarrow{v_1(y_1)v_2(y_2)} \rangle + \langle \overrightarrow{y_1y_2}, \eta \overrightarrow{w_1(x_1)w_2(x_2)} \rangle \geq 0.
\end{aligned}$$

□

This completes the proof of the lemma.

By Lemma 8, we have two resolvent mappings,

$$B_{\gamma}^{D} = (I + \gamma D)^{-1} \text{ and } B_{\gamma}^{D'} = (I + \gamma D')^{-1},$$

where $D' : dom(F') \times dom(K') \to CB(E)$ is defined by

$$D'(Ax, Ay) := \left(\overrightarrow{F'AxAy}\right) \times (K'Ay \oplus Ax)$$

$\forall Ax, Ay \in dom(F') \times dom(K') = \left\{(\overrightarrow{v'(Ay)Ay}, w'(Ax) \oplus Ax) : v'(Ay) \in F'u(Ay), w'(Ax) \in K'v'(Ax)\right\}.$

Now D and D' are maximal monotone by Lemma 8. When $U = D$ and $S = D'$ in Theorem 1, the algorithm (1) becomes

$$\begin{cases} y_n = B_{\gamma_n}^{D}\left(\alpha_n x_n \oplus (1 - \alpha_n)\lambda_n A^* \overrightarrow{P_{\overrightarrow{AC_n Ax_n}} B_{\gamma}^{D'}(Ax_n) Ax_n}\right) \\ x_{n+1} = \beta_n f(x_n) \oplus (1 - \beta_n) z_n,\ z_n \in \{T_n y_n\}, n \geq 1, \end{cases}$$

and its strong convergence is guaranteed, which solves the split Hammerstein integral inclusion problem and fixed point problem for the mappings involved in this scheme.

Author Contributions: All authors contributed equally, and they also read and finalized manuscript.

Funding: This research was co-funded by DEANSHIP OF SCIENTIFIC RESEARCH (DSR) at KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS by Grant Project No. IN151014 and BASQUE GOVERNMENT by Grant IT1207-19.

Acknowledgments: The author A. R. Khan would like to acknowledge the support provided by the Deanship of Scientific Research (DSR) at King Fahd University of Petroleum and Minerals (KFUPM) for funding this work through project No. IN151014.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Berg, I.D.; Nikolaev, I.G. Quasilinearization and curvature of Alexandrov spaces. *Geom. Dedic.* **2008**, *133*, 195–218. [CrossRef]
2. Bridson, M.R.; Haefliger, A. Metric spaces of non-positive survature. In *Grundlehren der Mathematischen Wissenschaften*; Springer: Berlin, Germany, 1999; Volume 319.
3. Brown, K.S. *Buildings*; Springer: New York, NY, USA, 1989.
4. Chang, S.S.; Wang, L.; Tang, Y.K.; Yang, L. The split common fixed point problem for total asymptotically strictly pseudocontractive mappings. *J. Appl. Math.* **2012**. [CrossRef]
5. Chidume, C.E. *Applicable Functional Analysis*; Ibadan University Press, University of Ibadan: Ibadan, Nigeria, 2014.

6. Dhompongsa, S.; Kirk, W.A.; Sims, B. Fixed point of uniformly lipschitzian mappings. *Nonlinear Anal.* **2006**, *65*, 762–772. [CrossRef]
7. Dhompongsa, S.; Panyanak, B. On Δ-convergence theorems in $CAT(0)$ spaces. *Comput. Math. Appl.* **2008**, *56*, 2572–2579. [CrossRef]
8. Espinola, R.; Fernandez-Leon, A. CAT(k)-spaces, weak convergence and fixed points. *J. Math. Anal. Appl.* **2009**, *353*, 410–427. [CrossRef]
9. Espinola, R.; Kirk, W.A. Fixed point theorems in R-trees with applications to graph theory. *Topol. Appl.* **2006**, *153*, 1046–1055. [CrossRef]
10. Goebel, K.; Reich, S. Uniform convexity, hyperbolic geometry and nonexpansive mappings. In *Monographs and Textbooks in Pure and Applied Mathematics;* Marcel Dekker Inc.: New York, NY, USA, 1984; Volume 83.
11. Hammerstein, A. Nichtlineare integralgleichungen nebst anwendungen. *Acta Math.* **1930**, *54*, 117–176. [CrossRef]
12. Hossein, D.; Jamal, R. A characterization of metric projection in $CAT(0)$ spaces. *arXiv* **2013**, arXiv:1311.4174.
13. Kakavandi, B.A. Weak topologies in complete $CAT(0)$ Metric spaces. *Proc. Am. Math. Soc.* **2013**, *141*, 1029–1039. [CrossRef]
14. Karapinar, E.; Vaezpour, S.M.; Salahifard, H. Demiclosedness principle for total asymptotically nonexpansive mappings in $CAT(0)$ spaces. *J. Appl. Math.* **2013**, 738150.
15. Khan, A.R.; Abbas, M.; Shehu, Y. A general convergence theorem for multiple-set split feasibilty problem in Hilbert spaces. *Carpathian J. Math.* **2015**, *31*, 349–357.
16. Khan, A.R.; Shukri, S.A. Best proximity points in the Hilbert ball. *J. Nonlinear Convex Anal.* **2016**, *17*, 1083–1094.
17. Khatibzadeh, H.; Ranjbar, S. Monotone operators and the proximal point algorithm in complete $CAT(0)$ metric space. *J. Aust. Math. Soc.* **2017**, *103*, 70–90. [CrossRef]
18. Kirk, W.A. Fixed point theorems in $CAT(0)$ spaces and R-trees. *Fixed Point Theory Appl.* **2004**, *2004*, 309–316. [CrossRef]
19. Kirk, W.A. Geodesic geometry and fixed point theory. In *Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Coleccion Abierta;* University of Seville, Secretary of Publications: Seville, Spain, 2003; Volume 64, pp. 195–225.
20. Kirk, W.A. Some recent results in metric fixed point theory. *J. Fixed Point Theory Appl.* **2007**, *2*, 195–207. [CrossRef]
21. Kirk, W.A.; Panyanak, B. A concept of convergence in geodesic spaces. *Nonlinear Anal.* **2008**, *68*, 3689–3696.
22. Lim, T.C. Remarks on some fixed point theorems. *Proc. Am. Math. Soc.* **1976**, *60*, 179–182.
23. Minjibir, M.S.; Mohammed, I. Iterative algorithms for solutions of Hammerstein integral inclusions. *Appl. Math. Comput.* **2018**, *320*, 389–399. [CrossRef]
24. Moudafi, A. A note on the split common fixed point problem for quasi nonexpansive operators. *Nonlinear Anal.* **2011**, *74*, 4083–4087. [CrossRef]
25. Moudafi, A. Split monotone variational inclusions. *J. Optim. Theory Appl.* **2011**, *150*, 275–283. [CrossRef]
26. Moudafi, A. Viscosity approximation methods for fixed points problems. *J. Math. Anal. Appl.* **2000**, *241*, 46–55. [CrossRef]
27. Nanjaras, B.; Panyanak, B. Demiclosed principle for asymptotically nonexpansive mappings in $CAT(0)$ spaces. *Fixed Point Theory Appl.* **2010**, *2010*, 268780. [CrossRef]
28. Nimit, N.; Narin, P. Viscosity approximation methods for split variational inclusion and fixed point problems in hilbert spaces. In Proceedings of the International Multi-Conference of Engineers and Computer Scientists 2014, IMECS 2014, Hong Kong, China, 12–14 March 2014; Volume 2.
29. Sahin, A.; Basarır, M. On the strong and Δ-convergence theorems for nonself mappings on a $CAT(0)$ space. In Proceedings of the 10th ICFPTA, Cluj-Napoca, Romania, 9–18 July 2012; pp. 227–240.
30. Schopfer, F. *Iterative Methods for the Solution of the Split Feasibility Problem in Banach Spaces*; Naturwissenschaftlich-Technischen Fakultaten, Universitat des Saarlandes: Saarbrucken, Germany, 2007.
31. Semple, C.; Steel, M. Phylogenetics. In *Oxford Lecture Series in Mathematics and Its Applications*; Oxford University Press: Oxford, UK, 2003; Volume 24.
32. Takahashi, W. The split feasibility problem in Banach spaces. *J. Nonlinear Convex Anal.* **2014**, *15*, 1349–1355.

33. Tang, J.; Chang, S.S.; Wang, L.; Wang, X. On the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings in Banach spaces. *J. Inequal. Appl.* **2015**, *23*, 205–221. [CrossRef]
34. Wangkeeree, R.; Preechasilp, P. Viscosity approximation methods for nonexpansive mappings in $CAT(0)$ spaces. *J. Inequal. Appl.* **2013**, *2013*, 93. [CrossRef]