

Article

On Some New Weighted Inequalities for Differentiable Exponentially Convex and Exponentially Quasi-Convex Functions with Applications

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Abstract: In this article, we aim to establish several inequalities for differentiable exponentially convex and exponentially quasi-convex mapping, which are connected with the famous Hermite–Hadamard (HH) integral inequality. Moreover, we have provided applications of our findings to error estimations in numerical analysis and higher moments of random variables.

Keywords: convex function; exponentially convex functions; HH inequality, r^{th} -moment

1. Introduction

Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, then

$$\Psi\left(\frac{\sigma_1 + \sigma_2}{2}\right) \leq \int_{\sigma_1}^{\sigma_2} \Psi(v) dv \leq \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2}.$$

We call the above double inequality a Hermite–Hadamard (HH) inequality. Equality holds in either side only for affine functions. This result of Hermite and Hadamard is very simple in nature but very powerful. Interestingly, both sides of the above integral inequality characterize convex functions. For some interesting details and applications of HH inequality, we refer readers to [1–19].

There are many famous results known in the theory of inequalities which can be obtained using functions having the convexity property. One of them is Hermite–Hadamard’s inequality that has wide application in the field. Many researchers have used different novel and innovative ideas in obtaining new generalizations of classical inequalities, see [20–26]. The inequality theory has developed and provided a rapid development of generalizations, improvements and refinements of the classical concept of convexity. For details, see [2,15–17].

Now, we refresh our memories by giving some preliminary definitions and concepts as follows:

Definition 1. Suppose that K is a subset of \mathbb{R} . A function $\Psi : K \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called a convex function if the following inequality

$$\Psi(s\sigma_1 + (1-s)\sigma_2) \leq s\Psi(\sigma_1) + (1-s)\Psi(\sigma_2)$$

holds for all $\sigma_1, \sigma_2 \in K$ and $s \in [0, 1]$.

Recently, the definition of exponentially convex functions has been given and studied by Awan et al. [2].

Definition 2 ([2]). Suppose that K is a subset of \mathbb{R} . The mapping $\Psi : K \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be exponentially convex, if

$$\Psi(s\sigma_1 + (1-s)\sigma_2) \leq se^{\theta\sigma_1}\Psi(\sigma_1) + (1-s)e^{\theta\sigma_2}\Psi(\sigma_2) \quad (1)$$

for all $\sigma_1, \sigma_2 \in K$, $s \in [0, 1]$ and $\theta \in \mathbb{R}$. One can say that Ψ is exponentially concave, in the case that in (1) the reverse inequality holds.

For example, the function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$, defined by $\Psi(v) = -v^2$ is a concave function, thus this function is exponentially convex for all $\theta > 0$.

Exponentially convex functions are used to manipulate for statistical learning, sequential prediction and stochastic optimization. Exponentially convex functions are very useful due to their interesting properties. An exponentially convex function on a closed interval is bounded, it also satisfies the Lipschitzian condition on any closed interval $[\sigma_1, \sigma_2] \subset \overset{\circ}{I}$ (interior of I). Therefore an exponentially convex function is absolutely continuous on $[\sigma_1, \sigma_2] \subset \overset{\circ}{I}$ and continuous on $\overset{\circ}{I}$. Now we introduce exponentially quasi-convex functions.

Definition 3 ([10]). A mapping $\Psi : K \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be exponentially quasi-convex, if

$$\Psi(s\sigma_1 + (1-s)\sigma_2) \leq \max \{e^{\theta\sigma_1}\Psi(\sigma_1), e^{\theta\sigma_2}\Psi(\sigma_2)\}$$

for all $\sigma_1, \sigma_2 \in K$, $s \in [0, 1]$ and $\theta \in \mathbb{R}$.

Here we recall some of the results for convex and quasi-convex functions which are closely related to the research of our paper.

Theorem 1 ([5]). Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$, where $\sigma_1, \sigma_2 \in I$ with $\sigma_1 < \sigma_2$, and $\Psi' \in L([\sigma_1, \sigma_2])$. If $|\Psi'|$ is convex on $[\sigma_1, \sigma_2]$, then

$$\left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} - \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \Psi(v) dv \right| \leq \frac{(\sigma_2 - \sigma_1)(|\Psi'(\sigma_1)| + |\Psi'(\sigma_2)|)}{8}. \quad (2)$$

Theorem 2 ([5]). Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$, where $\sigma_1, \sigma_2 \in I$ with $\sigma_1 < \sigma_2$, and $\Psi' \in L(\sigma_1, \sigma_2)$. If $|\Psi'|^{\frac{p}{p-1}}$ is convex on $[\sigma_1, \sigma_2]$, then

$$\left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} - \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \Psi(v) dv \right| \leq \frac{(\sigma_2 - \sigma_1)}{2(p+1)^{\frac{p}{p-1}}} \left[\frac{(|\Psi'(\sigma_1)|^{\frac{p}{p-1}} + |\Psi'(\sigma_2)|^{\frac{p}{p-1}})}{2} \right], \quad (3)$$

where $p > 1$ and $p^{-1} + q^{-1} = 1$.

In [24], Pearce and Pecaric gave an upgrading and overview of upper bounds as follows. It is clear that the upper bound of (4) is less than the one in the inequality (3).

Theorem 3 ([24]). Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$, where $\sigma_1, \sigma_2 \in I$ with $\sigma_1 < \sigma_2$, and $\Psi' \in L([\sigma_1, \sigma_2])$. If $|\Psi'|^{\frac{p}{p-1}}$ is convex on $[\sigma_1, \sigma_2]$, then

$$\left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} - \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \Psi(v) dv \right| \leq \frac{(\sigma_2 - \sigma_1)}{4} \left[\frac{(|\Psi'(\sigma_1)|^q + |\Psi'(\sigma_2)|^q)}{2} \right]^{\frac{1}{q}}, \quad (4)$$

where $p > 1$ and $p^{-1} + q^{-1} = 1$.

Recently in 2011, Hwang [27] derived the following identity and presented certain useful results via this identity.

Lemma 1 ([27]). Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$, where $\sigma_1, \sigma_2 \in I$ with $\sigma_1 < \sigma_2$, and let $u : [\sigma_1, \sigma_2] \rightarrow [0, \infty)$ be a differentiable mapping. If $\Psi' \in L([\sigma_1, \sigma_2])$, then

$$\begin{aligned} & \frac{1}{2} \left([u(\sigma_2) - 2u(\sigma_1)] \Psi(\sigma_1) + u(\sigma_2) \Psi(\sigma_2) \right) - \int_{\sigma_1}^{\sigma_2} \Psi(v) u'(v) dv \\ &= \frac{\sigma_2 - \sigma_1}{4} \left\{ \int_0^1 \left[2u\left(\frac{1+s}{2}\sigma_1 + \frac{1-s}{2}\sigma_2\right) - u(\sigma_2) \right] \Psi'\left(\frac{1+s}{2}\sigma_1 + \frac{1-s}{2}\sigma_2\right) ds \right. \\ & \quad \left. + \int_0^1 \left[2u\left(\frac{1-s}{2}\sigma_1 + \frac{1+s}{2}\sigma_2\right) - u(\sigma_2) \right] \Psi'\left(\frac{1-s}{2}\sigma_1 + \frac{1+s}{2}\sigma_2\right) ds \right\}. \end{aligned} \quad (5)$$

Theorem 4 ([27]). Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$, where $\sigma_1, \sigma_2 \in I$ with $\sigma_1 < \sigma_2$, and $h : [\sigma_1, \sigma_2] \rightarrow [0, \infty)$ be continuous and symmetric with respect to $\frac{\sigma_1 + \sigma_2}{2}$, where $\sigma_1, \sigma_2 \in \overset{\circ}{I}$ with $\sigma_1 < \sigma_2$.

(1) If $\Psi' \in L([\sigma_1, \sigma_2])$ and $|\Psi'|$ is convex on $[\sigma_1, \sigma_2]$, then

$$\left| \left[\frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \right] \int_{\sigma_1}^{\sigma_2} h(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(x) h(v) dv \right| \leq \frac{\sigma_2 - \sigma_1}{4} [|\Psi'(\sigma_1)| + |\Psi'(\sigma_2)|] \int_0^1 \int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} h(v) dv ds. \quad (6)$$

(2) If $\Psi' \in L([\sigma_1, \sigma_2])$ and $|\Psi'|$ is convex on $[\sigma_1, \sigma_2]$ for $q \geq 1$, then

$$\left| \left[\frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \right] \int_{\sigma_1}^{\sigma_2} h(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) h(v) dv \right| \leq \frac{\sigma_2 - \sigma_1}{2} \left[\frac{|\Psi'(\sigma_1)|^q + |\Psi'(\sigma_2)|^q}{2} \right]^{\frac{1}{q}} \int_0^1 \int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} h(v) dv ds. \quad (7)$$

In order to derive new results and generalizations in inequality theory studies sometimes it may be necessary for additional features to be added to the function, while sometimes some constraints can be needed in the conditions of functions. Functions may provide various features at the same time or a function class may look like another function class by means of some features. In our study, we can see that inequalities can be provided also for different convexity classes for special conditions while proving various integral inequalities for various convex functions. So, we aim to contribute to the literature by proving some new estimations.

In the present paper, firstly, we consider the identities obtained by Hwang [27] for the classical convex functions. Secondly, using these results for convex and quasi-convex functions, we establish some new weighted HH type inequalities for exponentially convex and exponentially quasi-convex functions. Finally, applications of our findings have been given for numerical analysis and the r^{th} moment of random variables.

For the sake of brevity, let the notation $\chi_1(\sigma_1, \sigma_2, s) = \frac{1+s}{2}\sigma_1 + \frac{1-s}{2}\sigma_2$ and $\chi_2(\sigma_1, \sigma_2, s) = \frac{1-s}{2}\sigma_1 + \frac{1+s}{2}\sigma_2$.

2. New Estimations for Exponentially Convex Functions

We prove new integral inequalities via Lemma 1.

The exponentially convex version of a weighted HH inequality can be represented as follows.

Theorem 5. Let $\Psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$ and $\mathfrak{h} : [\sigma_1, \sigma_2] \rightarrow \mathbb{R}$ be continuous and symmetric with respect to $\frac{\sigma_1 + \sigma_2}{2}$, where $\sigma_1, \sigma_2 \in \overset{\circ}{I}$ with $\sigma_1 < \sigma_2$. If $\Psi' \in L([\sigma_1, \sigma_2])$ and $|\Psi'|$ is exponentially convex on $[\sigma_1, \sigma_2]$, then

$$\begin{aligned} & \left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \\ & \leq \frac{\sigma_2 - \sigma_1}{4} \left[|e^{\theta\sigma_1} \Psi'(\sigma_1)| + |e^{\theta\sigma_2} \Psi'(\sigma_2)| \right] \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) ds. \end{aligned} \quad (8)$$

Proof. If we set $u(s) = \int_{\sigma_1}^s \mathfrak{h}(v) dv$ for all $s \in [\sigma_1, \sigma_2]$, in Lemma 1, one obtains

$$\begin{aligned} & \left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \\ & \leq \frac{\sigma_2 - \sigma_1}{4} \left[\int_0^1 \left| 2u\left(\frac{1+s}{2}\sigma_1 + \frac{1-s}{2}\sigma_2\right) - u(\sigma_2) \right| \left| \Psi'\left(\frac{1+s}{2}\sigma_1 + \frac{1-s}{2}\sigma_2\right) \right| ds \right. \\ & \quad \left. + \int_0^1 \left| 2u\left(\frac{1-s}{2}\sigma_1 + \frac{1+s}{2}\sigma_2\right) - u(\sigma_2) \right| \left| \Psi'\left(\frac{1-s}{2}\sigma_1 + \frac{1+s}{2}\sigma_2\right) \right| ds \right] \end{aligned} \quad (9)$$

Since $\mathfrak{h}(v)$ is symmetric with respect to $v = \frac{\sigma_1 + \sigma_2}{2}$, we have

$$\begin{aligned} & \left| 2u\left(\frac{1+s}{2}\sigma_1 + \frac{1-s}{2}\sigma_2\right) - u(\sigma_2) \right| = \int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \\ & \text{and} \\ & \left| 2u\left(\frac{1-s}{2}\sigma_1 + \frac{1+s}{2}\sigma_2\right) - u(\sigma_2) \right| = \int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \end{aligned} \quad (10)$$

for all $s \in [0, 1]$.

By (9) and (10), we have

$$\begin{aligned} & \left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \\ & \leq \frac{\sigma_2 - \sigma_1}{4} \left[\int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \left| \Psi'\left(\frac{1+s}{2}\sigma_1 + \frac{1-s}{2}\sigma_2\right) \right| ds \right. \\ & \quad \left. + \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \left| \Psi'\left(\frac{1-s}{2}\sigma_1 + \frac{1+s}{2}\sigma_2\right) \right| ds \right]. \end{aligned} \quad (11)$$

Using the exponential convexity of $|\Psi'|$, we have

$$\begin{aligned}
& \left[\int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \left| \Psi' \left(\frac{1+s}{2} \sigma_1 + \frac{1-s}{2} \sigma_2 \right) \right| ds + \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \left| \Psi' \left(\frac{1-s}{2} \sigma_1 + \frac{1+s}{2} \sigma_2 \right) \right| ds \right. \\
& \leq \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \times \left(\frac{1+s}{2} |e^{\theta \sigma_1} \Psi'(\sigma_1)| + \frac{1-s}{2} |e^{\theta \sigma_2} \Psi'(\sigma_2)| + \frac{1-s}{2} |e^{\theta \sigma_1} \Psi'(\sigma_1)| + \frac{1+s}{2} |e^{\theta \sigma_2} \Psi'(\sigma_2)| \right) ds \\
& = \left[|e^{\theta \sigma_1} \Psi'(\sigma_1)| + |e^{\theta \sigma_2} \Psi'(\sigma_2)| \right] \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) ds.
\end{aligned} \tag{12}$$

A combination of (11) and (12), we have (8). This ends the proof. \square

Corollary 1. If $\mathfrak{h}(x) = 1$ in Theorem 5, then we have

$$\left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} - \int_{\sigma_1}^{\sigma_2} \Psi(v) dv \right| \leq \frac{\sigma_2 - \sigma_1}{4} \left[|e^{\theta \sigma_1} \Psi'(\sigma_1)| + |e^{\theta \sigma_2} \Psi'(\sigma_2)| \right].$$

Remark 1. In Theorem 5:

- (i) If we choose $\theta = 0$, then we attain inequality (6) in [27],
- (ii) If $\mathfrak{h}(x) = 1$, $\theta = 0$, then we obtain inequality (2) in [5].

Theorem 6. Under conditions of Theorem 5 and $q \geq 1$. If $|\Psi'|^q$ is convex on $[\sigma_1, \sigma_2]$, then

$$\begin{aligned}
& \left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \\
& \leq \frac{\sigma_2 - \sigma_1}{2} \left[\frac{|e^{\theta \sigma_1} \Psi'(\sigma_1)|^q + |e^{\theta \sigma_2} \Psi'(\sigma_2)|^q}{2} \right]^{\frac{1}{q}} \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) ds.
\end{aligned} \tag{13}$$

Proof. Using Hölder's inequality for (11) in the proofs of Theorem 5, one has

$$\begin{aligned}
& \left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \\
& \leq \frac{\sigma_2 - \sigma_1}{4} \left[\left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \right\}^{1-\frac{1}{q}} \times \left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \left| \Psi' \left(\frac{1+s}{2} \sigma_1 + \frac{1-s}{2} \sigma_2 \right) \right|^q ds \right\}^{\frac{1}{q}} \right. \\
& \quad \left. + \left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \right\}^{1-\frac{1}{q}} \times \left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \left| \Psi' \left(\frac{1-s}{2} \sigma_1 + \frac{1+s}{2} \sigma_2 \right) \right|^q ds \right\}^{\frac{1}{q}} \right] \\
& \leq \frac{\sigma_2 - \sigma_1}{4} \left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \right\}^{1-\frac{1}{q}} \times \left[\left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \left| \Psi' \left(\frac{1+s}{2} \sigma_1 + \frac{1-s}{2} \sigma_2 \right) \right|^q ds \right\}^{\frac{1}{q}} \right. \\
& \quad \left. + \left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv \right) \left| \Psi' \left(\frac{1-s}{2} \sigma_1 + \frac{1+s}{2} \sigma_2 \right) \right|^q ds \right\}^{\frac{1}{q}} \right].
\end{aligned} \tag{14}$$

By an application of the discrete power-mean inequality ($\sigma_1^a + \sigma_2^a < 2^{1-a}(\sigma_1 + \sigma_2)^a$) for $\sigma_1 > 0, \sigma_2 > 0$ and $a < 1$), one has

$$\begin{aligned} & \left[\left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(\omega) d\omega \right) \left| \Psi' \left(\frac{1+s}{2} \sigma_1 + \frac{1-s}{2} \sigma_2 \right) \right|^q ds \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + \left\{ \int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(\omega) d\omega \right) \left| \Psi' \left(\frac{1-s}{2} \sigma_1 + \frac{1+s}{2} \sigma_2 \right) \right|^q ds \right\}^{\frac{1}{q}} \right] \\ & \leq 2^{1-\frac{1}{q}} \left[\int_0^1 \left(\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(\omega) d\omega \right) \left[\left| \Psi' \left(\frac{1+s}{2} \sigma_1 + \frac{1-s}{2} \sigma_2 \right) \right|^q + \left| \Psi' \left(\frac{1-s}{2} \sigma_1 + \frac{1+s}{2} \sigma_2 \right) \right|^q \right] ds \right]^{\frac{1}{q}}. \end{aligned} \quad (15)$$

From definition of the exponential convexity of $|\Psi|^q$ on $[\sigma_1, \sigma_2]$, we have

$$\begin{aligned} & \left| \Psi' \left(\frac{1+s}{2} \sigma_1 + \frac{1-s}{2} \sigma_2 \right) \right|^q + \left| \Psi' \left(\frac{1-s}{2} \sigma_1 + \frac{1+s}{2} \sigma_2 \right) \right|^q \\ & \leq \frac{1+s}{2} |e^{\theta \sigma_1} \Psi'(\sigma_1)|^q + \frac{1-s}{2} |e^{\theta \sigma_2} \Psi'(\sigma_2)|^q + \frac{1-s}{2} |e^{\theta \sigma_1} \Psi'(\sigma_1)|^q + \frac{1+s}{2} |e^{\theta \sigma_2} \Psi'(\sigma_2)|^q \\ & = |e^{\theta \sigma_1} \Psi'(\sigma_1)|^q + |e^{\theta \sigma_2} \Psi'(\sigma_2)|^q. \end{aligned} \quad (16)$$

A combination of (14)–(16) gives the desired inequality (13). \square

Corollary 2. If we choose $\mathfrak{h}(v) = 1$, then Theorem 6 reduces to

$$\begin{aligned} & \left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \\ & \leq \frac{\sigma_2 - \sigma_1}{4} \left[\frac{|e^{\theta \sigma_1} \Psi'(\sigma_1)|^q + |e^{\theta \sigma_2} \Psi'(\sigma_2)|^q}{2} \right]^{\frac{1}{q}}. \end{aligned}$$

Remark 2. In Theorem 6:

- (i) If we choose $\theta = 0$, then we attain the inequality (7) in [27].
- (ii) If $\mathfrak{h}(v) = 1$, $\theta = 0$, then we get inequality (3) in [5].

3. Hermite–Hadamard’s Inequalities for Exponentially Quasi-Convex Functions

For obtaining new results, we deal with the exponential quasi-convexity of Ψ' as follows:

Theorem 7. Under conditions of Theorem 5. If the mapping $|\Psi'|$ is exponentially quasi-convex on $[\sigma_1, \sigma_2]$, then

$$\begin{aligned} & \left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \leq \frac{\sigma_2 - \sigma_1}{4} \\ & \times \left[\max \left\{ |e^{\theta \sigma_1} \Psi'(\sigma_1)|, \left| e^{\theta \left(\frac{\sigma_1 + \sigma_2}{2} \right)} \Psi' \left(\frac{\sigma_1 + \sigma_2}{2} \right) \right| \right\} + \max \left\{ |e^{\theta \sigma_2} \Psi'(\sigma_2)|, \left| e^{\theta \left(\frac{\sigma_1 + \sigma_2}{2} \right)} \Psi' \left(\frac{\sigma_1 + \sigma_2}{2} \right) \right| \right\} \right] \\ & \times \int_0^1 \int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(\omega) d\omega ds. \end{aligned} \quad (17)$$

Proof. Using the inequality (11) in the proofs of Theorem 5 and by exponential quasi-convexity of $|\Psi'|$, we have

$$\left| \Psi' \left(\frac{1+s}{2} \sigma_1 + \frac{1-s}{2} \sigma_2 \right) \right| = \max \left\{ |e^{\theta \sigma_1} \Psi'(\sigma_1)|, \left| e^{\theta \left(\frac{\sigma_1 + \sigma_2}{2} \right)} \Psi' \left(\frac{\sigma_1 + \sigma_2}{2} \right) \right| \right\} \quad (18)$$

and

$$\left| \Psi' \left(\frac{1-s}{2} \sigma_1 + \frac{1+s}{2} \sigma_2 \right) \right| = \max \left\{ |e^{\theta \sigma_2} \Psi'(\sigma_2)|, \left| e^{\theta \left(\frac{\sigma_1 + \sigma_2}{2} \right)} \Psi' \left(\frac{\sigma_1 + \sigma_2}{2} \right) \right| \right\}. \quad (19)$$

A combination of (11), (18) and (19) gives the required inequality (17). \square

Remark 3. In Theorem 7:

- (i) If we choose $\theta = 0$, then we get Theorem 2.8 in [27],
- (ii) If we choose $\theta = 0$ along with $\mathfrak{h}(v) = 1$, then we get Theorem 2.2 in [1].

Corollary 3. Let Ψ as in Theorem 7, if in addition

- (1) $|\Psi'|$ is increasing, then we have

$$\left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \leq \frac{\sigma_2 - \sigma_1}{4} \times \left\{ |e^{\theta\sigma_2} \Psi'(\sigma_2)| + \left| e^{\theta\left(\frac{\sigma_1+\sigma_2}{2}\right)} \Psi'\left(\frac{\sigma_1+\sigma_2}{2}\right) \right| \right\} \int_0^1 \int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv ds, \quad (20)$$

- (2) $|\Psi'|$ is decreasing, then we have

$$\left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \leq \frac{\sigma_2 - \sigma_1}{4} \times \left\{ |e^{\theta\sigma_1} \Psi'(\sigma_1)| + \left| e^{\theta\left(\frac{\sigma_1+\sigma_2}{2}\right)} \Psi'\left(\frac{\sigma_1+\sigma_2}{2}\right) \right| \right\} \int_0^1 \int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv ds. \quad (21)$$

Remark 4. In Corollary 3:

- (i) If we choose $\theta = 0$, then we get Remark 2.9 in [27].
- (ii) If we choose $\theta = 0$ along with $\mathfrak{h}(v) = 1$, then we get Corollary 2.1 in [1].

Theorem 8. Under conditions of Theorem 5 and $q \geq 1$. If $|\Psi'|^q$ is exponentially quasi-convex on $[\sigma_1, \sigma_2]$, then

$$\left| \frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} \int_{\sigma_1}^{\sigma_2} \mathfrak{h}(v) dv - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \leq \frac{\sigma_2 - \sigma_1}{4} \times \left[\left(\max \left\{ |e^{\theta\sigma_1} \Psi'(\sigma_1)|^q, \left| e^{\theta\left(\frac{\sigma_1+\sigma_2}{2}\right)} \Psi'\left(\frac{\sigma_1+\sigma_2}{2}\right) \right|^q \right\} \right)^{\frac{1}{q}} + \left(\max \left\{ |e^{\theta\sigma_2} \Psi'(\sigma_2)|^q, \left| e^{\theta\left(\frac{\sigma_1+\sigma_2}{2}\right)} \Psi'\left(\frac{\sigma_1+\sigma_2}{2}\right) \right|^q \right\} \right)^{\frac{1}{q}} \right] \times \int_0^1 \int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} \mathfrak{h}(v) dv ds. \quad (22)$$

Proof. Using the inequality (11) in the proofs of Theorem 5 and by exponential quasi-convexity of $|\Psi'|^q$, we have

$$\left| \Psi'\left(\frac{1+s}{2}\sigma_1 + \frac{1-s}{2}\sigma_2\right) \right|^q = \max \left\{ |e^{\theta\sigma_1} \Psi'(\sigma_1)|^q, \left| e^{\theta\left(\frac{\sigma_1+\sigma_2}{2}\right)} \Psi'\left(\frac{\sigma_1+\sigma_2}{2}\right) \right|^q \right\} \quad (23)$$

and

$$\left| \Psi'\left(\frac{1-s}{2}\sigma_1 + \frac{1+s}{2}\sigma_2\right) \right|^q = \max \left\{ |e^{\theta\sigma_2} \Psi'(\sigma_2)|^q, \left| e^{\theta\left(\frac{\sigma_1+\sigma_2}{2}\right)} \Psi'\left(\frac{\sigma_1+\sigma_2}{2}\right) \right|^q \right\}. \quad (24)$$

A combination of (11), (23) and (24) gives the required inequality (22). \square

Remark 5. In Theorem 8:

- (i) If we choose $q = 1$, then we attain Theorem 7 in the present paper,
- (ii) If we choose $\theta = 0$, then we attain Theorem 2.10 in [27],
- (iii) If we choose $\theta = 0$ along with $\mathfrak{h} = 1$, then we attain Theorem 2.4 of [1].

4. Error Estimations with the Trapezoidal Formula

In this part of the article, results related to the trapezoidal rule, which has an important place in numerical analysis, will be given. In the numerical analysis, our findings suggest an approach for the error term are in the nature of confirming the results obtained previously and the findings regarding their special cases are included. Let p be the partition $\sigma_1 = \omega_0 < \omega_1 < \dots < \omega_{n-1} < \omega_n = \sigma_2$ of $[\sigma_1, \sigma_2]$, and recall the quadrature formula as

$$\int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv = T(\Psi, \mathfrak{h}, p) + E(\Psi, \mathfrak{h}, p), \quad (25)$$

where

$$T(\Psi, \mathfrak{h}, p) = \sum_{i=0}^{n-1} \frac{\Psi(v_i) + \Psi(v_{i+1})}{2} \int_{v_i}^{v_{i+1}} \mathfrak{h}(v) dv$$

for the trapezoidal version and $E(\Psi, \mathfrak{h}, p)$ is approximation error term.

Proposition 1. Under conditions of Theorem 6 and using $|\Psi'|^q$ is exponentially convex on $[\sigma_1, \sigma_2]$, then in (25), for every partition p of $[\sigma_1, \sigma_2]$, then

$$|E(\Psi, \mathfrak{h}, p)| \leq \frac{1}{2} \sum_{i=0}^{n-1} (v_{i+1} - v_i) \left(\frac{|e^{\theta v_i} \Psi'(v_i)|^q + |e^{\theta v_{i+1}} \Psi'(v_{i+1})|^q}{2} \right)^{\frac{1}{q}} \int_0^1 \int_{\chi_1(v_i, v_{i+1}, s)}^{\chi_2(v_i, v_{i+1}, s)} \mathfrak{h}(v) dv ds$$

.

Proof. By taking into account Theorem 6 on the intervals $[v_i, v_{i+1}]$ ($i = 0, 1, \dots, n-1$) of the partition p , we get

$$\begin{aligned} & \left| \frac{\Psi(v_i) + \Psi(v_{i+1})}{2} \int_{v_i}^{v_{i+1}} \mathfrak{h}(v) dv - \int_{v_i}^{v_{i+1}} \Psi(v) \mathfrak{h}(v) dv \right| \\ & \leq \frac{v_{i+1} - v_i}{2} \left[\frac{|e^{\theta v_i} \Psi'(v_i)|^q + |e^{\theta v_{i+1}} \Psi'(v_{i+1})|^q}{2} \right]^{\frac{1}{q}} \int_0^1 \int_{\chi_1(v_i, v_{i+1}, s)}^{\chi_2(v_i, v_{i+1}, s)} \mathfrak{h}(v) dv ds. \end{aligned}$$

By summation over i from 0 to $n-1$ and applying exponential convexity of $|\Psi'|^q$ and by the triangle inequality, we deduce that

$$\begin{aligned} & \left| T(\Psi, \mathfrak{h}, p) - \int_{\sigma_1}^{\sigma_2} \Psi(v) \mathfrak{h}(v) dv \right| \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} (v_{i+1} - v_i) \left(\frac{|e^{\theta v_i} \Psi'(v_i)|^q + |e^{\theta v_{i+1}} \Psi'(v_{i+1})|^q}{2} \right)^{\frac{1}{q}} \int_0^1 \int_{\chi_1(v_i, v_{i+1}, s)}^{\chi_2(v_i, v_{i+1}, s)} \mathfrak{h}(v) dv ds. \end{aligned}$$

This completes the proof. \square

Remark 6. If we set $\theta = 0$ in Proposition 1, then we have Proposition 3.1 in [27].

Proposition 2. Suppose the hypothesis of Theorem 8 is satisfied and using $|\Psi'|^q$ is exponentially quasi-convex on $[\sigma_1, \sigma_2]$, then in (25), for every partition p of $[\sigma_1, \sigma_2]$, then

$$|E(\Psi, h, p)| \leq \frac{1}{4} \sum_{i=0}^{n-1} (v_{i+1} - v_i) \left[\left[\max \left\{ |\Psi'(v_i)|^q, \left| \Psi' \left(\frac{v_i + v_{i+1}}{2} \right) \right|^q \right\} \right]^{\frac{1}{q}} \right. \\ \left. + \left[\max \left\{ |\Psi'(v_{i+1})|^q, \left| \Psi' \left(\frac{v_i + v_{i+1}}{2} \right) \right|^q \right\} \right]^{\frac{1}{q}} \right] \int_0^1 \int_{\chi_1(v_i, v_{i+1}, s)}^{\chi_2(v_i, v_{i+1}, s)} h(v) dv ds.$$

Proof. Applying Theorem 8 on the intervals $[v_i, v_{i+1}] (i = 0, 1, \dots, n-1)$ of the partition p , we get

$$\left| \frac{\Psi(v_i) + \Psi(v_{i+1})}{2} \int_{v_i}^{v_{i+1}} h(v) dv - \int_{v_i}^{v_{i+1}} \Psi(v) h(v) dv \right| \\ \leq \frac{v_{i+1} - v_i}{4} \left[\left[\max \left\{ |e^{\theta v_i} \Psi'(v_i)|^q, |e^{\theta \left(\frac{v_i + v_{i+1}}{2} \right)} \Psi' \left(\frac{v_i + v_{i+1}}{2} \right)|^q \right\} \right]^{\frac{1}{q}} \right. \\ \left. + \left[\max \left\{ |e^{\theta v_{i+1}} \Psi'(v_{i+1})|^q, |e^{\theta \left(\frac{v_i + v_{i+1}}{2} \right)} \Psi' \left(\frac{v_i + v_{i+1}}{2} \right)|^q \right\} \right]^{\frac{1}{q}} \right] \int_0^1 \int_{\chi_1(v_i, v_{i+1}, s)}^{\chi_2(v_i, v_{i+1}, s)} h(v) dv ds.$$

By summation over i from 0 to $n-1$ and by definition of $|\Psi'|^q$, also by using the triangle inequality, we obtain that

$$|T(\Psi, h, p) - \int_{\sigma_1}^{\sigma_2} \Psi(v) h(v) dv| \\ \leq \frac{v_{i+1} - v_i}{4} \left[\left[\max \left\{ |e^{\theta v_i} \Psi'(v_i)|^q, |e^{\theta \left(\frac{v_i + v_{i+1}}{2} \right)} \Psi' \left(\frac{v_i + v_{i+1}}{2} \right)|^q \right\} \right]^{\frac{1}{q}} \right. \\ \left. + \left[\max \left\{ |e^{\theta v_{i+1}} \Psi'(v_{i+1})|^q, |e^{\theta \left(\frac{v_i + v_{i+1}}{2} \right)} \Psi' \left(\frac{v_i + v_{i+1}}{2} \right)|^q \right\} \right]^{\frac{1}{q}} \right] \int_0^1 \int_{\chi_1(v_i, v_{i+1}, s)}^{\chi_2(v_i, v_{i+1}, s)} h(v) dv ds.$$

This completes the proof. \square

Remark 7. If we set $\theta = 0$ in Proposition 2, then we get Proposition 3.3 in [27].

Remark 8. If $|\Psi'|$ is nondecreasing in Proposition 2, then

$$|E(\Psi, h, d)| \leq \frac{1}{4} \sum_{i=0}^{n-1} (v_{i+1} - v_i) \left[|e^{\theta \left(\frac{\omega_{i+1} + \omega_i}{2} \right)} \Psi' \left(\frac{\omega_{i+1} + \omega_i}{2} \right)| + |e^{\theta v_{i+1}} \Psi'(v_{i+1})| \right] \int_0^1 \int_{\chi_1(\omega_i, \omega_{i+1}, s)}^{\chi_2(\omega_i, \omega_{i+1}, s)} h(v) dv ds$$

and if $|\Psi'|$ is nonincreasing in proposition 2, then

$$|E(\Psi, h, d)| \leq \frac{1}{4} \sum_{i=0}^{n-1} (v_{i+1} - v_i) \left[|e^{\theta \left(\frac{v_{i+1} + v_i}{2} \right)} \Psi' \left(\frac{v_{i+1} + v_i}{2} \right)| + |e^{\theta v_i} \Psi'(v_i)| \right] \int_0^1 \int_{\chi_1(\omega_i, \omega_{i+1}, s)}^{\chi_2(\omega_i, \omega_{i+1}, s)} h(v) dv ds.$$

5. Application to Random Variables

In this section, by giving various applications to the results of our study, we will prove that the findings obtained are effective. In addition, approaches to the expected value function will be obtained with the help of the probability density function in the field of statistics. Let X be a random variable in $[\sigma_1, \sigma_2]$, with the probability density function $h : [\sigma_1, \sigma_2] \rightarrow [0, \infty)$, and symmetric with respect to $\frac{\sigma_1 + \sigma_2}{2}$ with $0 < \sigma_1 < \sigma_2, r \in \mathbb{R}$, then the r^{th} -moment

$$E_r(v) := \int_{\sigma_1}^{\sigma_2} t^r h(s) ds,$$

which is supposed to be finite.

Theorem 9. *The inequality*

$$\left| \frac{\sigma_1^r + \sigma_2^r}{2} - E_r(v) \right| \leq \frac{r(\sigma_2 - \sigma_1)}{4} [e^{\theta\sigma_1} \sigma_1^{r-1} + e^{\theta\sigma_2} \sigma_2^{r-1}],$$

holds for $0 < \sigma_1 < \sigma_2$ and $r \geq 2$.

Proof. Let $\Psi(s) = s^r$ on $[\sigma_1, \sigma_2]$ for $r \geq 2$, we have $|\Psi'(s)| = rs^{r-1}$ is exponentially convex. Since

$$\int_{\sigma_1}^{\sigma_2} \Psi(v) h(v) dv = E_r(v),$$

and

$$\int_{\chi_1(\sigma_1, \sigma_2, s)}^{\chi_2(\sigma_1, \sigma_2, s)} h(v) dv \leq \int_{\sigma_1}^{\sigma_2} h(v) dv = 1, \quad \forall s \in [0, 1].$$

From (8), one has

$$\frac{\Psi(\sigma_1) + \Psi(\sigma_2)}{2} = \frac{\sigma_1^r + \sigma_2^r}{2}$$

and

$$|e^{\theta\sigma_1} \Psi'(\sigma_1)| + |e^{\theta\sigma_2} \Psi'(\sigma_2)| = r(e^{\theta\sigma_1} \sigma_1^{r-1} + e^{\theta\sigma_2} \sigma_2^{r-1}).$$

By the inequality (8), the desires are obtained immediately. \square

Remark 9. In Theorem 9, we have the following assumptions:

- (1) If we choose $r = 1$ and $h(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(v-\mu)^2}{2\sigma^2}}$, for $-\infty < v < \infty$, and $\sigma > 0$ is normally distributed, where μ is the mean, σ is the standard deviation and $e (= 2.71828...)$ are constants, then we have inequality

$$\left| \frac{\sigma_1 + \sigma_2}{2} - \mu \right| \leq \frac{\sigma_2 - \sigma_1}{4} [e^{\theta\sigma_1} + e^{\theta\sigma_2}],$$

which holds for $0 < \sigma_1 < \sigma_2$.

- (2) If we choose $r = 1$ and $h(v) = \lambda e^{-\lambda v}$ for $\omega > 0$ with parameter λ is exponentially distributed, then we have inequality

$$\left| \frac{\sigma_1 + \sigma_2}{2} - \frac{1}{\lambda} \right| \leq \frac{\sigma_2 - \sigma_1}{4} [e^{\theta\sigma_1} + e^{\theta\sigma_2}],$$

which holds for $0 < \sigma_1 < \sigma_2$.

By applying Theorems 6–8, similar relations can be established; we have omitted the details here.

Remark 10. Applications can be given based on the obtained results to special means, and we omit the details.

6. Conclusions

In this article, we have provided several new weighted HH inequalities for exponentially convex and exponentially quasi-convex functions. Our findings can be considered as refinements and significant improvements to the new classes of convex functions by extraordinary choices of θ . It is clear that our new results can be reduced for $\theta = 0$ to previously known results. Also, we have presented their applications to the Trapezoidal formula and in statistics for the r^{th} moment for the derived results. The obtained results can be extended for different kinds of convex functions. These ideas may stimulate further research in this captivating field.

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