mathematics

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Modified Proximal Algorithms for Finding Solutions of the Split Variational Inclusions

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Received: 27 June 2019; Accepted: 2 August 2019; Published: 6 August 2019

Abstract: We investigate the split variational inclusion problem in Hilbert spaces. We propose efficient algorithms in which, in each iteration, the stepsize is chosen self-adaptive, and proves weak and strong convergence theorems. We provide numerical experiments to validate the theoretical results for solving the split variational inclusion problem as well as the comparison to algorithms defined by Byrne et al. and Chuang, respectively. It is shown that the proposed algorithms outrun other algorithms via numerical experiments. As applications, we apply our method to compressed sensing in signal recovery. The proposed methods have as a main advantage that the computation of the Lipschitz constants for the gradient of functions is dropped in generating the sequences.

Keywords: split variational inclusion problem; compressed sensing; proximal algorithm; hilbert spaces

1. Introduction

Let H be a real Hilbert space. Then, $B : H \to 2^H$ is called monotone if $\langle u - v, x - y\rangle \geq 0$ for each $u \in Bx$, $v \in By$. Moreover, B is maximal monotone provided its graph is not properly included in the graph of other monotone mappings. Many problems in optimization can be reduced to finding $x^* \in H$ such that $0 \in Bx^*$. Martinet [1] and Rockafellar [2] suggested the proximal method for solving this problem. They construct the sequence $\{x_n\} \subset H$ by choosing $x_1 \in H$ and putting

$$x_{n+1} = J^B_{\beta_n} x_n,\ n \in \mathbb{N}, \tag{1}$$

where $\{\beta_n\} \subseteq (0, \infty)$, B is a set-valued maximal monotone operator and J^B_β is defined by $J^B_\beta = (I + \beta B)^{-1}$ for each $\beta > 0$. We see that Equation (1) is equivalent to $x_n - x_{n+1} \in \beta_n B x_{n+1}$, $n \in \mathbb{N}$.

The split variational inclusion problem (SVIP) was first investigated by Moudafi [3]. The problem consists of finding $x^* \in H_1$ such that

$$0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*), \tag{2}$$

where H_1 and H_2 are real Hilbert spaces, B_1 and B_2 are set-valued mappings on H_1 and H_2. In addition, $A : H_1 \to H_2$ is a bounded and linear operator and A^* is the adjoint of A. We know that the SVIP is a generalization of the split feasibility problem that was investigated by Censor and Elfving [4] in Euclidean spaces. See [4–9]. In this paper, we denote by Ω the solution set of SVIP. Suppose that Ω is nonempty.

In 2011, Byrne et al. [6] established a weak convergence theorem for SVIP as follows:

Theorem 1. *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators. Let $\beta > 0$ and $\gamma \in (0, \frac{2}{\|A\|^2})$. Let $\{x_n\}$ be generated by*

$$x_{n+1} = J_{\beta}^{B_1}(x_n - \gamma A^*(I - J_{\beta}^{B_2})Ax_n),\ n \in \mathbb{N}. \tag{3}$$

Then, $\{x_n\}$ converges weakly to x^* in Ω.

In 2015, Chuang [10] introduced the following iteration for SVIP in Hilbert spaces.
Chuang [10] established its convergence as follows:

Theorem 2. *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded and linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators. Choose $\delta \in (0,1)$ and let $\{\beta_n\} \subseteq (0, \infty)$ and $\{\gamma_n\} \subseteq (0, \frac{\delta}{\|A\|^2})$ and assume that*

$$\sum_{n=1}^{\infty} \gamma_n = \infty,\ \sum_{n=1}^{\infty} \gamma_n^2 < \infty,\ \liminf_{n\to\infty} \beta_n > 0. \tag{4}$$

If H_1 is finite dimensional, then $\lim_{n\to\infty} x_n = x^* \in \Omega$.

Chuang [10] also provided the following result.

Theorem 3. *Let H_1 and H_2 be infinite dimensional Hilbert spaces, $A : H_1 \to H_2$ be a bounded and linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone mappings. Choose $\delta \in (0,1)$ and let $\{\beta_n\} \subseteq (0, \infty)$, $\liminf_{n\to\infty} \beta_n > 0$ and $\{\gamma_n\} \subseteq (0, \frac{\delta}{\|A\|^2})$ with $\inf_{n\in\mathbb{N}} \gamma_n > 0$. Then, $x_n \to x^* \in \Omega$.*

In 2013, Chuang [11] proved strong convergence theorem for SVIP using the following algorithm.

Algorithm 1:

[11]
For $n \in \mathbb{N}$, set y_n as

$$y_n = J_{\beta_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\beta_n}^{B_2})Ax_n), \tag{5}$$

where $\gamma_n > 0$ is chosen such that

$$\gamma_n \|A^*(I - J_{\beta_n}^{B_2})Ax_n - A^*(I - J_{\beta_n}^{B_2})Ay_n\| \le \delta \|x_n - y_n\|,\ 0 < \delta < 1. \tag{6}$$

The iterative x_{n+1} is generated by

$$x_{n+1} = J_{\beta_n}^{B_1}(x_n - \alpha_n D(x_n, \gamma_n)), \tag{7}$$

where

$$D(x_n, \gamma_n) = x_n - y_n + \gamma_n(A^*(I - J_{\beta_n}^{B_2})Ay_n - A^*(I - J_{\beta_n}^{B_2})Ax_n) \tag{8}$$

and

$$\alpha_n = \frac{\langle x_n - y_n, D(x_n, \gamma_n)\rangle}{\|D(x_n, \gamma_n)\|^2}. \tag{9}$$

Theorem 4. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded and linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two set-valued maximal monotone operators. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + b_n + c_n + d_n = 1$ and $0 < a_n < 1$ for each $n \in \mathbb{N}$.*

Let $\{\beta_n\} \subseteq (0,\infty)$ and let $\{\gamma_n\} \subseteq (0, \frac{2}{\|A\|^2+1})$. Let $\{v_n\}$ be a bounded sequence in H_1. Fix $u \in H_1$ and let the sequence $\{x_n\} \subseteq H_1$ be generated by

$$x_{n+1} = a_n u + b_n x_n + c_n J_{\beta_n}^{B_1}(x_n - \gamma_n A^*(I - J_{\beta_n}^{B_2})Ax_n) + d_n v_n \tag{10}$$

for each $n \in \mathbb{N}$. Suppose that

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{d_n}{a_n} = 0$; $\sum_{n=1}^{\infty} a_n = \infty$; $\sum_{n=1}^{\infty} d_n < \infty$;
(ii) $\liminf_{n\to\infty} c_n\gamma_n > 0$, $\liminf_{n\to\infty} b_n c_n > 0$, $\liminf_{n\to\infty} \beta_n > 0$.
Then, $\lim_{n\to\infty} x_n = x^*$, where $x^* = P_\Omega u$ and $P_\Omega u$ is nearest to u.

We aim to find the approximate algorithms with a new step size which is self-adaptive (see López et al. [8]) for solving our SVIP and prove its convergence. We present numerical examples and the comparison to algorithms of Byne et al. [6] and algorithms of Chuang [10,11]. We also obtain the result for split feasibility problem (SFP) and its applications to compressed sensing in signal recovery. It reveals that our methods have a better convergence than those of Byrne et al. [6] and Chuang [10,11].

2. Preliminaries

We next provide some basic concepts for our proof. In what follows, we shall use the following symbols:

- $\rightharpoonup$ stands for the weak convergence,
- $\to$ stands for the strong convergence.

Recall that a mapping $T : H \to H$ is called

(1) nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\| \leq \|x - y\|. \tag{11}$$

(2) firmly-nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle. \tag{12}$$

It is clear that $I - T$ is also firmly-nonexpansive when T is firmly-nonexpansive. We know that, for each $x, y \in H$,

$$\langle x, y\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2 \tag{13}$$

and

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2 \tag{14}$$

for all $x, y \in H$ and for all $t \in [0,1]$.

The following lemma can be found in [12].

Lemma 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. If $x_n \rightharpoonup x \in C$ and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, then $x = Tx$.*

We use $Fix(T)$ by the fixed point set of a mapping T, that is, $Fix(T) = \{x \in H : x = Tx\}$ and $D(T)$ by the domain of a mapping T, i.e., $D(T) = \{x \in H : T(x) \neq \emptyset\}$.

The following lemma can be found in [11,13].

Lemma 2. *Let H be a real Hilbert space and let $B : H \to 2^H$ be a maximal monotone operator. Then,*

(i) J^B_β *is single-valued and firmly nonexpansive for each* $\beta > 0$;

(ii) $\mathcal{D}(J^B_\beta) = H$ *and* $Fix(J^B_\beta) = \{x \in \mathcal{D}(B) : 0 \in Bx\}$;

(iii) $\|x - J^B_\beta x\| \leq \|x - J^B_\gamma x\|$ *for all* $0 < \beta \leq \gamma$ *and for all* $x \in H$;

(iv) *If* $B^{-1}(0) \neq \emptyset$, *then we have* $\|x - J^B_\beta x\|^2 + \|J^B_\beta x - x^*\|^2 \leq \|x - x^*\|^2$ *for all* $x \in H$, *each* $x^* \in B^{-1}(0)$, *and each* $\beta > 0$;

(v) *If* $B^{-1}(0) \neq \emptyset$, *then we have* $\langle x - J^B_\beta x, J^B_\beta x - w\rangle \geq 0$ *for all* $x \in H$, *each* $w \in B^{-1}(0)$, *and each* $\beta > 0$.

Lemma 3. *Let* H_1 *and* H_2 *be real Hilbert spaces. Let* $A : H_1 \to H_2$ *be a bounded and linear operator. Let* $\beta > 0$, $\gamma > 0$, $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *be maximal monotone operators. Let* $x^* \in H_1$.

(i) *If* x^* *is a solution of (SVIP), then* $J^{B_1}_\beta(x^* - \gamma A^*(I - J^{B_2}_\beta)Ax^*) = x^*$.

(ii) *Suppose that* $J^{B_1}_\beta(x^* - \gamma A^*(I - J^{B_2}_\beta)Ax^*) = x^*$ *and the solution set of (SVIP) is nonempty. Then,* x^* *is a solution of (SVIP).*

Lemma 4. *Let* H_1 *and* H_2 *be real Hilbert spaces. Let* $A : H_1 \to H_2$ *be a bounded and linear operator and* $\beta > 0$. *Let* $B : H_2 \to 2^{H_2}$ *be a maximal monotone operator. Define a mapping* $T : H_1 \to H_1$ *by* $Tx := A^*(I - J^B_\beta)Ax$ *for each* $x \in H_1$. *Then,*

(i) $\|(I - J^B_\beta)Ax - (I - J^B_\beta)Ay\|^2 \leq \langle Tx - Ty, x - y\rangle$ *for all* $x, y \in H_1$;

(ii) $\|A^*(I - J^B_\beta)Ax - A^*(I - J^B_\beta)Ay\|^2 \leq \|A\|^2 \cdot \langle Tx - Ty, x - y\rangle$ *for all* $x, y \in H_1$.

The following lemma can be found in [14].

Lemma 5. *Let* C *be a nonempty subset of a Hilbert space* H. *Let* $\{x_n\}$ *be a sequence in* H *that satisfies the following assumptions:*

(i) $\lim\limits_{n\to\infty} \|x_n - x\|$ *exists for each* $x \in C$;

(ii) *every sequential weak limit point of* $\{x_n\}$ *is in* C.

Then, $\{x_n\}$ *weakly converges to a point in* C.

The following lemma can be found in [15].

Lemma 6. *Assume* $\{s_n\} \subseteq (0, \infty)$ *such that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\delta_n, \ n \geq 1, \tag{15}$$

$$s_{n+1} \leq s_n - \lambda_n + \varphi_n, \ n \geq 1, \tag{16}$$

where $\{\alpha_n\} \subseteq (0, 1)$, $\{\lambda_n\} \subseteq (0, 1)$ *and* $\{\delta_n\}$ *and* $\{\varphi_n\}$ *are real sequences such that*

(i) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim\limits_{n\to\infty} \varphi_n = 0$;

(iii) $\lim_{k\to\infty} \lambda_{n_k} = 0$ *implies* $\limsup\limits_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$.

Then, $\lim\limits_{n\to\infty} s_n = 0$.

3. Weak Convergence Result

Let, H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded and linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators.

Let Ω be a solution set of problem (SVIP) and assume that $\Omega \neq \emptyset$. We remark that the stepsize sequence $\{\gamma_n\}$ does not depend on the norm of an operator A as introduced by Byrne et al. [6] and Chuang [10,11].

Theorem 5. *Suppose that* $\liminf_{n\to\infty} \beta_n > 0$, $\inf_n \rho_n(4-\rho_n) > 0$ *and* $\lim_{n\to\infty} \theta_n = 0$. *Then,* $\{x_n\}$ *defined by Algorithm 2 converges weakly to a solution in* Ω.

Algorithm 2:

Choose $x_1 \in H_1$ and define

$$x_{n+1} = J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)), \tag{17}$$

where

$$\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2 + \theta_n}, \quad 0 < \rho_n < 4, \quad 0 < \theta_n < 1, \quad \beta_n > 0, \tag{18}$$

and

$$f(x_n) = \frac{1}{2}\|(I - J_{\beta_n}^{B_2})Ax_n\|^2, \quad g(x_n) = A^*(I - J_{\beta_n}^{B_2})Ax_n. \tag{19}$$

Proof. Let $z \in \Omega$. Then, $z \in B_1^{-1}(0)$ and $Az \in B_2^{-1}(0)$. Thus, we have $J_{\beta_n}^{B_2}Az = Az$. Using Lemma 4 (i), we have

$$\begin{aligned}
\langle x_n - z, g(x_n)\rangle &= \langle x_n - z, g(x_n) - g(z)\rangle \\
&= \langle x_n - z, A^*(I - J_{\beta_n}^{B_2})Ax_n - A^*(I - J_{\beta_n}^{B_2})Az\rangle \\
&= \langle Ax_n - Az, (I - J_{\beta_n}^{B_2})Ax_n - (I - J_{\beta_n}^{B_2})Az\rangle \\
&\geq \|(I - J_{\beta_n}^{B_2})Ax_n\|^2 \\
&= 2f(x_n).
\end{aligned} \tag{20}$$

From Equation (20), Lemma 2 (iv) and the defining formulas for Algorithm 2

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - z\|^2 \\
&\leq \|x_n - \gamma_n g(x_n) - z\|^2 - \|x_{n+1} - x_n + \gamma_n g(x_n)\|^2 \\
&= \|x_n - z\|^2 + \gamma_n^2\|g(x_n)\|^2 - 2\gamma_n\langle x_n - z, g(x_n)\rangle - \|x_{n+1} - x_n + \gamma_n g(x_n)\|^2 \\
&\leq \|x_n - z\|^2 + \gamma_n^2\|g(x_n)\|^2 - 4\gamma_n f(x_n) - \|x_{n+1} - x_n + \gamma_n g(x_n)\|^2 \\
&= \|x_n - z\|^2 + \frac{\rho_n^2 f^2(x_n)}{(\|g(x_n)\|^2 + \theta_n)^2}\|g(x_n)\|^2 - \frac{4\rho_n f^2(x_n)}{\|g(x_n)\|^2 + \theta_n} \\
&\quad - \|x_{n+1} - x_n + \gamma_n g(x_n)\|^2 \\
&\leq \|x_n - z\|^2 + \frac{\rho_n^2 f^2(x_n)}{\|g(x_n)\|^2 + \theta_n} - \frac{4\rho_n f^2(x_n)}{\|g(x_n)\|^2 + \theta_n} - \|x_{n+1} - x_n + \gamma_n g(x_n)\|^2 \\
&= \|x_n - z\|^2 - \rho_n(4 - \rho_n)\frac{f^2(x_n)}{\|g(x_n)\|^2 + \theta_n} - \|x_{n+1} - x_n + \gamma_n g(x_n)\|^2.
\end{aligned} \tag{21}$$

This implies that, since $0 < \rho_n < 4$,

$$\|x_{n+1} - z\| \leq \|x_n - z\|. \tag{22}$$

Thus, $\lim_{n\to\infty} \|x_n - z\|$ exists. It follows that $\{x_n\}$ is bounded. Again, by Equation (21), we get

$$\rho_n(4 - \rho_n)\frac{f^2(x_n)}{\|g(x_n)\|^2 + \theta_n} \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2, \tag{23}$$

which yields by our assumptions that

$$\lim_{n\to\infty} \frac{f^2(x_n)}{\|g(x_n)\|^2} = 0. \tag{24}$$

By Lemma 3 (ii), it can be checked that g is a Lipschitzian mapping and thus $\{\|g(x_n)\|\}$ is bounded. Hence, we get $\lim_{n\to\infty} f(x_n) = 0$. This means

$$\lim_{n\to\infty} \|(I - J_{\beta_n}^{B_2})Ax_n\| = 0. \tag{25}$$

Furthermore, by Equation (21), we also have

$$\lim_{n\to\infty} \|x_{n+1} - x_n + \gamma_n g(x_n)\| = 0. \tag{26}$$

We note that

$$\gamma_n\|g(x_n)\| = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2 + \theta_n}\|g(x_n)\| \to 0, \text{ as } n \to \infty. \tag{27}$$

Hence, by Equations (26) and (27), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{28}$$

From Equation (25) and Lemma 2 (iii), we get

$$\lim_{n\to\infty} \|Ax_n - J_\beta^{B_2} Ax_n\| \le \lim_{n\to\infty} \|Ax_n - J_{\beta_n}^{B_2} Ax_n\| = 0, \tag{29}$$

for some $\beta > 0$ such that $\beta_n \ge \beta > 0$ for all $n \in \mathbb{N}$. From Equation (27), we see that

$$\begin{aligned} \|x_{n+1} - J_{\beta_n}^{B_2} x_n\| &= \|J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - J_{\beta_n}^{B_1} x_n\| \\ &\le \|x_n - \gamma_n g(x_n) - x_n\| \\ &= \gamma_n \|g(x_n)\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{30}$$

From Equations (28) and (30), we have

$$\begin{aligned} \|x_n - J_{\beta_n}^{B_1} x_n\| &= \|x_n - x_{n+1} + x_{n+1} - J_{\beta_n}^{B_1} x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - J_{\beta_n}^{B_1} x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{31}$$

Lemma 2 (iii) gives

$$\lim_{n\to\infty} \|x_n - J_\beta^{B_1} x_n\| \le \lim_{n\to\infty} \|x_n - J_{\beta_n}^{B_1} x_n\| = 0. \tag{32}$$

Since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x^* \in H_1$ with $x_{n_k} \rightharpoonup x^*$. We also have $Ax_{n_k} \rightharpoonup Ax^*$. By Equations (29) and (32), Lemmas 1 and 2 (ii), we obtain $x^* \in \Omega$. Using Lemma 5, we obtain that $\{x_n\}$ converges weakly to a solution in Ω. □

4. Strong Convergence Result

Theorem 6. *Assume that $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\theta_n\}$ satisfy the assumptions:*

(a1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(a2) $\inf_n \rho_n(4-\rho_n) > 0$;

(a3) $\lim_{n\to\infty} \theta_n = 0$;

(a4) $\liminf_{n\to\infty} \beta_n > 0$.

Then, $\{x_n\}$ defined by Algorithm 3 converges strongly to $z = P_\Omega u$ and $P_\Omega u$ is closest to u.

Algorithm 3:

Choose $x_1 \in H_1$ and let $u \in H_1$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$. Let $\{x_n\}$ be iteratively generated by

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)), \tag{33}$$

where

$$\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2 + \theta_n}, \quad 0 < \rho_n < 4, \quad 0 < \theta_n < 1, \quad \beta_n > 0 \tag{34}$$

and

$$f(x_n) = \frac{1}{2}\|(I - J_{\beta_n}^{B_2})Ax_n\|^2, \quad g(x_n) = A^*(I - J_{\beta_n}^{B_2})Ax_n. \tag{35}$$

Proof. Set $z = P_\Omega u \in \Omega$. Using the line of proof as for Theorem 5, we have

$$\begin{aligned} \|J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - z\|^2 \quad &\leq \quad \|x_n - z\|^2 - \rho_n(4-\rho_n)\frac{f^2(x_n)}{\|g(x_n)\|^2 + \theta_n} \\ &\quad -\|J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - x_n + \gamma_n g(x_n)\|^2. \end{aligned} \tag{36}$$

Then,

$$\begin{aligned} \|x_{n+1} - z\|^2 \quad &= \quad \|\alpha_n(u-z) + (1-\alpha_n)(J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - z)\|^2 \\ &\leq \quad (1-\alpha_n)\|J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{37}$$

Combining Equations (36) and (37), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 \quad &\leq \quad (1-\alpha_n)\|x_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{f^2(x_n)}{\|g(x_n)\|^2 + \theta_n} \\ &\quad -(1-\alpha_n)\|J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - x_n + \gamma_n g(x_n)\|^2 \\ &\quad +2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{38}$$

Next, we will show that $\{x_n\}$ is bounded. Again, using Equation (36),

$$\begin{aligned} \|x_{n+1} - z\| \quad &= \quad \|\alpha_n u + (1-\alpha_n)J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - z\| \\ &\leq \quad \alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\|. \end{aligned} \tag{39}$$

Thus, $\{x_n\}$ is bounded. Employing Lemma 6, from Equation (38), we set

$$\begin{aligned} s_n \quad &= \quad \|x_n - z\|^2; \\ \varphi_n \quad &= \quad 2\alpha_n\langle u - z, x_{n+1} - z\rangle; \\ \delta_n \quad &= \quad 2\langle u - z, x_{n+1} - z\rangle; \\ \lambda_n \quad &= \quad (1-\alpha_n)\rho_n(4-\rho_n)\frac{f^2(x_n)}{\|g(x_n)\|^2 + \theta_n} \\ &\quad +(1-\alpha_n)\|J_{\beta_n}^{B_1}(x_n - \gamma_n g(x_n)) - x_n + \gamma_n g(x_n)\|^2. \end{aligned} \tag{40}$$

Thus, Equation (38) reduces to the inequalities

$$\begin{aligned} s_{n+1} &\leq (1-\alpha_n)s_n + \alpha_n \delta_n, n \geq 1, &(41)\\ s_{n+1} &\leq s_n - \lambda_n + \varphi_n. &(42) \end{aligned}$$

Let $\{n_k\} \subseteq \{n\}$ be such that

$$\lim_{k\to\infty} \lambda_{n_k} = 0. \tag{43}$$

Then, we have

$$\begin{aligned} &\lim_{k\to\infty} \Big((1-\alpha_{n_k})\rho_{n_k}(4-\rho_{n_k})\frac{f^2(x_{n_k})}{\|g(x_{n_k})\|^2+\theta_{n_k}} \\ &\quad + (1-\alpha_{n_k})\|J^{B_1}_{\beta_{n_k}}(x_{n_k}-\gamma_{n_k}g(x_{n_k})) - x_{n_k} + \gamma_{n_k}g(x_{n_k})\|^2\Big) = 0, \end{aligned} \tag{44}$$

which, by using our assumptions, implies

$$\frac{f^2_{n_k}(x_{n_k})}{\|g_{n_k}(x_{n_k})\|^2} \to 0 \text{ as } k \to \infty, \tag{45}$$

and

$$\|J^{B_1}_{\beta_{n_k}}(x_{n_k}-\gamma_{n_k}g(x_{n_k})) - x_{n_k} + \gamma_{n_k}g(x_{n_k})\| \to 0 \text{ as } k\to\infty. \tag{46}$$

Since $\{\|g_{n_k}(x_{n_k})\|\}$ is bounded, it follows that $f_{n_k}(x_{n_k}) \to 0$ as $k\to\infty$. Thus, we get

$$\lim_{k\to\infty} \|(I - J^{B_1}_{\beta_{n_k}})Ax_{n_k}\| = 0. \tag{47}$$

As the same proof in Theorem 5, we can show that there is $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightharpoonup x^* \in \Omega$. From Lemma 2 (v), we obtain

$$\begin{aligned} \limsup_{k\to\infty}\langle u-z, x_{n_k}-z\rangle &= \lim_{i\to\infty}\langle u-z, x_{n_{k_i}}-z\rangle \\ &= \langle u-z, x^*-z\rangle \\ &\leq 0. \end{aligned} \tag{48}$$

We see that

$$\begin{aligned} \|x_{n_k+1}-x_{n_k}\| &= \|\alpha_{n_k}u + (1-\alpha_{n_k})J^{B_1}_{\beta_{n_k}}(x_{n_k}-\gamma_{n_k}g(x_{n_k})) - x_{n_k}\| \\ &\leq \alpha_{n_k}\|u-x_{n_k}\| + (1-\alpha_{n_k})\|J^{B_1}_{\beta_{n_k}}(x_{n_k}-\gamma_{n_k}g(x_{n_k})) - x_{n_k}\| \\ &\leq \alpha_{n_k}\|u-x_{n_k}\| + (1-\alpha_{n_k})\|J^{B_1}_{\beta_{n_k}}(x_{n_k}-\gamma_{n_k}g(x_{n_k})) - x_{n_k} + \gamma_{n_k}g(x_{n_k})\| \\ &\quad + (1-\alpha_{n_k})\gamma_{n_k}\|g(x_{n_k})\| \\ &\to 0 \text{ as } k\to\infty. \end{aligned} \tag{49}$$

From Equations (48) and (49), it follows that

$$\limsup_{k\to\infty}\langle u-z, x_{n_k+1}-z\rangle \leq 0. \tag{50}$$

Hence, we get

$$\limsup_{k\to\infty} \delta_{n_k} \leq 0. \tag{51}$$

Thus, $\{x_n\}$ converges strongly to $z = P_\Omega u$ by Lemma 6. □

5. Numerical Experiments

We present numerical experiments for our main results.

First, we give a comparison among Theorems 1–3 and 5 for a weak convergence theorem.

The following example is introduced in [10].

Example 1. *Let $B_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $B_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}, B_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix}, B_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & 2 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{52}$$

We aim to find $x^* = (x_1^*, x_2^*)^T \in \mathbb{R}^2$ such that $B_1(x^*) = (0,0)^T$ and $B_2(Ax^*) = (0,0,0)^T$. In this case, we know that $x_1^* = 1.5$ and $x_2^* = -0.5$.

We set $\gamma_n = 0.001$ in Theorem 1, $\gamma_n = \frac{1}{2n\|A\|^2}$ in Theorem 2, $\gamma_n = \frac{\delta}{2\|A\|^2}$ in Theorem 3 and $\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2+\theta_n}$, $\theta_n = \frac{1}{n^5}$ in Theorem 5. The stopping criterion is given by $\|x_n - x^*\|_2 < \varepsilon$.

We test by the following cases:

Case 1: $x_1 = [1,1]$, $\beta_n = 1$, $\rho_n = \frac{1.5n}{n+1}$, and $\delta = \frac{1}{3}$,
Case 2: $x_1 = [4,-2]$, $\beta_n = 2$, $\rho_n = \frac{3.5n}{n+1}$, and $\delta = \frac{1}{2}$,
Case 3: $x_1 = [-5,-3]$, $\beta_n = 3$, $\rho_n = 2.8$, and $\delta = \frac{1}{4}$,
Case 4: $x_1 = [-2,-7]$, $\beta_n = 4$, $\rho_n = 3.9$, and $\delta = \frac{1}{5}$.

From Table 1, we see that Theorem 5 using Algorithm 2 has a better convergence rate than other algorithms.

Table 1. Comparison for Theorems 1–3 and 5 for each case.

	Method	$\varepsilon = 10^{-4}$ CPU	Iter	$\varepsilon = 10^{-5}$ CPU	Iter
Case 1	Theorem 1	0.1091	3657	0.2763	7688
	Theorem 2	0.0078	131	0.0778	1272
	Theorem 3	0.0452	777	0.0699	1186
	Theorem 5	0.0017	66	0.0023	86
Case 2	Theorem 1	0.1565	4645	0.5374	8388
	Theorem 2	0.0276	454	0.3143	4357
	Theorem 3	0.0368	609	0.0487	860
	Theorem 5	0.0011	39	0.0028	48
Case 3	Theorem 1	0.1390	4572	0.3172	8219
	Theorem 2	0.0280	471	0.2936	4510
	Theorem 3	0.0635	1048	0.0905	1541
	Theorem 5	0.0014	45	0.0016	55
Case 4	Theorem 1	0.1189	4069	0.2849	7668
	Theorem 2	0.0213	345	0.2092	3307
	Theorem 3	0.0686	1159	0.1046	1768
	Theorem 5	0.0011	34	0.0011	43

Second, we give a comparison between Theorems 4 and 6 for a strong convergence theorem by using Example 1.

Choose $a_n = \frac{1}{n+1}$, $b_n = \frac{1}{5}$, $c_n = 1 - a_n - b_n$, $d_n = 0$ and $\gamma_n = \frac{1}{\|A\|^2+1}$ in Theorem 4 and set $\theta_n = \frac{1}{n^5}$, $\alpha_n = \frac{1}{n+1}$ and $\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2+\theta_n}$ in Theorem 6. In this case, we let $u = [2,2]$.

We test by the following cases:

Case 1: $x_1 = [1,1]$, $\beta_n = 1$ and $\rho_n = \frac{1.5n}{n+1}$,

Case 2: $x_1 = [4, -2]$, $\beta_n = 2$ and $\rho_n = \frac{3.5n}{n+1}$,
Case 3: $x_1 = [-5, -3]$, $\beta_n = 3$ and $\rho_n = 2.8$,
Case 4: $x_1 = [-2, -7]$, $\beta_n = 4$ and $\rho_n = 3.9$.

From Table 2, we observe that, in each case, the convergence behavior of Theorem 4 is worse than that Theorem 6.

Table 2. Comparing results for Theorems 4 and 6 for each case.

	Method	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
		CPU	**Iter**	**CPU**	**Iter**
Case 1	Theorem 4	0.0310	714	0.0871	2256
	Theorem 6	0.0068	273	0.0240	860
Case 2	Theorem 4	0.0225	685	0.0790	2166
	Theorem 6	0.0038	142	0.0117	448
Case 3	Theorem 4	0.0204	675	0.0727	2135
	Theorem 6	0.0043	156	0.0132	492
Case 4	Theorem 4	0.0241	671	0.0677	2120
	Theorem 6	0.0038	140	0.0156	441

6. Split Feasibility Problem

Let H_1 and H_2 be real Hilbert spaces. We next study the split feasibility problem (SFP) that is to seek $x^* \in H_1$ such that

$$x^* \in C \text{ and } Ax^* \in Q, \tag{53}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator with the adjoint operator A^*. Many authors introduced various algorithms for solving the SFP [16–19].

Let H be a Hilbert space and let $g : H \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. The subdifferential ∂g of g is defined by

$$\partial g(x) = \{z \in H : g(x) + \langle z, y - x\rangle \leq g(y), \forall y \in H\} \tag{54}$$

for all $x \in H$. Let C be a nonempty closed convex subset of H, and ι_C be the indicator function of C defined by

$$\iota_C x = \begin{cases} 0 & x \in C, \\ \infty & x \notin C. \end{cases} \tag{55}$$

The normal cone $N_C u$ of C at u is defined by

$$N_C u = \{z \in H : \langle z, v - u\rangle \leq 0, \forall v \in C\}. \tag{56}$$

Then, ι_C is a proper, lower semicontinuous and convex function on H. See [20,21]. Moreover, the subdifferential $\partial\iota_C$ of ι_C is a maximal monotone mapping. In this connection, we can define the resolvent $J_\lambda^{\partial\iota_C}$ of $\partial\iota_C$ for $\lambda > 0$ by

$$J_\lambda^{\partial\iota_C} x = (I + \lambda\partial\iota_C)^{-1} x \tag{57}$$

for all $x \in H$. Hence, we see that

$$\begin{aligned} \partial\iota_C x &= \{z \in H : \iota_C x + \langle z, y - x\rangle \leq \iota_C y, \forall y \in H\} \\ &= \{z \in H : \langle z, y - x\rangle \leq 0, \forall y \in C\} \\ &= N_C x \end{aligned} \tag{58}$$

for all $x \in C$. Hence, for each $\beta > 0$, we obtain the following relation:

$$\begin{aligned} u = J_\beta^{\partial \iota_C} x \quad &\Leftrightarrow \quad x \in u + \beta \partial \iota_C u \\ &\Leftrightarrow \quad x - u \in \beta N_C u \\ &\Leftrightarrow \quad \langle x - u, y - u \rangle \le 0, \forall y \in C \\ &\Leftrightarrow \quad u = P_C x. \end{aligned} \tag{59}$$

Consequently, we obtain the following results which are deduced from Algorithm 2.

Theorem 7. *Assume that* $\inf_n \rho_n(4 - \rho_n) > 0$ *and* $\lim_{n \to \infty} \theta_n = 0$. *Choose* $x_1 \in H_1$ *and let* $\{x_n\}$ *be defined by*

$$x_{n+1} = P_C(x_n - \gamma_n g(x_n)), \tag{60}$$

where

$$\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2 + \theta_n}, \ \ 0 < \rho_n < 4, \ \ 0 < \theta_n < 1 \tag{61}$$

and

$$f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2, \ \ g(x_n) = A^*(I - P_Q)Ax_n. \tag{62}$$

Then, $\{x_n\}$ *converges weakly to a solution in* Ω.

By Theorem 1, we obtain the result of Byrne et al. [6].

Theorem 8. *Let* $\{x_n\}$ *be generated by*

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \ n \in \mathbb{N}, \tag{63}$$

where H_1 *and* H_2 *are Hilbert spaces,* $A : H_1 \to H_2$ *be a bounded and linear operator and* $\gamma \in (0, \frac{2}{\|A\|^2})$. *Then,* $\{x_n\}$ *converges weakly to* $x^* \in \Omega$.

Using Chuang's results in Algorithm 1, we have

Theorem 9. *Let* H_1 *and* H_2 *be infinite dimensional Hilbert spaces,* $A : H_1 \to H_2$ *be a bounded and linear operator. Choose* $\delta \in (0, 1)$ *and* $\{\gamma_n\} \subseteq (0, \frac{\delta}{\|A\|^2})$ *with* $\inf_{n \in \mathbb{N}} \gamma_n > 0$. *Choose* $x_1 \in H_1$. *For* $n \in \mathbb{N}$, *set* y_n *as*

$$y_n = P_C(x_n - \gamma_n A^*(I - P_Q)Ax_n), \tag{64}$$

where $\gamma_n > 0$ *satisfies*

$$\gamma_n \|A^*(I - P_Q)Ax_n - A^*(I - P_Q)Ay_n\| \le \delta \|x_n - y_n\|, \ \ 0 < \delta < 1. \tag{65}$$

Construct x_{n+1} *by*

$$x_{n+1} = P_C(x_n - \alpha_n D(x_n, \gamma_n)), \tag{66}$$

where

$$D(x_n, \gamma_n) = x_n - y_n + \gamma_n(A^*(I - P_Q)Ay_n - A^*(I - P_Q)Ax_n) \tag{67}$$

and

$$\alpha_n = \frac{\langle x_n - y_n, D(x_n, \gamma_n) \rangle}{\|D(x_n, \gamma_n)\|^2}. \tag{68}$$

Then, the sequence $\{x_n\}$ *converges weakly to* $x^* \in \Omega$.

From Algorithm 3 and Theorem 6, we have

Theorem 10. *Assume that $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\theta_n\}$ satisfy the assumptions:*

(a1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(a2) $\inf\limits_{n} \rho_n(4-\rho_n) > 0$;

(a3) $\lim\limits_{n\to\infty} \theta_n = 0$.

Choose $x_1 \in H_1$ and define $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)P_C(x_n - \gamma_n g(x_n)), \tag{69}$$

where

$$\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2 + \theta_n}, \quad 0 < \rho_n < 4, \;\; 0 < \theta_n < 1, \;\; 0 < \alpha_n < 1, \tag{70}$$

and

$$f(x_n) = \frac{1}{2}\|(I-P_Q)Ax_n\|^2, \;\; g(x_n) = A^*(I-P_Q)Ax_n. \tag{71}$$

Then, $\{x_n\}$ converges strongly to $z = P_\Omega u$.

We also have the following result.

Theorem 11. *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + b_n + c_n + d_n = 1$ and $0 < a_n < 1$ for each $n \in \mathbb{N}$. Let $\{v_n\}$ be a bounded sequence in H_1. Let $u \in H_1$ be fixed and $\{\gamma_n\} \subseteq (0, \frac{2}{\|A\|^2+1})$. Let $\{x_n\}$ be defined by*

$$x_{n+1} = a_n u + b_n x_n + c_n P_C(x_n - \gamma_n A^*(I-P_Q)Ax_n) + d_n v_n \tag{72}$$

for each $n \in \mathbb{N}$. Suppose that

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{d_n}{a_n} = 0$; $\sum_{n=1}^{\infty} a_n = \infty$; $\sum_{n=1}^{\infty} d_n < \infty$;

(ii) $\liminf_{n\to\infty} c_n\gamma_n > 0$ *and* $\liminf_{n\to\infty} b_n c_n > 0$.

Then, $\lim_{n\to\infty} x_n = x^*$, where $x^* = P_\Omega u$, $A : H_1 \to H_2$ be a bounded and linear operator. Then, $\{x_n\}$ converges strongly to a point in Ω.

7. Applications to Compressed Sensing

In signal processing, we consider the following linear equation:

$$y = Ax + \varepsilon, \tag{73}$$

where $x \in \mathbb{R}^N$ is a sparse vector that has m nonzero components, $y \in \mathbb{R}^M$ is the observed data with noisy ε, and $A : \mathbb{R}^N \to \mathbb{R}^M$ $(M < N)$. It can be seen that Equation (73) relates to the LASSO problem [22]

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|y - Ax\|_2^2 \text{ subject to } \|x\|_1 \le t, \tag{74}$$

where $t > 0$. In particular, if $C = \{x \in \mathbb{R}^N : \|x\|_1 \le t\}$ and $Q = \{y\}$, then the LASSO problem can be considered as the SFP Equation (53).

The vector $x \in \mathbb{R}^N$ is generated by the uniform distribution in $[-2, 2]$ with m nonzero components. Let A be an $M \times N$ matrix that is generated by the normal distribution with mean zero and the variance one. The observed data y is generated by white Gaussian noise with signal-to-noise ratio (SNR)40. The process is started with $t = m$ and initial point $x_1 = 0$.

The stopping error is defined by

$$E_n = \frac{1}{N}\|x_n - x\|_2^2 < \kappa, \tag{75}$$

where x_n is an estimated signal of x.

We give some numerical results of Theorems 7–9. Choose $\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2+\theta_n}$, $\rho_n = 3$, $\theta_n = \frac{1}{n^5}$ in Theorem 7 and $\gamma_n = \frac{\delta}{2\|A\|^2}$ in Theorem 8 and $\delta = 0.8$, $\gamma_n = \frac{\delta}{2\|A\|^2}$ in Theorem 9.

Tables 3 and 4 show that both the number of iterations and the CPU time in our algorithm in Theorem 7 are less than algorithms in Theorems 8 and 9 have in their computations. Next, we test numerical experiments in signal recovery in the case $N = 512$, $M = 256$ and $N = 2048$, $M = 1024$, respectively.

Table 3. Numerical results for the LASSO problem in case $M = 256$, $N = 512$.

***m*-Sparse**	**Method**	$\kappa = 10^{-3}$ **CPU**	**Iter**	$\kappa = 10^{-4}$ **CPU**	**Iter**
$m = 10$	Theorem 8	0.9662	44	3.6208	132
	Theorem 9	1.3204	58	4.2151	170
	Theorem 7	0.0054	26	0.0111	63
$m = 15$	Theorem 8	1.3082	57	2.8470	124
	Theorem 9	1.8984	84	3.7938	170
	Theorem 7	0.0058	36	0.0099	72
$m = 20$	Theorem 8	1.4928	65	3.5994	161
	Theorem 9	2.7294	122	5.7801	251
	Theorem 7	0.0070	42	0.0143	99
$m = 25$	Theorem 8	2.2008	98	6.0600	275
	Theorem 9	4.1730	183	18.6269	824
	Theorem 7	0.0107	67	0.0323	227

Table 4. Numerical results for the LASSO problem in case $M = 2048$, $N = 1024$.

***m*-Sparse**	**Method**	$\kappa = 10^{-3}$ **CPU**	**Iter**	$\kappa = 10^{-4}$ **CPU**	**Iter**
$m = 30$	Theorem 8	47.6530	41	119.9776	101
	Theorem 9	67.6869	57	157.1087	134
	Theorem 7	0.0807	25	0.1899	58
$m = 40$	Theorem 8	47.7347	41	151.0891	117
	Theorem 9	93.1898	79	306.8623	240
	Theorem 7	0.1007	31	0.2880	82
$m = 50$	Theorem 8	65.1771	55	136.1508	115
	Theorem 9	99.0021	83	188.9366	158
	Theorem 7	0.1227	35	0.2203	67
$m = 60$	Theorem 8	76.7457	64	163.8805	138
	Theorem 9	127.5520	106	209.5990	177
	Theorem 7	0.1401	43	0.2449	75

Finally, we discuss the strong convergence of Theorems 10 and 11. We set $a_n = \frac{1}{n+1}$, $b_n = \frac{1}{5}$, $c_n = 1 - a_n - b_n$, $d_n = 0$ and $\gamma_n = \frac{1}{\|A\|^2+1}$ in Theorem 11 and set $\rho_n = 2$, $\theta_n = \frac{1}{n^5}$, $\alpha_n = \frac{1}{n+1}$ and $u = [1, 1, \ldots, 1]$ in Theorem 10.

Tables 5 and 6 show that our proposed algorithm in Theorem 10 has a better convergence behavior than the algorithm defined in Theorem 11 in iterations and CPU time.

Table 5. Numerical results for the LASSO problem in case $M = 512$, $N = 256$.

m-Sparse	Method	$\kappa = 10^{-3}$ CPU	Iter	$\kappa = 10^{-4}$ CPU	Iter
$m = 10$	Theorem 11	5.8869	237	28.2850	863
	Theorem 10	0.0296	157	0.1232	551
$m = 15$	Theorem 11	6.1204	245	38.9049	1561
	Theorem 10	0.0260	155	0.1550	950
$m = 20$	Theorem 11	9.3238	376	116.7590	4613
	Theorem 10	0.0377	233	0.4484	2730
$m = 25$	Theorem 11	9.3206	379	32.8208	1255
	Theorem 10	0.0420	252	0.1578	858

Table 6. Numerical results for the LASSO problem in case $M = 2048$, $N = 1024$.

m-Sparse	Method	$\kappa = 10^{-3}$ CPU	Iter	$\kappa = 10^{-4}$ CPU	Iter
$m = 10$	Theorem 11	131.3365	111	578.6894	490
	Theorem 10	0.2419	74	0.9969	305
$m = 20$	Theorem 11	184.8031	157	616.7051	526
	Theorem 10	0.3274	101	1.0929	339
$m = 30$	Theorem 11	262.3976	224	1.3220×10^3	503
	Theorem 10	0.4633	141	1.4516	339
$m = 40$	Theorem 11	282.6013	237	1.6136×10^3	1326
	Theorem 10	0.5393	158	2.6758	791

We next provide some experiments in recovering the signal.

From Figures 1–4, we observe that our algorithms can be applied to solve the LASSO problem. Moreover, the proposed algorithms have a better convergence behavior than other methods.

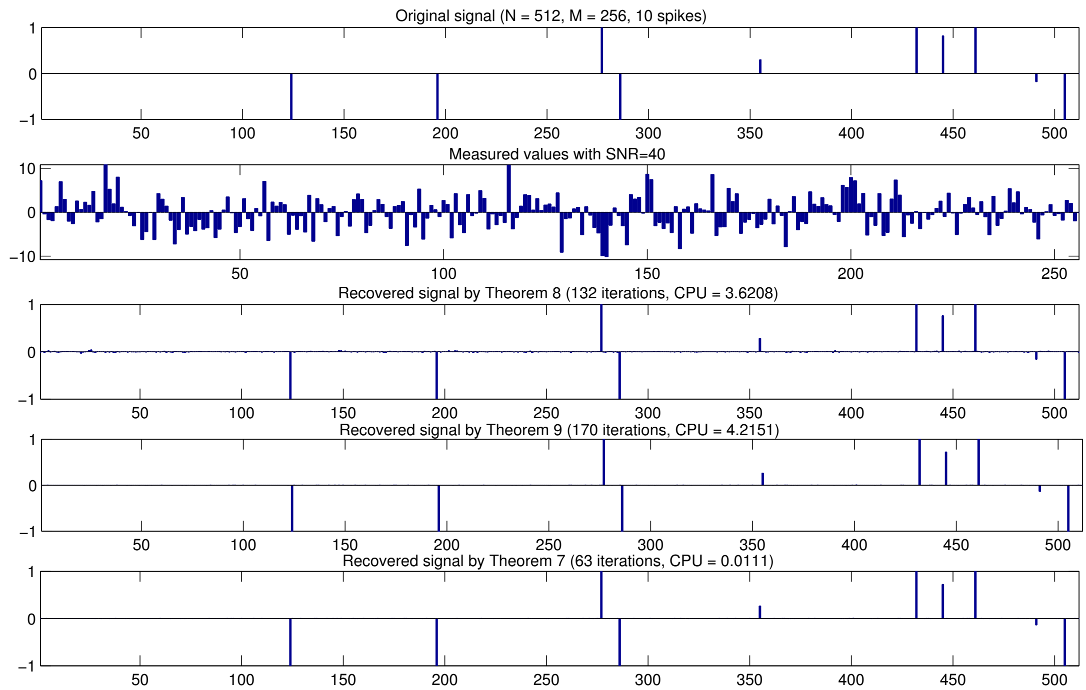

Figure 1. From top to bottom: original signal, measured values, recovered signal by Theorem 8, Theorem 9 and Theorem 7 with $N = 512$, $M = 256$ and $m = 10$.

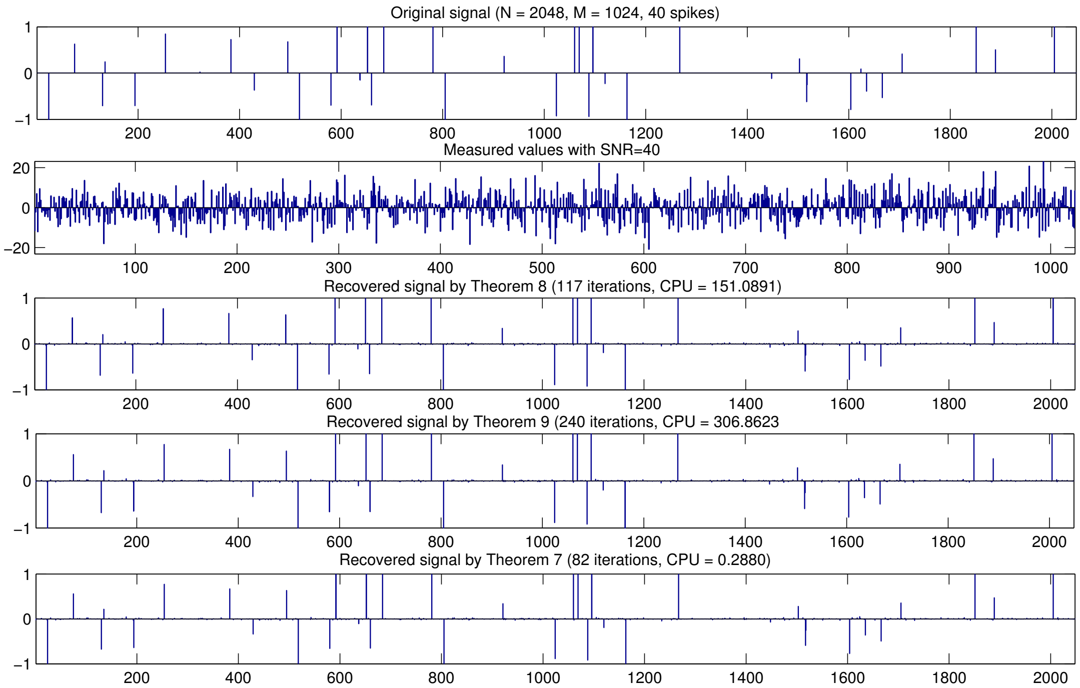

Figure 2. From top to bottom: original signal, measured values, recovered signal by Theorem 8, Theorem 9 and Theorem 7 with $N = 2048$, $M = 1024$ and $m = 40$.

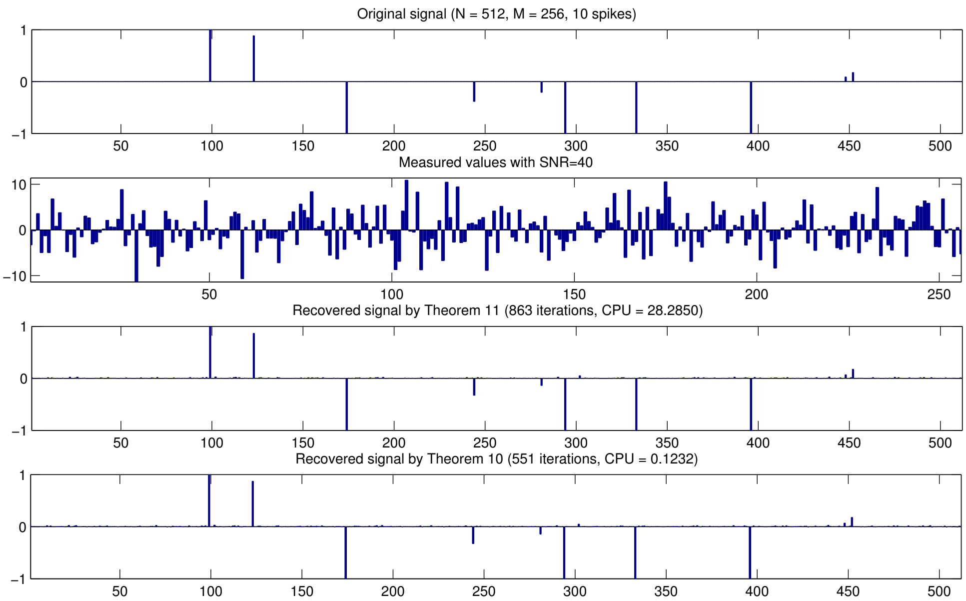

Figure 3. From top to bottom: original signal, measured values, recovered signal by Theorem 11 and Theorem 10 with $N = 512$, $M = 256$ and $m = 10$.

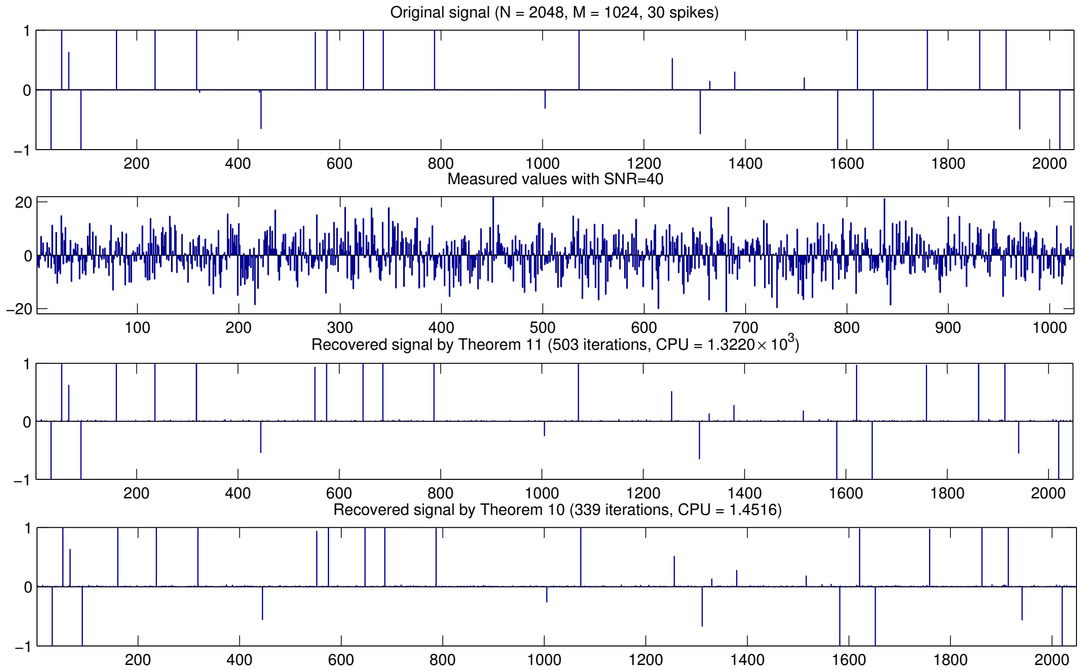

Figure 4. From top to bottom: original signal, measured values, recovered signal by Theorem 11 and Theorem 10 with $N = 2048$, $M = 1024$ and $m = 30$.

8. Conclusions

In the present work, we introduce a new approximation algorithm with a new stepsize that involves the self adaptive method for SVIP. The stepsize does not use the Lipschitz constant and the norm of operators in computing. We show its convergence analysis, which was proved under some suitable assumptions. The numerical results showed the efficiency of our algorithms. It is reported that the performance of our algorithms outruns those of Byrne et al. [6] and Chuang [10,11] through experiments.

Author Contributions: S.S.; Funding acquisition and Supervision, S.K. and P.C.; Writing-review & editing and Software.

Funding: This research was funded by Chiang Mai University.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Martinet, B. Régularisation d'inéquations variationnelles par approximations successives. *Revue Francaise d'Informatique et de Recherche Operationelle* **1970**, *4*, 154–158.
2. Rockafellar, R.T. Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.* **1976**, *14*, 877–898. [CrossRef]
3. Moudafi, A. Split monotone variational inclusions. *J. Optim. Theory Appl.* **2011**, *150*, 275–283. [CrossRef]
4. Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer Algor.* **1994**, *8*, 221–239. [CrossRef]
5. Byrne, C. Iterative oblique projection onto convex sets and the split feasibility problem. *Inverse Probl.* **2002**, *18*, 441–453. [CrossRef]
6. Byrne, C.; Censor, Y.; Gibali, A. Weak and strong convergence of algorithms for the split common null point problem. *J. Nonlinear Convex Anal.* **2011**, *13*, 759–775.
7. Censor, Y.; Bortfeld, T.; Martin, B. A unified approach for inversion problems in intensity-modulated radiation therapy. *Phys. Med. Biol.* **2006**, *51*, 2353–2365. [CrossRef] [PubMed]

8. López, G.; Martín-Márquez, V.; Xu, H.K. Iterative algorithms for the multiple-sets split feasibility problem. In *Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems*; Censor, Y., Jiang, M., Wang, G., Eds.; Medical Physics Publishing: Madison, WI, USA, 2010; pp. 243–279.
9. Stark, H. *Image Recovery: Theory and Applications*; Academic Press: San Diego, CA, USA, 1987.
10. Chuang, C.S. Algorithms with new parameter conditions for split variational inclusion problems in Hilbert spaces with application to split feasibility problem. *Optimization* **2016**, *65*, 859–876. [CrossRef]
11. Chuang, C.S. Strong convergence theorems for the split variational inclusion problem in Hilbert spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 350. [CrossRef]
12. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Cambridge studies in advanced mathematics; Cambridge University Press: Cambridge, UK, 1990; Volume 28.
13. Marino, G.; Xu, H.K. Convergence of generalized proximal point algorithms. *Commun. Pure Appl. Anal.* **2004**, *3*, 791–808.
14. Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–597. [CrossRef]
15. He, S.; Yang, C. Solving the variational inequality problem defined on intersection of finite level sets. *Abstr. Appl. Anal.* **2013**. [CrossRef]
16. Dang, Y.; Gao, Y. The strong convergence of a KM-CQ-like algorithm for a split feasibility problem. *Inverse Prob.* **2011**, *27*, 015007. [CrossRef]
17. Wang, F.; Xu, H.K. Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem. *J. Inequal. Appl.* **2010**, *2010*, 102085. [CrossRef]
18. Xu, H.K. Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. *Inverse Prob.* **2010**, *26*, 105018. [CrossRef]
19. Zhao, J.; Zhang, Y.; Yang, Q. Modified projection methods for the split feasibility problem and multiple-sets feasibility problem. *Appl. Math. Comput.* **2012**, *219*, 1644–1653. [CrossRef]
20. Agarwal, R.P.; O'Regan, D.; Sahu, D.R. *Fixed Point Theory for Lipschitzian-Type Mapping with Applications*; Springer: New York, NY, USA, 2009; Volume 6.
21. Takahashi, W. *Introduction to Nonlinear and Convex Analysis*; Yokohama Tokohama Publishers, Inc.: Yokohama, Japan, 2009.
22. Tibshirani, R. Regression shrinkage and selection via the lasso. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **1996**, *58*, 267–288. [CrossRef]